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Posted Date: 3 January 2024

doi: 10.20944/preprints202401.0244.v1

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Fixed Point Theorems for Generalized Set Valued F-Contractions in Metric Spaces

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Abstract: In this paper, a positive answer to the open question of Fabino et al. is given, and the concept of Işik type set valued contractions is introduced and fixed point theorem for such contractions is established. Examples to support main theorems are given, and an application to integral inclusion is given.

Keywords: fixed point; contraction; generalized contraction; set valued contraction; metric space

MSC: 47H10; 54H25

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1. Introduction

Wardowski [21] introduced the notion of F-contraction mappings and generalized Banach contraction principle by proving that every F-contractions on complete metric spaces have only one fixed point, where $F:(0,\infty)\to(-\infty,\infty)$ is a function such that

(F1) F is stlictly increasing;
(F2) $\forall\{s_n\}\subset(0,\infty)$,

$$\lim_{n\to\infty} s_n=0 \Longleftrightarrow \lim_{n\to\infty} F(s_n)=-\infty;$$

(F3) $\exists q\in(0,1):\lim_{t\to 0^+} t^q F(t)=0$.

Among several results ([1–4,7,8,10–13,15,16,18–20,22,23]) generalizing Wardowski's result, Piri and Kumam [17] introduced the concept of Suzuki type F-contractions and obtained related fixed point results in complete metric spaces, where $F:(0,\infty)\to(-\infty,\infty)$ is a stlictly increasing function such that

(F4) $\inf F=-\infty$;
(F5) F is continuous on $(0,\infty)$.

Note that for a function $F:(0,\infty)\to(-\infty,\infty)$, the following are equivalent.

(1) (F2) is satisfied;
(2) (F4) is satisfied;
(3) $\lim_{t\to 0^+} F(t)=-\infty$.

Hence, we have that

$$\lim_{n\to\infty} s_n=0 \Rightarrow \lim_{n\to\infty} F(s_n)=-\infty$$

whenever (F4) holds.

Very recently, Fabiano et al. [9] gave a generalization of Wardowski's result [21] by reducing the condition on function $F:(0,\infty)\to(-\infty,\infty)$ and by using the right limit of function $F:(0,\infty)\to(-\infty,\infty)$. They proved the following theorem.

Theorem 1. *(Theorem 2.3 [9]) Let (E,ρ) be a complete metric space. Suppose that $T:E\to E$ is a map such that for all $x,y\in E$ with $\rho(Tx,Ty)>0$*

$$\tau+F(\rho(Tx,Ty))\le F(\rho(x,y))$$

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where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function If $(F1)$ is satisfied, then T possesses only one fixed point.

In [9], Fabiano et al. asked the following question:

Question.(Question 4.3 [9]) Can conditions for the function F be reduced to $(F1)$ and $(F2)$, and can the proof be made simpler in some results for multivalued mappings in the same way as it was presented in [9] for single-valued mappings?

In this paper, we give a positive answer to the above question by extending above theorem to set-valued maps and obtain a fixed point result for Işik type set valued contractions. We give examples to interpret main results and an application to integral inclusion.

Let (E, ρ) be a metric space. We denote by $CL(E)$ the family of all nonempty closed subsets of E.

Let $H(\cdot, \cdot)$ be the *generalized Pompeiu-Hausdorff distance* [5] on $CL(E)$, i.e., for all $A, B \in CL(E)$,

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} \rho(a,B), \sup_{b\in B} \rho(b,A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\rho(a, B) = \inf\{\rho(a, b) : b \in B\}$ is the distance from the point a to the subset B.
Let $\delta(A, B) = \sup\{\rho(a, b) : a \in A, b \in B\}$. When $A = \{x\}$, we denote $\delta(A, B)$ by $\delta(x, B)$.

