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New Tanimoto Similarity and Distance Measures for Pythagorean Fuzzy Sets with Applications

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Abstract: Currently, Pythagorean fuzzy sets (PFSs) have been widely applied in various fields due to their substantial advantages in expressing and dealing with uncertainty. However, measuring the similarity and difference between PFSs effectively remains an unresolved issue. Inspired by Tanimoto similarity, we propose a novel set of similarity and distance measures for PFSs. We delve into the theoretical properties of the proposed measures and compare it with existing PFSs measures. Numerous numerical examples validate their rationality and effectiveness. Furthermore, our experimental findings suggest that in contrast to existing measures, the introduced measures successfully circumvent various counter-intuitive issues encountered by current measures, and yield more pronounced outcomes in the discrimination of different fuzzy sets. This enhances the uniqueness and superiority of our measures. Finally, we developed two decision models based on the proposed measures and validated their applicability in three applications.

Keywords: Pythagorean fuzzy sets; Tanimoto similarity measure; distance measure; pattern recognition; medical diagnosis; multi-attribute decision-making

1. Introduction

Decision-making is a common behavior. Ideally, the data required for decision-making would be precise and comprehensive; unfortunately, in reality, the majority of the information we have for decision-making is either uncertain or incomplete, implying the presence of uncertainty [1,2]. Uncertainty is the primary factor impeding correct decision-making, and it has become increasingly pervasive in various fields because of the complexity of practical issuses and the natural bounds of understanding. This ambiguity always manifests itself randomly and in uncertain manners, which leads to the difficulties of accurate depiction and decision-making process. Therefore, how to accurately describe uncertainty has become a paramount challenge [3,4]. Currently, numerous innovative theories and techniques have been suggested for depicting uncertainty in real-world scenarios, including fuzzy sets [5], Intuitionistic fuzzy sets (IFSs) [6], evidence theory [7,8], rough sets [9–11] and R-numbers [12]. Among the array of theories, the IFSs stands out for its proficiency in capturing ambiguity and indeterminacy through the delineation of membership and nonmembership intervals for elements. This distinctive characteristic has rendered IFSs an indispensable instrument across various disciplines for grappling with uncertainties [13–15]. However, in some cases, the condition that the sum of the membership degree and nonmembership degree required by the International Federations must be less than or equal to 1 may be violated. Inspired by this, Yager [16] first introduced Pythagorean fuzzy sets (PFSs) as an evolution of IFSs in 2013. The hallmark of this model is its employment of the Pythagorean membership function, which introduces a degree of hesitation into the parameters of IFSs, including their membership degree, nonmembership degree and indeterminacy degree. PFSs stipulate that the sum of squares of membership and nonmembership is not more than 1, making it more effective in representing uncertain information. Subsequently, Zhang and Xu [17] introduced the concept of Pythagorean fuzzy numbers (PFNs) and subsequently proposed the PF-TOPSIS method for multi-attribute decision-making (MADM) problems. Wei and Gao [18,19] developed the Pythagorean fuzzy interactive aggregation operator based on arithmetic and topological measures. As a means of simplifying supplier selection, Li [20] proposed a fuzzy Hamy mean operator. Gao [21] proposed a Pythagorean fuzzy Hamacher priority aggregation operator for MADM based on existing priority aggregation operators [22]. Wei and Lu [23] developed a Pythagorean fuzzy Maclaurin symmetric mean operator, which enabled the capture of relationships between multiple parameters.

A similarity measure is a mathematical function that assesses the degree of resemblance between two objects. In the context of fuzzy sets, similarity measures are of utmost importance in solving clustering, classification and decision-making problems [24–26]. At present, various similarity measures have been proposed for Pythagorean fuzzy sets:

- Wei [27] proposed some similarity measures between PFSs based on cosine function. To address the problem of MADM, Garg [28] developed a PFSs-based correlation measure and Wang [29] proposed a generalized Dice similarity measure. Zhang [30] introduced exponential functions and proposed several new similarity measures for Pythagorean fuzzy sets. Li [31] proposed a new similarity measure for PFSs based on spherical arc distance from a geometric perspective and constructed a MADM method in a Pythagorean fuzzy environment. Hussian and Yang [32] proposed new similarity measures for PFSs based on the Hausdorff measures and applied them to solve MADM problems. Li and Lu [33] proposed new similarity measures by extending the Hamming distance and Hausdorff distance. Zeng, Li and Yin [34] developed a novel similarity measure for PFSs and applied it to analyzing decision makers' preferences. Zhang [35] presented a similarity measure and proposed a method for approximately calculating experts' weights when their weights are entirely unknown. In addition, for more similarity measures for PFSs, please refer to the papers[36–38].

The concept of a similarity measure is used to express the degree of resemblance between individuals, while a distance measure represents the degree of divergence between individuals:

- Hussian and Yang [32] developed a measure to calculate the distance between PFSs using the Hasudorff measures. Li and Lu [33] proposed some novel distance while Xu [17] proposed a Hamming distance measure. Moreover, Ren, Xu and Gou [39] proposed a novel distance measure that builds upon the Euclidean distance model. Chen [40] has developed a novel method based on VIKOR for multi-criteria decision-making tasks involving Pythagorean fuzzy information. Simultaneously, Li and Zeng [41] proposed multiple distance measures after considering the four parameters of PFSs. Zeng, Li and Yin [34] proposed a series of modified distance measures by taking into account the importance of incorporating ambiguity into the equation. In addition, Chen [42] defined a novel generalized distance measure and devised a distance-based compromise method for decision analysis based on multiple criteria. There are more existing distance measures for PFSs [43–45].

In many cases, similarity and distance measures are used interchangeably, with distance being seen as the inverse of similarity, and vice versa. In practical applications, for distance calculations, the shortest distance is observed between the closest observation points, while for similarity calculations, the highest level of similarity is observed between the closest observation points.

Currently, the utilization of PFSs is exceedingly well-received [46–52], yet the existing similarity and distance measures may yield counter-intuitive outcomes in certain situations. Therefore, the exploration and utilization of similarity and distance measures for PFSs is a highly valuable area of inquiry. This paper proposes several novel similarity measures and distance measures for PFSs, inspired by Tanimoto similarity [53]. Our examples illustrate the properties of these measures. As well, a pair of models is also proposed for using these measures in patterns recognition, medical diagnosis, and MADM issues in Pythagorean fuzzy environments. As part of our experiment, we compared our proposed measures with a number of existing ones. In addition to overcoming numerous counter-intuitive scenarios in existing measures, our measures provide more differentiated measurement results when distinguishing between PFSs. These qualities exemplify the superior nature of our proposed measures.

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The main contributions of this paper are as follows:

(1) We propose several similarity and distance measures for PFSs based on the Tanimoto similarity measure and prove their basic properties;

(2) Two models based on the Tanimoto measures are proposed and applied to pattern recognition, medical diagnosis and MADM problems to verify their effectiveness;

(3) Several experiments demonstrate that the proposed measures overcome counter-intuitive limitations of existing measures for PFSs and tend to produce more significant results when distinguishing PFSs.

Section 2 provides a brief review of the basic ideas and mathematical foundations of fuzzy sets. Several new similarity and distance measures for fuzzy sets are introduced in Section 3 and their mathematical properties are established. Section 4 presents two models based on the proposed measures to address pattern recognition, medical diagnosis and MADM problems, respectively. Section 5 draws conclusions and provides future research directions.

2. Preliminaries

In this section, some basic concepts related to fuzzy sets, Tanimoto measure and several existing measures for PFSs will be given.

2.1. Intuitionistic Fuzzy Sets

Definition 1 ([6])**.** *We utilize the symbol Z to denote a finite set. An Intuitionistic fuzzy set I is given by:*

$$I = \{\langle z, \rho_I(z), \sigma_I(z)\rangle : z \in Z\} \tag{1}$$

where $\rho_I(z) : Z \to [0,1]$ *signifies the membership degree of z, and* $\sigma_I(z) : Z \to [0,1]$ *expresses the nonmembership degree of z.* $\forall z$, $\rho_I(z)$ *and* $\sigma_I(z)$ *satisfy:*

$$0 \leq \rho_I(z) + \sigma_I(z) \leq 1 \tag{2}$$

$\forall z$, *the indeterminacy degree of z is:*

$$\theta_I(z) = 1 - \rho_I(z) - \sigma_I(z) \tag{3}$$

2.2. Pythagorean fuzzy sets

Definition 2 ([16])**.** *The Pythagorean fuzzy set P is defined as:*

$$P = \{\langle z, \rho_P(z), \sigma_P(z)\rangle : z \in Z\} \tag{4}$$

where $\rho_P(z) : Z \to [0,1]$ *and* $\sigma_P(z) : Z \to [0,1]$ *denote, respectively, the membership degree and nonmembership degree of z.* $\forall z$, $\rho_P(z)$ *and* $\sigma_P(z)$ *satisfy:*

$$0 \leq \rho_P^2(z) + \sigma_P^2(z) \leq 1 \tag{5}$$

$\forall z$, *the indeterminacy degree of z is:*

$$\theta_P(z) = \sqrt{1 - \rho_P^2(z) - \sigma_P^2(z)} \tag{6}$$

Table 1 presents a number of classic measures between PFSs.

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Table 1. Exiting similarity and distance measures for PFSs.

Ref.	*Measures*
Zhang [30]	$SM^1_{PFS}(F,G)=\frac{1}{n}\sum_{i=1}^{n}\left[2^{1-\left(\left\|\rho_F^2(z_i)-\rho_G^2(z_i)\right\|\vee\left\|\sigma_F^2(z_i)-\sigma_G^2(z_i)\right\|\right)}-1\right]$
Zhang [30]	$SM^2_{PFS}(F,G)=\frac{1}{n}\sum_{i=1}^{n}\left[2^{1-\frac{1}{2}\left(\left\|\rho_F^2(z_i)-\rho_G^2(z_i)\right\|+\left\|\sigma_F^2(z_i)-\sigma_G^2(z_i)\right\|\right)}-1\right]$
Wang [29]	$D_{PFS}(F,G)=\frac{1}{n}\sum_{i=1}^{n}\frac{2\left(\rho_F^2(z_i)\rho_G^2(z_i)+\sigma_F^2(z_i)\sigma_G^2(z_i)\right)}{\left(\left(\rho_F^2(z_i)+\sigma_F^2(z_i)\right)+\left(\rho_G^2(z_i)+\sigma_G^2(z_i)\right)\right)}$
Li [31]	$G_{PFS}(F,G)=1-\frac{2\left(\arccos\left(\rho_F^2(z_i)\rho_G^2(z_i)+\sigma_F^2(z_i)\sigma_G^2(z_i)+\theta_F^2(z_i)\theta_G^2(z_i)\right)\right)}{\pi}$
Wei [27]	$C^1_{PFS}(F,G)=\frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\left(\left\|\rho_F^2(z_i)-\rho_G^2(z_i)\right\|\vee\left\|\sigma_F^2(z_i)-\sigma_G^2(z_i)\right\|\right)\right]$
Wei [27]	$C^2_{PFS}(F,G)=\frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\left(\left\|\rho_F^2(z_i)-\rho_G^2(z_i)\right\|+\left\|\sigma_F^2(z_i)-\sigma_G^2(z_i)\right\|\right)\right]$
Li [31]	$DH_{PFS}(F,G)=$ $\frac{1}{5n}\sum_{i=1}^{n}\left(\left\|\rho_F(z_i)-\rho_G(z_i)\right\|+\left\|\sigma_F(z_i)-\sigma_G(z_i)\right\|+\left\|\theta_F(z_i)-\theta_G(z_i)\right\|+\left\|d_F(z_i)-d_G(z_i)\right\|+\left\|l_F(z_i)-l_G(z_i)\right\|\right);$ $\varphi_P=\cos\theta_P;\ \mu_P=\arctan\frac{\sigma_P(z_i)}{\rho_P(z_i)};\ d_P=1-\frac{2\mu_P}{\pi};\ l_P=\frac{2\varphi_P}{\pi}$
Ejegwa [54]	$DE_{PFS}(F,G)=$ $\frac{1}{4n}\sum_{i=1}^{n}\left[\left\|\rho_F(z_i)-\rho_G(z_i)\right\|+\left\|\left\|\rho_F(z_i)-\sigma_F(z_i)\right\|-\left\|\rho_G(z_i)-\sigma_G(z_i)\right\|\right\|+\left\|\left\|\rho_F(z_i)-\theta_F(z_i)\right\|-\left\|\rho_G(z_i)-\theta_G(z_i)\right\|\right\|\right]$

2.3. Tanimoto measure

Definition 3 ([53])**.** *For two probability sets $A=\{a_1,a_2,\dots,a_n\}$ and $B=\{b_1,b_2,\dots,b_n\}$, the Tanimoto measure can be defined as:*

$$T(A,B)=\frac{\sum_{i=1}^{n}a_ib_i}{\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2-\sum_{i=1}^{n}a_ib_i}\tag{7}$$

3. Some novel tanimoto similarity and distance measures for PFSs

This section presents the Pythagorean fuzzy version of the Tanimoto measure, which includes both similarity and distance measures. Furthermore, we demonstrate several outstanding properties satisfied by the proposed measures. Comparative experiments with existing measures are conducted to verify their effectiveness and superiority.

3.1. Novel similarity measures

Definition 4. *For two PFSs, $F=\{(z,[\rho_F(z_i,),\sigma_F(z_i)]):z_i\in Z\}$ and $G=\{(z,[\rho_G(z_i,),\sigma_G(z_i)]):z_i\in Z\}$, where $Z=\{z_1,z_2,\dots,z_n\}$, the Tanimoto similarity measure for them can be given by:*

$$T_{PFS}(F,G)=\frac{\sum_{i=1}^{n}\left(\rho_F^2(z_i)\rho_G^2(z_i)+\sigma_F^2(z_i)\sigma_G^2(z_i)\right)}{\sum_{i=1}^{n}\left(\rho_F^4(z_i)+\rho_G^4(z_i)-\rho_F^2(z_i)\rho_G^2(z_i)\right)+\sum_{i=1}^{n}\left(\sigma_F^4(z_i)+\sigma_G^4(z_i)-\sigma_F^2(z_i)\sigma_G^2(z_i)\right)}\tag{8}$$

Theorem 1. *For any two PFSs F and G, the $T_{PFS}(F,G)$ satisfies:*

1. $0\le T_{PFS}(F,G)\le 1$;
2. $T_{PFS}(F,G)=T_{PFS}(G,F)$;
3. $T_{PFS}(F,G)=1$, *if* $F=G(\rho_F(z_i)=\rho_G(z_i),\sigma_F(z_i)=\sigma_G(z_i))$.

Proof of Theorem 1. 1. Considering the ith item of the summation in Equation 8:

$$T_{PFS}(F,G) = \frac{\rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i)}{\rho_F^4(z_i) + \rho_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^4(z_i) + \sigma_G^4(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i)} \tag{9}$$

According to $0 \leq \rho(z_i) \leq 1$ and $0 \leq \sigma(z_i) \leq 1$, we can get $\rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i) \geq 0$. According to the inequality $a^2 + b^2 \geq 2ab$, $\rho_F^4(z_i) + \rho_G^4(z_i) + \sigma_F^4(z_i) + \sigma_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i) \geq \rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i)$. Therefore, $0 \leq T_{PFS}(F,G) \leq 1$. From the Equation 8, the sumation of n terms is $0 \leq T_{PFS}(F,G) \leq 1$.