For $A, B \in CL(E)$, let $D(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A} \inf_{y\in B} d(x, y)$.
Then, we have that for all $A, B \in CL(E)$

$$D(A, B) \le H(A, B) \le \delta(A, B).$$

Lemma 1. *Let $l > 0$, and let $\{t_n\}, \{s_n\} \subset (l, \infty)$ be non-increasing sequences such that*

$$t_n < s_n, \forall n = 1, 2, 3, \cdots \text{ and } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l.$$

If $F : (0, \infty) \to (-\infty, \infty)$ is non-decreasing, then we have

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} F(s_n) = F(l^+) > 0$$

where $F(l^+)$ denotes $\lim_{t\to l^+} F(t)$.

Proof. Since $\{t_n\}$ and $\{s_n\}$ are non-increasing, it follows from the nondecreasingness of F that

$$\lim_{t\to l^+} F(t) = \lim_{n\to\infty} F(t_{n+1}) \le \lim_{n\to\infty} F(t_n) \le \lim_{n\to\infty} F(s_n) \le \lim_{n\to\infty} F(s_{n-1}) \le \lim_{t\to l^+} F(t).$$

Thus we have

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} F(s_n) = \lim_{t\to l^+} F(t) = F(l^+) > 0.$$

□

Lemma 2. *[6] Let (E, ρ) be a metric space. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which*

$$m(k) > n(k) > k, \ \rho(x_{m(k)}, x_{n(k)}) \ge \epsilon \text{ and } \rho(x_{m(k)-1}, x_{n(k)}) < \epsilon. \quad (1.1)$$

Further if

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0,$$

 doi:10.20944/preprints202401.0244.v1

then we have that

$$\lim_{k\to\infty}\rho(x_{n(k)},x_{m(k)})=\lim_{k\to\infty}\rho(x_{n(k)+1},x_{m(k)})$$
$$=\lim_{k\to\infty}\rho(x_{n(k)},x_{m(k)+1})=\lim_{k\to\infty}\rho(x_{n(k)+1},x_{m(k)+1})=\epsilon. \quad (1.2)$$

Lemma 3. *Let (E,ρ) be a metric space, and let $A,B\in CL(E)$.*
If $a\in A$ and $\rho(a,B)<c$, then there exists $b\in B$ such that $\rho(a,b)<c$.

Proof. Let $\epsilon=c-\rho(a,B)$.
It follows from definition of infimum that there exists $b\in B$ such that $\rho(a,b)<\rho(a,B)+\epsilon$. Hence, $\rho(a,b)<c$. □

Lemma 4. *Let (E,ρ) be a metric space, and let $A,B\in CL(E)$ and $\phi:[0,\infty)\to[0,\infty)$ be a strictly increasing function. If $a\in A$ and $\rho(a,B)+\phi(\rho(a,B))<c$, then there exists $b\in B$ such that $\rho(a,b)+\phi(\rho(a,b))<c$.*

Proof. Since ϕ is strictly increasing,

$$\rho(a,B)<\phi^{-1}(c-\rho(a,B)).$$

By Lemma 3, there exists $b'\in B$ such that

$$\rho(a,b')<\phi^{-1}(c-\rho(a,B))$$

which yields

$$\rho(a,B)<c-\phi(\rho(a,b')).$$

Again, by applying Lemma 3, there exists $b''\in B$ such that

$$\rho(a,b'')<c-\phi(\rho(a,b')).$$

Let $\min\{\rho(a,b'),\rho(a,b'')\}=\rho(a,b)$.
Then, we have that

$$\rho(a,b)+\phi(\rho(a,b))<c.$$

□

Lemma 5. *If (E,ρ) is a metric space, then $K(E)\subset CL(E)$, where $K(E)$ is the family of nonempty compact subsets of E.*

Proof. Let $A\in K(E)$, and let $\{x_n\}\subset A$ be a sequence such that

$$\lim_{n\to\infty}\rho(x_n,x)=0, \text{ where } x\in E.$$

It follows from compactness of A that there exists a convergent subsequence $\{x_{n(k)}\}$ of $\{x_n\}$. Let

$$\lim_{n\to\infty}\rho(x_{n(k)},a)=0, \text{ for } a\in A.$$

Since

$$\lim_{n\to\infty}\rho(x_{n(k)},x)=0,$$

$x=a\in A$. Hence $A\in CL(E)$. □

2. Fixed point results

Let (E,ρ) be a metric space.