2.

$$\begin{aligned} T_{PFS}(F,G) &= \frac{\sum_{i=1}^{n}(\rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i))}{(\sum_{i=1}^{n}(\rho_F^4(z_i) + \rho_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i)) + \sum_{i=1}^{n}(\sigma_F^4(z_i) + \sigma_G^4(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i))} \\ &= \frac{\sum_{i=1}^{n}(\rho_G^2(z_i)\rho_F^2(z_i) + \sigma_G^2(z_i)\sigma_F^2(z_i))}{(\sum_{i=1}^{n}(\rho_G^4(z_i) + \rho_F^4(z_i) - \rho_G^2(z_i)\rho_F^2(z_i)) + \sum_{i=1}^{n}(\sigma_G^4(z_i) + \sigma_F^4(z_i) - \sigma_G^2(z_i)\sigma_F^2(z_i))} \\ &= T_{PFS}(G,F) \end{aligned}$$

3. When $F = G$, there are $\rho_F(z_i) = \rho_G(z_i)$, $(\sigma_F(z_i) = \sigma_G(z_i)$, for $i = 1, 2, \ldots n$. So, there is

$$\begin{aligned} T_{PFS}(F,G) &= \frac{\sum_{i=1}^{n}(\rho_F^2(z_i)\rho_F^2(z_i) + \sigma_F^2(z_i)\sigma_F^2(z_i))}{(\sum_{i=1}^{n}(\rho_F^4(z_i) + \rho_F^4(z_i) - \rho_F^2(z_i)\rho_F^2(z_i)) + \sum_{i=1}^{n}(\sigma_F^4(z_i) + \sigma_F^4(z_i) - \sigma_F^2(z_i)\sigma_F^2(z_i))} \\ &= \frac{\sum_{i=1}^{n}(\rho_F^2(z_i)\rho_F^2(z_i) + \sigma_F^2(z_i)\sigma_F^2(z_i))}{\sum_{i=1}^{n}(\rho_F^2(z_i)\rho_F^2(z_i) + \sigma_F^2(z_i)\sigma_F^2(z_i)} \\ &= 1 \end{aligned}$$

Therefore, we have finished the proofs. □

If we consider the weights of z_i, a weighted Tanimoto similarity measure between PFSs F and G is proposed as fllows:

Definition 5. *For $z_i \in Z$, take the weight ω_i. The weighted Tanimoto measure $T^{\omega}_{PFS}(F,G)$ is described as:*

$$T^{\omega}_{PFS}(F,G) = \frac{\sum_{i=1}^{n}\omega_i^4(\rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i))}{\sum_{i=1}^{n}\omega_i^4(\rho_F^4(z_i) + \rho_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i)) + \sum_{i=1}^{n}\omega_i^4(\sigma_F^4(z_i) + \sigma_G^4(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i))} \tag{10}$$

Similar to the Proof of Theorem 1, we can get:

Theorem 2. *For any two PFSs F and G, the $T^{\omega}_{PFS}(F,G)$ satisfies:*

1. *$0 \leq T^{\omega}_{PFS}(F,G) \leq 1$;*
2. *$T^{\omega}_{PFS}(F,G) = T^{\omega}_{PFS}(G,F)$;*
3. *$T^{\omega}_{PFS}(F,G) = 1$, if $F = G(\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.*

For $z_i \in Z$, taking the indeterminacy degree θ_i, we can get:

$$T^{\theta}_{PFS}(F,G) = \frac{\sum_{i=1}^{n} (\rho_F^2(z_i)\rho_G^2(z_i)+\sigma_F^2(z_i)\sigma_G^2(z_i)+\theta_F^2(z_i)\theta_G^2(z_i))}{\sum_{i=1}^{n} (\rho_F^4(z_i) + \rho_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i)) + \sum_{i=1}^{n} (\sigma_F^4(z_i) + \sigma_G^4(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i)) + \sum_{i=1}^{n} (\theta_F^4(z_i) + \theta_G^4(z_i) - \theta_F^2(z_i)\theta_G^2(z_i))} \quad (11)$$

$$T^{\omega\theta}_{PFS}(F,G) = \frac{\sum_{i=1}^{n} \omega_i^4(\rho_F^2(z_i)\rho_G^2(z_i) + \sigma_F^2(z_i)\sigma_G^2(z_i) + \theta_F^2(z_i)\theta_G^2(z_i))}{\sum_{i=1}^{n} \omega_i^4(\rho_F^4(z_i) + \rho_G^4(z_i) - \rho_F^2(z_i)\rho_G^2(z_i)) + \sum_{i=1}^{n} \omega_i^4(\sigma_F^4(z_i) + \sigma_G^4(z_i) - \sigma_F^2(z_i)\sigma_G^2(z_i)) + \sum_{i=1}^{n} \omega_i^4(\theta_F^4(z_i) + \theta_G^4(z_i) - \theta_F^2(z_i)\theta_G^2(z_i))} \quad (12)$$

3.2. Novel distance measures

Definition 6. *For two PFSs, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ and $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$, where $Z = \{z_1, z_2, \ldots, z_n\}$, the Tanimoto distance measure for them can be given by:*

$$DT_{PFS}(F,G) = 1 - T_{PFS}(F,G) \quad (13)$$

The larger the $DT_{PFS}(F,G)$ is, the greater the difference between two PFSs.

Theorem 3. *For any two PFSs F and G, the $DT_{PFS}(F,G)$ satisfies the conditions:*

1. $0 \leq DT_{PFS}(F,G) \leq 1$;
2. $DT_{PFS}(F,G) = DT_{PFS}(G,F)$;
3. $DT_{PFS}(F,G) = 0$, *if* $F = G(\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.

If we consider the weights of z_i, a weighted Tanimoto distance measure between PFSs F and G is proposed as fllows:

Definition 7. *For $z_i \in Z$, take the weight ω_i. The weighted Tanimoto measure $DT^{\omega}_{PFS}(F,G)$ is described as:*

$$DT^{\omega}_{PFS}(F,G) = 1 - T^{\omega}_{PFS}(F,G) \quad (14)$$

Theorem 4. *For any two PFSs F and G, the $DT^{\omega}_{PFS}(F,G)$ satisfies the conditions:*

1. $0 \leq DT^{\omega}_{PFS}(F,G) \leq 1$;
2. $DT^{\omega}_{PFS}(F,G) = DT^{\omega}_{PFS}(G,F)$;
3. $DT^{\omega}_{PFS}(F,G) = 0$, *if* $F = G(\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.

Taking the the indeterminacy degree into consideration, then:

$$DT^{\theta}_{PFS}(F,G) = 1 - T^{\theta}_{PFS}(F,G) \quad (15)$$

$$DT^{\omega\theta}_{PFS}(F,G) = 1 - T^{\omega\theta}_{PFS}(F,G) \quad (16)$$

3.3. Numerical experiments

Example 1. *Let F_1, F_2, F_3 be three PFSs in $Z = \{z_1, z_2\}$, which are expressed as:*

$$F_1 = \{\langle z_1, 0.4, 0.3\rangle, \langle z_2, 0.5, 0.2\rangle\}$$
$$F_2 = \{\langle z_1, 0.4, 0.3\rangle, \langle z_2, 0.5, 0.2\rangle\}$$
$$F_3 = \{\langle z_1, 0.7, 0.5\rangle, \langle z_2, 0.3, 0.4\rangle\}$$

Based on the equations presented before, the Tanimoto measures have been computed and shown in Table 2 and Table 3.

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Table 2. The outcomes using Tanimoto similarity measures in Example 1

Measures	F_1F_2	F_1F_3	F_3F_1
T_{PFS}	1.0000	0.4266	0.4266
T^{θ}_{PFS}	1.0000	0.6732	0.6732

Table 3. The outcomes using Tanimoto distance measures in Example 1

Measures	F_1F_2	F_1F_3	F_3F_1
DT_{PFS}	0.0000	0.5734	0.5734
DT^{θ}_{PFS}	0.0000	0.3268	0.3268

Taking the weights $\omega = \{0.3, 0.7\}$, the weighted Tanimoto measures between PFSs F_1, F_2, and F_3 are shown in Table 4 and Table 5.

Table 4. The outcomes using weighted Tanimoto similarity measures in Example 1

Measures	F_1F_2	F_1F_3	F_3F_1
T^{ω}_{PFS}	1.0000	0.4204	0.4204
$T^{\omega\theta}_{PFS}$	1.0000	0.9133	0.9133

Table 5. The outcomes using weighted Tanimoto distance measures in Example 1

Measures	F_1F_2	F_1F_3	F_3F_1
DT^{ω}_{PFS}	0.0000	0.5796	0.5796
$DT^{\omega\theta}_{PFS}$	0.0000	0.0867	0.0867

According to the results above, it is discernible that when $F_1 = F_2$, the Tanimoto measures between F_1andF_2 , $T_{PFS}(F_1, F_2) = 1$ and $DT_{PFS}(F_1, F_2) = 0$, which satisfies the Property 3 in Definition 4 and Property 3 in Definition 6. Besides, $T_{PFS}(F_1, F_3) = T_{PFS}(F_3, F_1)$ and $DT_{PFS}(F_1, F_3) = DT_{PFS}(F_3, F_1)$, which satisfy the Property 2 in Definition 4 and Property 2 in Definition 6.

Example 2. *F_1 and F_2 are two PFSs in z, where*

$$F_1 = \{\langle z, \rho, \sigma \rangle\}, F_2 = \{\langle z, \sigma, \rho \rangle\}$$

. In this example, Figure 1 illustrates the distribution of the proposed similarity measure under changes in both membership degree and nonmembership degree. As shown in Figure 1 (c) and Figure 1 (d), with variations in ρ and σ, the value of T_{PFS} remains within the range of $[0, 1]$. Additionally, when ρ equals σ, the similarity measure between F_1 and F_2 achieves its maximum value of 1, and when ρ equals 1 and σ equals 0 or when ρ equals 0 and σ equals 1, it reaches its minimum value of 0. This confirms that the Tanimoto measure satisfies Property 1 as defined in Definition 4.

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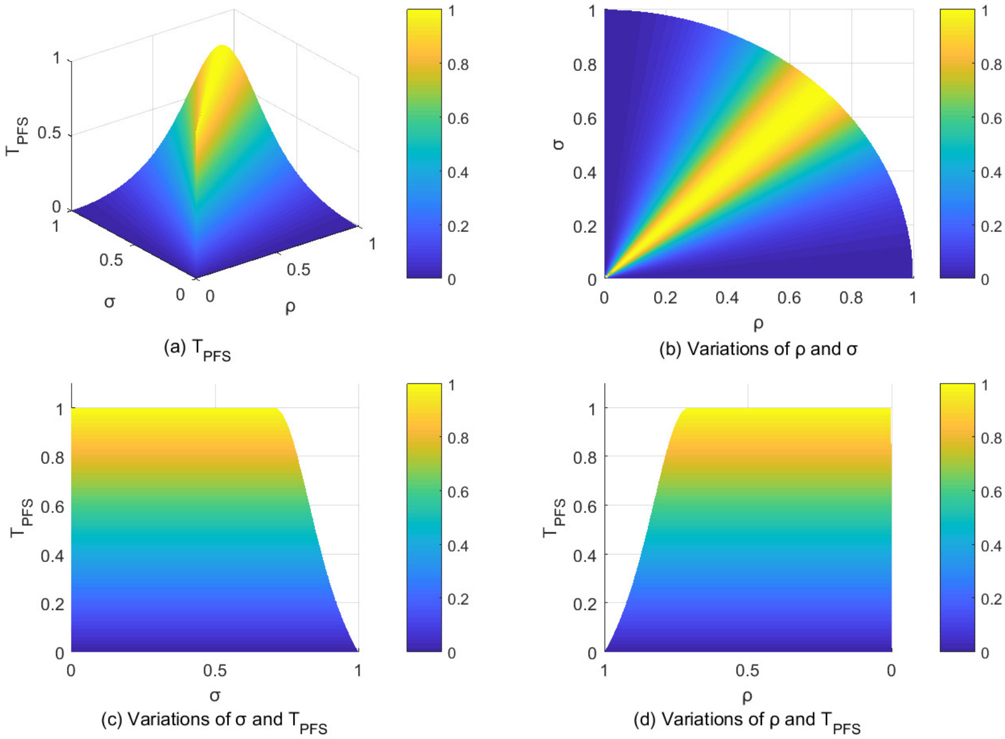

Figure 1. The value of Tanimoto similarity varying with ρ and σ in example 2.

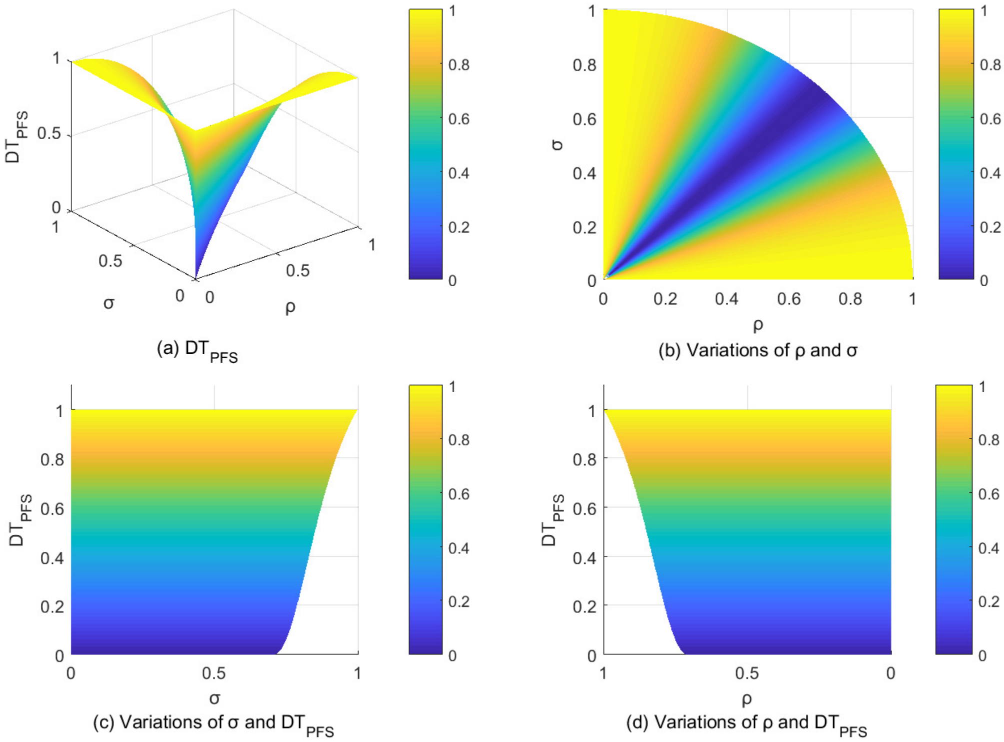

Figure 2. The value of Tanimoto distance varying with ρ and σ in example 2.

Example 3. *F and G are two PFSs under different cases in* $Z = \{z_1, z_2\}$ *and shown in Table 6.*

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As Table 6 indicates, the F of Case 1 is equivalent to the F of Case 2, while their Gs differ. Hence, the similarity between F and G in Case 1 ought to differ from that of F and G in Case 2. Similar observations hold for other cases. Table 7 presents the outcomes of employing diverse similarity measures to assess the similarity between F and G across different cases. In each case group, we employed distinct colors to highlight the counter-intuitive outcomes generated by each measure. Specifically, red denotes counter-intuitive results obtained in Case 1 and 2, blue represents those in Case 3 and 4, and green indicates the counter-intuitive results observed in Case 5 and 6. Notably, most similarity measures yield counter-intuitive measure outcomes. Specifically, SM^2_{PFS}, D_{PFS}, G_{PFS}, C^1_{PFS} *and* C^2_{PFS} *produce counter-intuitive outcomes on Case 1 and Case 2;* SM^2_{PFS}, D_{PFS} *and* G_{PFS} *yield counter-intuitive outcomes on Case 3 and Case 4; and,* SM^1_{PFS}, G_{PFS}, C^1_{PFS} *and* C^2_{PFS} *yield counter-intuitive outcomes on Case 5 and Case 6. However, the proposed Tanimoto similarity measures perform correctly in all cases, demonstrating the superiority of our measures.*

Table 6. PFSs F and G under different cases in Example 3

PFSs	$Case1$	$Case2$
F	$\{\langle z_1, 0.476, 0.543\rangle, \langle z_2, 0.221, 0.465\rangle\}$	$\{\langle z_1, 0.476, 0.543\rangle, \langle z_2, 0.221, 0.465\rangle\}$
G	$\{\langle z_1, 0.379, 0.536\rangle, \langle z_2, 0.645, 0.497\rangle\}$	$\{\langle z_1, 0.079, 0.604\rangle, \langle z_2, 0.326, 0.288\rangle\}$
PFSs	$Case3$	$Case4$
F	$\{\langle z_1, 0.451, 0.328\rangle, \langle z_2, 0.184, 0.775\rangle\}$	$\{\langle z_1, 0.451, 0.328\rangle, \langle z_2, 0.184, 0.775\rangle\}$
G	$\{\langle z_1, 0.290, 0.402\rangle, \langle z_2, 0.099, 0.881\rangle\}$	$\{\langle z_1, 0.404, 0.536\rangle, \langle z_2, 0.138, 0.861\rangle\}$
PFSs	$Case5$	$Case6$
F	$\{\langle z_1, 0.295, 0.768\rangle, \langle z_2, 0.302, 0.761\rangle\}$	$\{\langle z_1, 0.295, 0.768\rangle, \langle z_2, 0.302, 0.761\rangle\}$
G	$\{\langle z_1, 0.175, 0.376\rangle, \langle z_2, 0.274, 0.633\rangle\}$	$\{\langle z_1, 0.126, 0.914\rangle, \langle z_2, 0.679, 0.543\rangle\}$