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A set valued map $T : E \to CL(E)$ is called generalized F-contraction if the following condition holds:

for all $x, y \in E$ with $H(Tx, Ty) > 0$

$$\tau + F(H(Tx, Ty)) \le F(m(x, y)) \tag{2.1}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function, and

$$m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\}.$$

We now prove our main result.

Theorem 2. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a generalized F-contraction. If $(F1)$ is satisfied, then T possesses a fixed point.*

Proof. Let $x_0 \in E$ be a point, and let $x_1 \in Tx_0$.

If $x_1 \in Tx_1$, then the proof is completed.

Assume that $x_1 \notin Tx_1$.

Then, $\rho(x_1, Tx_1) > 0$, because $Tx_1 \in CL(X)$. Hence, $H(Tx_0, Tx_1) \ge d(x_1, Tx_1) > 0$. From (2.1) we have that

$$\tau + F(H(Tx_0, Tx_1)) \le F(m(x_0, x_1)). \tag{2.2}$$

We infer that

$$m(x_0, x_1) = \max\{\rho(x_0, x_1), \rho(x_0, Tx_0), \rho(x_1, Tx_1), \frac{1}{2}[\rho(x_0, Tx_1) + \rho(x_1, Tx_0)]\}$$
$$= \max\{\rho(x_0, x_1), \rho(x_1, Tx_1)\}, \text{ because that } \rho(x_0, Tx_0) \le \rho(x_0, x_1) \text{ and}$$
$$\frac{1}{2}[\rho(x_0, Tx_1) + \rho(x_1, Tx_0)] \le \frac{1}{2}[\rho(x_0, x_1) + \rho(x_1, Tx_1)].$$

If $m(x_0, x_1) = \rho(x_1, Tx_1)$, then from (2.2) we obtain that

$$F(\rho(x_1, Tx_1)) < \tau + F(H(Tx_0, Tx_1)) \le F(\rho(x_1, Tx_1))$$

which is a contradiction.

Thus, $m(x_0, x_1) = \rho(x_0, x_1)$. It follows from (2.2) that

$$\frac{1}{2}\tau + F(\rho(x_1, Tx_1)) < \tau + F(H(Tx_0, Tx_1)) \le F(\rho(x_0, x_1)). \tag{2.3}$$

Since (F1) is satisfied, we obtain that

$$\rho(x_1, Tx_1) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_0, Tx_1))).$$

Applying Lemma 3, there exists $x_2 \in Tx_1$ such that

$$\rho(x_1, x_2) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_0, Tx_1)))$$

which implies

$$F(\rho(x_1, x_2)) < \frac{1}{2}\tau + F(H(Tx_0, Tx_1)) \le F(\rho(x_0, x_1)) - \frac{1}{2}\tau. \tag{2.4}$$

 doi:10.20944/preprints202401.0244.v1

Again from (2.1) we have that

$$\frac{1}{2}\tau + F(\rho(x_2, Tx_2)) < \tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) \tag{2.5}$$

which implies

$$\rho(x_2, Tx_2) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_1, Tx_2))).$$

By Lemma 3, there exists $x_3 \in Tx_2$ such that

$$\rho(x_2, x_3) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_1, Tx_2))).$$

Hence, we obtain that

$$F(\rho(x_2, x_3)) < \frac{1}{2}\tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) - \frac{1}{2}\tau. \tag{2.6}$$

Inductively, we have that for all $n = 1, 2, 3, \cdots$,

$$x_n \in Tx_{n-1}$$

and

$$F(\rho(x_n, x_{n+1})) < \frac{1}{2}\tau + F(H(Tx_{n-1}, x_n)) \leq F(\rho(x_{n-1}, x_n)) - \frac{1}{2}\tau. \tag{2.7}$$

Because (F1) is satisfied,

$$\rho(x_n, x_{n+1}) < \rho(x_{n-1}, x_n), \ \forall n = 1, 2, 3, \cdots.$$

Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = r.$$

Assume that $r > 0$.