Table 7. The results of different similarity measures in Example 3

Measures	$Case1$	$Case2$	$Case3$	$Case4$	$Case5$	$Case6$
T_{PFS}	0.573	0.639	0.912	0.905	0.579	0.713
T_{PFS}	0.697	0.909	0.935	0.917	0.592	0.722
SM^1_{PFS}	0.719	0.770	0.806	0.791	0.617	0.617
SM^2_{PFS}	0.840	0.840	0.875	0.875	0.774	0.693
D_{PFS}	0.746	0.746	0.884	0.884	0.697	0.799
G_{PFS}	0.800	0.800	0.879	0.879	0.765	0.765
C^1_{PFS}	0.887	0.887	0.972	0.965	0.894	0.894
C^2_{PFS}	0.971	0.971	0.993	0.991	0.973	0.973

Example 4. *Similar to Example 3, we apply the proposed Tanimoto distance measures to the counter-intuitive cases of existing distance measures. As shown in Table 8, F and G are two PFSs under different cases, and Table 9 presents the measurement results. In each case group, we use different colors to highlight the counter-intuitive results generated by each measures. Specifically,* DH_{PFS} *produce counter-intuitive outcomes on Case 1 and Case 2;* DE_{PFS} *yield counter-intuitive outcomes on Case 3 and Case 4; and,* DH_{PFS} *and* DE_{PFS} *yield counter-intuitive outcomes on Case 5 and Case 6. However, the proposed Tanimoto distance measures perform correctly in all cases, demonstrating the superiority of our measures.*

Table 8. PFSs F and G under different cases in Example 4

PFSs	$Case1$	$Case2$
F	$\{\langle z_1, 0.887, 0.217\rangle, \langle z_2, 0.394, 0.289\rangle\}$	$\{\langle z_1, 0.887, 0.217\rangle, \langle z_2, 0.394, 0.289\rangle\}$
G	$\{\langle z_1, 0.184, 0.112\rangle, \langle z_2, 0.232, 0.165\rangle\}$	$\{\langle z_1, 0.297, 0.579\rangle, \langle z_2, 0.551, 0.394\rangle\}$
PFSs	$Case3$	$Case4$
F	$\{\langle z_1, 0.725, 0.319\rangle, \langle z_2, 0.189, 0.094\rangle\}$	$\{\langle z_1, 0.725, 0.319\rangle, \langle z_2, 0.189, 0.094\rangle\}$
G	$\{\langle z_1, 0.074, 0.004\rangle, \langle z_2, 0.557, 0.696\rangle\}$	$\{\langle z_1, 0.125, 0.633\rangle, \langle z_2, 0.692, 0.217\rangle\}$
PFSs	$Case5$	$Case6$
F	$\{\langle z_1, 0.791, 0.487\rangle, \langle z_2, 0.211, 0.510\rangle\}$	$\{\langle z_1, 0.791, 0.487\rangle, \langle z_2, 0.211, 0.510\rangle\}$
G	$\{\langle z_1, 0.288, 0.671\rangle, \langle z_2, 0.438, 0.755\rangle\}$	$\{\langle z_1, 0.232, 0.388\rangle, \langle z_2, 0.381, 0.267\rangle\}$

Table 9. The results of different distance measures in Example 4

Measures	$Case1$	$Case2$	$Case3$	$Case4$	$Case5$	$Case6$
DT_{PFS}	0.939	0.805	0.970	0.892	0.592	0.801
DT^{θ}_{PFS}	0.578	0.555	0.724	0.528	0.587	0.523
DH_{PFS}	0.269	0.269	0.445	0.321	0.283	0.283
DE_{PFS}	0.258	0.187	0.362	0.362	0.178	0.178

Example 5. *A, B and C are three random PFSs. We employ the proposed similarity measure to assess their similarity and subtract the minimum value from the maximum Tanimoto similarity value to obtain D_{value}. We have conducted 100 such experiments and computed the average of the final results to get the average distance between the maximum and minimum values of similarity results in these 100 experiments ($D_{AVG} = \frac{D_{value}}{100}$). Similarly, we will compare D_{AVG}s obtained using other measures, as shown in Figure 3.*

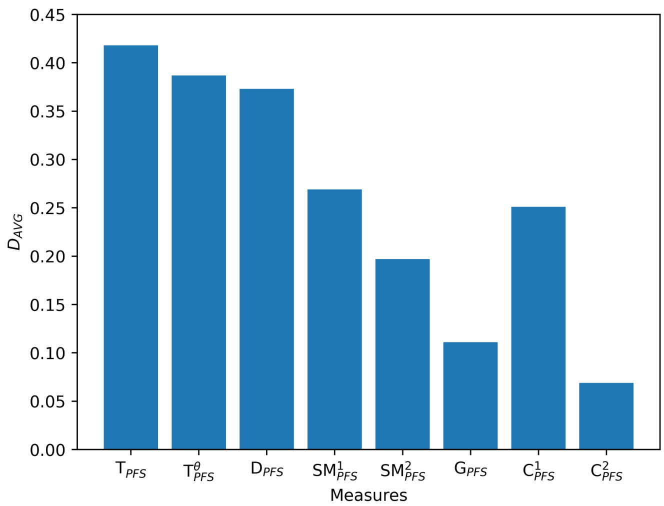

Figure 3. Difference between the highest and lowest values obtained by different measures in Example 5.

The findings in Figure 3 suggest that the Tanimoto similarity measure exhibits the maximum D_{AVG}, indicating that the proposed similarity measure tends to assign more differentiated similarity values when

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distinguishing between different fuzzy sets. This attribute endows it with greater discriminatory power to discern different levels of similarity and enhances its performance in distinguishing PFSs with high similarity. A wider range of similarity values assigned to different PFSs may enhance the credibility of PFSs classification and foster more confident decision-making. On the other hand, C^2_{PFS} *has the smallest* D_{AVG}*, which means that it is more likely to classify highly similar PFSs into the same category or exhibit more hesitation in selecting between different samples. This characteristic will be detrimental to high-precision decision-making. These results indicate that our proposed measures are superior to the existing measures. These characteristics will be further validated in the subsequent examples.*

4. Applications

In this section, we introduce two models that utilize the proposed measures in pattern recognition, medical diagnosis and MADM problems. Furthermore, we conducted a series of experiments and compared our models with existing measures to demonstrate their effectiveness and superiority.

4.1. A model for pattern recognition and medical diagnosis

There are some known patterns $F = \{F_1, F_2, \ldots, F_n\}$, denoted by PFSs $F_i = \{\langle z_l, \rho_{F_i}, \sigma_{F_i}\rangle : z_l \in Z\}(i = 1, 2, \ldots, n)$ represented by a finite universe of discourse $Z = \{z_1, z_2, \ldots, z_k\}$. There are some samples of unknown categories $S = \{S_1, S_2, \ldots, S_m\}$, denoted by $S_j = \{\left\langle z_l, \rho_{S_j}, \sigma_{S_j}\right\rangle : z_l \in Z\}(j = 1, 2, \ldots, m)$. Our goal is to classify S_j into the given patterns F_i. The recognition process is described in detail below:

Step 1 Calculate the Tanimoto similarity(or distance) between S_j and F_i.
Step 2 Obtain the maximum Tanimoto similarity $sm\left(F_o, S_j\right)$ using equation 17 or the minimum Tanimoto distance $d\left(F_o, S_j\right)$ using equation 18:

$$sm\left(F_o, S_j\right) = \max\{T_{PFS}(F_i, S_j)\} \tag{17}$$

$$d\left(F_o, S_j\right) = \min\{DT_{PFS}(F_i, S_j)\} \tag{18}$$

Step 3 If any pattern F_o has the highest Tanimoto similarity between S_i, then, S_i and F_o belong to the same category:

$$o = \arg\max\{sm\left(F_o, S_j\right)\}, S_j \to F_o \tag{19}$$

If distance measure is used as the standard of measure, then the following form would be applied:

$$o = \arg\min\{d\left(F_o, S_j\right)\}, S_j \to F_o \tag{20}$$

The pseudo code description of the model is as Algorithm 1.

Algorithm 1 Algorithm for pattern recognition and medical diagnosis

```
Input: F = {F_1, F_2, ..., F_n}, S = {S_1, S_2, ..., S_m};
Output: The classification of S_j;
for j = 1, j ≤ m do
    for i = 1, i ≤ n do
        Calculate the similarity between F_i and S_j using equation 8, or calculate the distance using equation 13;
    end for
    Obtain the maximum similarity between S_j and F using equation 17 or the minimum distance using equation 18;
    Obtain the pattern of S_j using equation 19 or 20;
end for
```

Example 6. *This example is centered on the recognition of patterns. It leverages PFSs for the delineation of three instances whose categories have been confirmed, denoted as* $F = \{F_1, F_2, F_3\}$*, as well as a solitary sample from an as yet unrecognized category, labeled S. The parameters* $z_i(i = 1, 2, 3, 4, 5)$ *that govern these samples are detailed in Table 10.*

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Table 10. Known PFSs and a simple S in Example 6

	z_1	z_2	z_3	z_4	z_5
F_1	$\langle 0.095, 0.267 \rangle$	$\langle 0.601, 0.401 \rangle$	$\langle 0.577, 0.215 \rangle$	$\langle 0.848, 0.512 \rangle$	$\langle 0.431, 0.433 \rangle$
F_2	$\langle 0.521, 0.468 \rangle$	$\langle 0.875, 0.472 \rangle$	$\langle 0.532, 0.701 \rangle$	$\langle 0.832, 0.120 \rangle$	$\langle 0.549, 0.070 \rangle$
F_3	$\langle 0.207, 0.258 \rangle$	$\langle 0.689, 0.060 \rangle$	$\langle 0.081, 0.411 \rangle$	$\langle 0.468, 0.736 \rangle$	$\langle 0.485, 0.056 \rangle$
S	$\langle 0.693, 0.587 \rangle$	$\langle 0.694, 0.232 \rangle$	$\langle 0.392, 0.913 \rangle$	$\langle 0.746, 0.198 \rangle$	$\langle 0.747, 0.179 \rangle$

In order to ascertain the categorical ascription of the unknown sample S, we employed the Tanimoto similarity measures as delineated in our methodological framework:

Step 1 *Calculate the Tanimoto similarity(or distance) between S_j and F_i, the results are presented in Table 11 and Table 12.*

Step 2 *According to the equation 17 and equation 18, the maximum similarity and minimum distance are shown as below:*

$$sm\left(F_2, S\right) = 0.804$$

$$d\left(F_2, S\right) = 0.196$$

Step 3 *According to the principle of maximum similarity and minimum distance, the pattern recognition result of S are as follows:*

$$S \rightarrow F_2$$

Furthermore, the measurement results derived from the extant methods are detailed in Table 13.

Table 11. The results of Tanimoto similarity measures in Example 6

Measures	F_1S	F_2S	F_3S
T_{PFS}	0.403	0.804	0.364
T^{θ}_{PFS}	0.387	0.713	0.373

Table 12. The results of Tanimoto distance measures in Example 6

Measures	F_1S	F_2S	F_3S
DT_{PFS}	0.597	0.196	0.636
DT^{θ}_{PFS}	0.613	0.287	0.627

Table 13. The results of different similarity measures in Example 6

Measures	F_1S	F_2S	F_3S
D_{PFS}	0.560	0.886	0.555
SM^1_{PFS}	0.540	0.689	0.536
SM^2_{PFS}	0.648	0.779	0.659
G_{PFS}	0.798	0.878	0.807
C^1_{PFS}	0.719	0.859	0.707
C^2_{PFS}	0.926	0.964	0.922

Utilizing the data shown in Table 13, we can identify that S and F_2 demonstrate the highest degree of similarity. This finding is consistently confirmed across the Tanimoto similarity measures before, validating the effectiveness of the proposed similarity measures. It is noteworthy that the similarity scores derived from the Tanimoto measures exhibit significant variability. We computed the Tanimoto similarity values for all PFSs

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with respect to S and subtracted the second highest value from the maximum one, denoted as D_d, and a similar analysis was conducted for other measures, as depicted in Figure 4. We can find that the Tanimoto similarity measure yields the highest D_d and ranks third in terms of indeterminacy. The similarity scores between S and F_2 as determined by our proposed metrics are substantially different from those between S and other PFSs. Consequently, we can assert with confidence that F_2 is a more suitable pattern. However, the similarity scores between S and the known samples derived from other measures are more closely aligned. Hence, relying on these measures for decision-making may result in greater hesitation. These findings align with the conclusions drawn in Example 5, suggesting that our proposed measures are superior in distinguishing between samples.

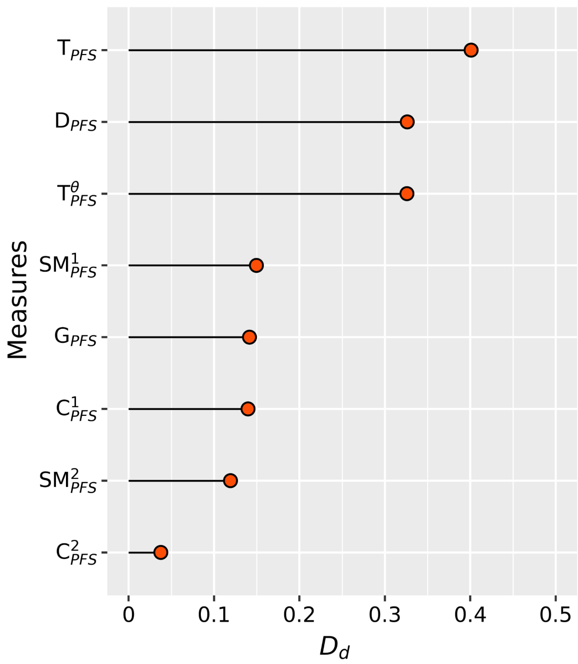

Figure 4. Difference between the highest and second-highest values obtained by different measures in Example 6 .

Example 7. *In this example, the aim is to recognize mineral categories through pattern recognition. Specifically, we consider six common mixtures of minerals that are represented by PFSs $F_i(i = 1,2,3,4,5,6)$, with each mixture consisting of six fundamental minerals which collectively form the universe of discourse $z_i(i = 1,2,3,4,5,6)$. Our primary goal is to employ the proposed measure in order to determine the category to which an unknown mixed mineral S pertains. Table 14 displays the known PFSs and the unidentified mineral S, whereas Table 15 summarizes the resulting outcomes.*

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Table 14. Known PFSs and a simple S in Example 7

	z_1	z_2	z_3	z_4	z_5	z_6
F_1	$\langle 0.425, 0.784\rangle$	$\langle 0.770, 0.605\rangle$	$\langle 0.375, 0.384\rangle$	$\langle 0.062, 0.928\rangle$	$\langle 0.559, 0.219\rangle$	$\langle 0.308, 0.553\rangle$
F_2	$\langle 0.434, 0.773\rangle$	$\langle 0.142, 0.026\rangle$	$\langle 0.048, 0.153\rangle$	$\langle 0.256, 0.598\rangle$	$\langle 0.631, 0.652\rangle$	$\langle 0.115, 0.263\rangle$
F_3	$\langle 0.527, 0.054\rangle$	$\langle 0.015, 0.897\rangle$	$\langle 0.210, 0.661\rangle$	$\langle 0.497, 0.054\rangle$	$\langle 0.469, 0.008\rangle$	$\langle 0.341, 0.060\rangle$
F_4	$\langle 0.252, 0.908\rangle$	$\langle 0.850, 0.136\rangle$	$\langle 0.519, 0.807\rangle$	$\langle 0.143, 0.288\rangle$	$\langle 0.354, 0.904\rangle$	$\langle 0.118, 0.270\rangle$
F_5	$\langle 0.156, 0.922\rangle$	$\langle 0.837, 0.088\rangle$	$\langle 0.612, 0.408\rangle$	$\langle 0.223, 0.548\rangle$	$\langle 0.782, 0.348\rangle$	$\langle 0.087, 0.580\rangle$
F_6	$\langle 0.100, 0.770\rangle$	$\langle 0.188, 0.494\rangle$	$\langle 0.111, 0.203\rangle$	$\langle 0.849, 0.163\rangle$	$\langle 0.652, 0.216\rangle$	$\langle 0.447, 0.122\rangle$
S	$\langle 0.837, 0.282\rangle$	$\langle 0.979, 0.099\rangle$	$\langle 0.407, 0.894\rangle$	$\langle 0.628, 0.242\rangle$	$\langle 0.193, 0.880\rangle$	$\langle 0.547, 0.574\rangle$

Table 15. Results of different similarity measures in Example 7

Measures	F_1S	F_2S	F_3S	F_4S	F_5S	F_6S
T_{PFS}	0.294	0.194	0.207	0.603	0.321	0.144
T^{θ}_{PFS}	0.310	0.233	0.315	0.613	0.395	0.198
D_{PFS}	0.452	0.303	0.462	0.591	0.470	0.304
SM^1_{PFS}	0.383	0.400	0.441	0.627	0.436	0.314
SM^2_{PFS}	0.518	0.518	0.573	0.703	0.546	0.495
G_{PFS}	0.775	0.746	0.759	0.848	0.790	0.733
C^1_{PFS}	0.638	0.610	0.566	0.811	0.699	0.541
C^2_{PFS}	0.904	0.895	0.880	0.951	0.921	0.875

Table 15 displays the similarity measures for mineral S and F_4, with the Tanimoto measure producing the highest similarity score. This result confirms that mineral S belongs to F_4, which is consistent with findings from other studies that have used various similarity measures. Likewise, Figure 5 illustrates the D_d among different measures, and the Tanimoto similarity measures rank the first and the second among the measures shown in the figure.

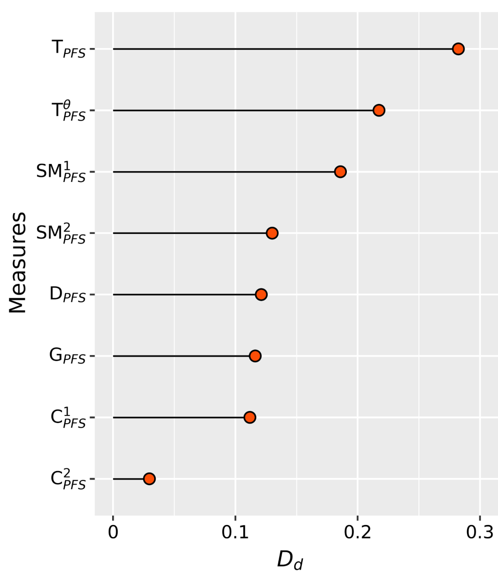

Figure 5. Difference between the highest and second-highest values obtained by different measures in Example 7 .

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Example 8 ([55]). *Assuming the presence of four patients, namely Ragu, Mathi, Velu and Karthi, denoted by* $I = \{I_1, I_2, I_3, I_4\}$*, and exhibiting symptoms including Headache, Acidity, Burning eyes, Back pain and Depression, represented as* $Z = \{Z_1, Z_2, Z_3, Z_4, Z_5\}$*. The set of possible diagnoses is denoted by* $D = \{D_1, D_2, D_3, D_4, D_5\}$*, and includes:* D_1*: Stress;* D_2*: Ulcer;* D_3*: Vision problem;* D_4*: Spinal problem;* D_5*: Blood pressure. The Pythagorean fuzzy relation* $I \to Z$ *is expressed by PFS, as shown in Table 16, while the Pythagorean fuzzy relation* $Z \to D$ *is represented by PFSs and listed in Table 17. Every entry in both tables is defined by the PFS, with the values indicating membership degree and nonmembership degree, respectively. The proposed similarity and distance measures are employed to evaluate the similarity and distance between each patient and potential diagnosis. Based on the principle of maximum similarity or minimum distance, each patient is diagnosed accordingly. Tables 18 and 19 present the similarity measure outcomes and distance measure outcomes of each patient I towards each diagnosis D, alongside the ultimate diagnosis results.*

Table 16. Symptomatic characteristic of the patient in Example 8

	z_1	z_2	z_3	z_4	z_5
I_1	$\langle 0.9, 0.1 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.2, 0.7 \rangle$
I_2	$\langle 0.0, 0.7 \rangle$	$\langle 0.4, 0.5 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.1, 0.2 \rangle$
I_3	$\langle 0.7, 0.1 \rangle$	$\langle 0.7, 0.1 \rangle$	$\langle 0.0, 0.5 \rangle$	$\langle 0.1, 0.7 \rangle$	$\langle 0.0, 0.6 \rangle$
I_4	$\langle 0.5, 0.1 \rangle$	$\langle 0.4, 0.3 \rangle$	$\langle 0.4, 0.5 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.3, 0.4 \rangle$

Table 17. Symptomatic characteristic of the diagnosis in Example 8

	z_1	z_2	z_3	z_4	z_5
D_1	$\langle 0.3, 0.0 \rangle$	$\langle 0.3, 0.5 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.2, 0.6 \rangle$
D_2	$\langle 0.0, 0.6 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.5, 0.0 \rangle$	$\langle 0.1, 0.8 \rangle$
D_3	$\langle 0.2, 0.2 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.1, 0.7 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.2, 0.8 \rangle$
D_4	$\langle 0.2, 0.8 \rangle$	$\langle 0.1, 0.5 \rangle$	$\langle 0.7, 0.0 \rangle$	$\langle 0.1, 0.0 \rangle$	$\langle 0.2, 0.7 \rangle$
D_5	$\langle 0.2, 0.8 \rangle$	$\langle 0.0, 0.7 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.1, 0.8 \rangle$	$\langle 0.8, 0.1 \rangle$

Table 18. Diagnostic results of the Tanimoto similarity measure in Example 8

	D_1	D_2	D_3	D_4	D_5	**Classification**
I_1	0.564	0.432	0.450	0.132	0.169	D_1
I_2	0.142	0.233	0.236	0.530	0.446	D_4
I_3	0.317	0.256	0.617	0.127	0.222	D_3
I_4	0.676	0.363	0.281	0.150	0.145	D_1

Table 19. Diagnostic results of the Tanimoto distance measure in Example 8

	D_1	D_2	D_3	D_4	D_5	**Classification**
I_1	0.436	0.568	0.550	0.868	0.831	D_1
I_2	0.858	0.767	0.764	0.470	0.554	D_4
I_3	0.683	0.744	0.383	0.873	0.778	D_3
I_4	0.324	0.637	0.719	0.850	0.855	D_1

Based on the findings presented in Tables 18 and 19, it is observed that I_1 *exhibits the highest Tanimoto similarity measure and the lowest Tanimoto distance measure towards* D_1*;* I_2 *displays the highest Tanimoto similarity measure and the lowest Tanimoto distance measure towards* D_4*;* I_3 *demonstrates the highest Tanimoto* *similarity measure and the lowest Tanimoto distance measure towards D_3; and I_4 showcases the highest Tanimoto similarity measure and the lowest Tanimoto distance measure towards D_1. Thus, we can conclude that Ragu is diagnosed with stress, Mathi with spinal problems, Velu with vision problems and Karthi with stress.*

In order to validate the effectiveness of our proposed measures, a comparison was conducted against other methods, and the outcomes have been summarized in Table 20. It is observed from Table 20 that our proposed measures provide diagnostic outcomes that are consistent with those obtained using methods proposed by Xiao and Ding [55], Zhou [56] and Deng [57]. The experimental results lend support to the practicability of our proposed similarity and distance measures.

Table 20. Comparisons of different methods in Example 8

Methods	I_1	I_2	I_3	I_4
T_{PFS}	Stress	Spinal problem	Vision problem	Stress
Xiao and Ding	Stress	Spinal problem	Vision problem	Stress
Zhou	Stress	Spinal problem	Vision problem	Stress
Deng	Stress	Spinal problem	Vision problem	Stress

4.2. The model for MADM

Suppose that $I = \{I_1, I_2, \ldots, I_m\}$ is a discrete set of alternatives, and $A = \{A_1, A_2, \ldots, A_n\}$ is the set of attributes, $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is the weighting vector of the attribute $A_j (j = 1, 2, \ldots, n)$, where $\omega > 0$, $\sum_{j=1}^{n} w_j = 1$. $T = \{t_1, t_2, \ldots, t_n\}$ is the experts evaluate, representing the type of each attribute, where 1 represents the benefit type and 0 represents the cost type. Suppose that $R = (\rho_{ij}, \sigma_{ij})_{m \times n}$ is the Pythagorean fuzzy matrix, where ρ_{ij} indicates the degree that the alternative I_i satisfies the attribute A_j and σ_{ij} indicates the degree that the alternative I_i does not satisfy the attribute A_j, $\rho_{ij} \subset [0, 1]$, $\sigma_{ij} \subset [0, 1]$, $(\rho_{ij})^2 + (\sigma_{ij})^2 \leq 1, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. The proposed model is described below:

Step 1 Defining the Pythagorean fuzzy positive ideal solution I^+:

$$(\rho^+, \sigma^+) = ((\rho_1^+, \sigma_1^+), (\rho_2^+, \sigma_2^+), \ldots, (\rho_n^+, \sigma_n^+)) = ((1, 0), (1, 0), \ldots, (1, 0)) \tag{21}$$

The equation for correction is as follows:

$$I^+ = ((\rho_j, \sigma_j)) \begin{cases} (\rho_j, \sigma_j) & for\ t_j = 1 \\ (\sigma_j, \rho_j) & for\ t_j = 0 \end{cases} \tag{22}$$

Step 2 Calculating the weighted Tanimoto similarity measure between $I_i (i = 1, 2, \ldots, m)$ and I^+ as follows:

$$T_{PFS}^{\omega}(I_i, I^+) = \frac{\sum_{i=1}^{n} \omega_i^4 (\rho_{ij}^2 (\rho_j^+)^2 + \sigma_{ij}^2 (\sigma_j^+)^2)}{(\sum_{i=1}^{n} \omega_i^4 (\rho_{ij}^4 + (\rho_j^+)^4 - \rho_{ij}^2 (\rho_j^+)^2) + \sum_{i=1}^{n} \omega_i^4 (\sigma_{ij}^4 + (\sigma_j^+)^4 - \sigma_{ij}^2 (\sigma_j^+)^2))} \tag{23}$$

If the standard of measure is distance, then additional distance measures must be calculated in the following way:

$$DT_{PFS}^{\omega}(I_i, I^+) = 1 - T_{PFS}^{\omega}(I_i, I^+) \tag{24}$$

Step 3 Obtain the maximum Tanimoto similarity $sm\left(I_o, I^+\right)$ using equation 25 or the minimum Tanimoto distance $d\left(I_o, I^+\right)$ using equation 26:

$$sm\left(I_o, I^+\right) = \max\{T_{PFS}(I_i, I^+)\} \tag{25}$$

$$d\left(I_o, I^+\right) = \min\{DT_{PFS}(I_i, I^+)\} \tag{26}$$

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Step 4 If any alternative I_o has the highest Tanimoto similarity between I^+, then, I_o is the most important alternative:

$$o = \arg\max\{sm\left(I_o, I^+\right)\}, I_o \to Result \tag{27}$$

If distance measure is used as the standard of measure, then the following form would be applied:

$$o = \arg\min\{d\left(I_o, I^+\right)\}, I_o \to Result \tag{28}$$

The pseudo code description of the model is as Algorithm 2.

Algorithm 2 Algorithm for MADM

Input: $R = \{I_1, I_2, \ldots, I_m\}, T = \{t_1, t_2, \ldots, t_n\}$;
Output: The most important alternative I_o;
Defining the Pythagorean fuzzy positive ideal solution I^+ using equation 21;
for j = 1, j ≤ n **do**
    The equation 22 is used to correct (ρ_j, σ_j);
end for
for i = 1, i ≤ m **do**
    Calculate the similarity between I_i and I^+ using equation 23, or the distance using equation 24;
end for
Obtain the maximum similarity between I_i and I^+ using equation 25 or the minimum distance using equation 26;
Obtain the most important result using equation 27 or 28;

Example 9. *A company is planning to purchase a batch of computers, with six alternative model schemes $I_i(i = 1, 2, 3, 4, 5, 6)$ to choose from, and has selected five attributes as selection criteria, including the materials used, reputation, response time, service life and price $A_j(j = 1, 2, 3, 4, 5)$. The weights of these attributes are denoted by $\omega_j(j = 1, 2, 3, 4, 5)$, where $\omega = (0.2, 0.4, 0.1, 0.1, 0.2)$. After expert evaluation, all attribute types are provided: $T = \{1, 1, 0, 1, 0\}$. Forming a Pythagorean fuzzy decision matrix $R_{6\times5}$:*

$$R_{6\times5} = \begin{bmatrix} (0.287, 0.955) & (0.846, 0.123) & (0.622, 0.776) & (0.214, 0.791) & (0.808, 0.524) \\ (0.232, 0.364) & (0.940, 0.241) & (0.564, 0.366) & (0.019, 0.805) & (0.207, 0.834) \\ (0.081, 0.645) & (0.026, 0.584) & (0.801, 0.266) & (0.870, 0.030) & (0.598, 0.554) \\ (0.565, 0.459) & (0.759, 0.416) & (0.860, 0.093) & (0.480, 0.145) & (0.622, 0.743) \\ (0.130, 0.768) & (0.613, 0.584) & (0.114, 0.653) & (0.048, 0.531) & (0.050, 0.440) \\ (0.210, 0.300) & (0.634, 0.176) & (0.146, 0.919) & (0.323, 0.702) & (0.812, 0.575) \end{bmatrix}$$

We will utilize the proposed MADM model to select the most appropriate solution.

Step 1 *Defining the Pythagorean fuzzy positive ideal solution I^+. As we want the response time and price to be as low as possible, these two attributes are defined as cost types and set to $(0, 1)$:*

$$\begin{aligned}(\rho^+, \sigma^+) &= ((\rho_1^+, \sigma_1^+), (\rho_2^+, \sigma_2^+), (\rho_3^+, \sigma_3^+), (\rho_4^+, \sigma_4^+), (\rho_5^+, \sigma_5^+)) \\ &= ((1, 0), (1, 0), (0, 1), (1, 0), (0, 1)).\end{aligned}$$

Step 2 *Calculating the weighted Tanimoto measure between $I_i(i = 1, 2, \ldots, 6)$ and I^+ as Table 21 and 22.*

Table 21. The results of Tanimoto similarity measures in Example 9

Measures	I_1I^+	I_2I^+	I_3I^+	I_4I^+	I_5I^+	I_6I^+
T^{ω}_{PFS}	0.747	0.913	0.019	0.699	0.381	0.474
$T^{\omega\theta}_{PFS}$	0.696	0.870	0.014	0.642	0.337	0.337

Table 22. The results of Tanimoto distance measures in Example 9

Measures	I_1I^+	I_2I^+	I_3I^+	I_4I^+	I_5I^+	I_6I^+
DT^{ω}_{PFS}	0.253	0.087	0.981	0.301	0.619	0.526
$DT^{\omega\theta}_{PFS}$	0.304	0.130	0.986	0.358	0.663	0.663

Step 3 *According to the equation 25 and equation 26, the maximum similarity and minimum distance are shown as below:*

$$sm\left(I_2, I^+\right) = 0.913$$
$$d\left(I_2, I^+\right) = 0.087$$

Step 4 *According to the equation 27 and equation 28, the most suitable computer type is I_2.*

Furthermore, we have obtained congruent outcomes by employing the MADM models proposed by Wang [29] and Zhang [30], respectively, which proves the effectiveness of the proposed model. The corresponding calculation outcomes have been delineated in Table 23 and Table 24, respectively. The parameter α is a positive constant and U^+ represents the positive ideal solution, which are calculated using Wang's method.

Table 23. The results of Dice similarity measure in Example 9

α	I_1U^+	I_2U^+	I_3U^+	I_4U^+	I_5U^+	I_6U^+
0.0	0.654	0.822	0.021	0.557	0.345	0.379
0.3	0.762	0.897	0.028	0.691	0.445	0.503
0.7	0.975	1.020	0.053	1.017	0.726	0.893
1.0	1.234	1.138	0.162	1.573	1.381	2.136

Table 24. The results of Exponential similarity measure in Example 9

Measures	I_1I^+	I_2I^+	I_3I^+	I_4I^+	I_5I^+	I_6I^+
SM^1_{ω}	0.366	0.479	0.122	0.357	0.185	0.273
SM^2_{ω}	0.450	0.629	0.326	0.505	0.398	0.486

5. Conclusions

In this study, we propose a set of innovative similarity and distance measures for PFSs, drawing inspiration from the Tanimoto coefficient. The experiment demonstrated that the proposed measures possess two outstanding characteristics: (1) avoiding numerous counter-intuitive outcomes caused by existing measures; and (2) providing more differentiated measurement results when distinguishing between different PFSs. Therefore, decision-making based on this measure can yield more confident results. Compared to some existing PFSs measures, the proposed measures are more effective and superior. We also designed two decision models based on the proposed measures and applied them to problems such as pattern recognition, medical diagnosis and multi-attribute decision-making to demonstrate their effectiveness. Future research will apply the various measures proposed to a wider range of scenario problems to confirm the potential of our measures and PFSs.

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