By Lemma 1, we have that

$$\lim_{n\to\infty} F(\rho(x_n, x_{n+1})) = \lim_{n\to\infty} F(\rho(x_{n-1}, x_n)) = \lim_{t\to r^+} F(t) = F(r^+). \tag{2.8}$$

Taking limit $n \to \infty$ in (2.7) and using (2.8), we obtain that

$$F(r^+) \leq F(r^+) - \frac{1}{2}\tau$$

which is a contradiction, because $\tau > 0$.

Thus, we obtain that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0. \tag{2.9}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Assume that $\{x_n\}$ is not a Cauchy sequence.

Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which (1.1) holds.

That is, the following are satisfied:

$$m(k) > n(k) > k, \ \rho(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } \rho(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

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It follows from (2.1) that

$$\begin{aligned} &F(\rho(x_{n(k)+1}, Tx_{m(k)}) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}) \\ \leq &\tau + F(H(Tx_{n(k)}, Tx_{m(k)}) \leq F(m(x_{n(k)}, x_{m(k)})). \end{aligned} \tag{2.10}$$

We infer that

$$\begin{aligned} &\epsilon \leq \rho(x_{n(k)}, x_{m(k)}) \leq m(x_{n(k)}, x_{m(k)}) \\ = &\max\{\rho(x_{n(k)}, x_{m(k)}), \rho(x_{n(k)}, Tx_{n(k)}), \rho(x_{m(k)}, Tx_{m(k)}), \\ &\frac{1}{2}[\rho(x_{n(k)}, Tx_{m(k)}) + \rho(x_{m(k)}, Tx_{n(k)})]\} \\ \leq &\max\{\rho(x_{n(k)}, x_{m(k)}), \rho(x_{n(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{m(k)+1}), \\ &\frac{1}{2}[\rho(x_{n(k)}, x_{m(k)+1}) + \rho(x_{m(k)}, x_{n(k)+1})]\} \end{aligned} \tag{2.11}$$

Taking limit as $k \to \infty$ on both sides of (2.11) and using (1.2), we obtain that

$$\lim_{k \to \infty} m(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.12}$$

Since (F1) is satisfied, from (2.10) we have that

$$\rho(x_{n(k)+1}, Tx_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).$$

By applying Lemma 3, there exists $y_{m(k)} \in Tx_{m(k)}$ such that

$$\rho(x_{n(k)+1}, y_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).$$

Hence,

$$F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}).$$

Thus, it follows from (2.10) that

$$\begin{aligned} &F(\rho(x_{n(k)+1}, y_{m(k)})) \\ < &\tau + F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}) \\ \leq &\tau + F(H(Tx_{n(k)}, Tx_{m(k)}) \\ \leq &F(m(x_{n(k)}, x_{m(k)})) \end{aligned} \tag{2.13}$$

which leads to

$$\rho(x_{n(k)+1}, y_{m(k)}) < m(x_{n(k)}, x_{m(k)}), \ \forall k = 1, 2, 3, \cdots. \tag{2.14}$$

By taking lim sup as $k \to \infty$ in (2.14) and using (2.12), we have that

$$\lim_{k \to \infty} \sup \rho(x_{n(k)+1}, y_{m(k)}) \leq \epsilon. \tag{2.15}$$

Since

$$\rho(x_{n(k)+1}, Tx_{m(k)}) \leq \rho(x_{n(k)+1}, y_{m(k)}),$$

$$\begin{aligned} &\rho(x_{n(k)+1}, x_{m(k)}) \\ \leq &\rho(x_{n(k)+1}, Tx_{m(k)}) + \rho(Tx_{m(k)}, x_{m(k)}) \\ \leq &\rho(x_{n(k)+1}, y_{m(k)}) + \rho(x_{m(k)+1}, x_{m(k)}) \end{aligned} \tag{2.16}$$

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Taking lim inf as $k \to \infty$ in (2.16) and using (1.2), we obtain that

$$\epsilon \leq \liminf_{k\to\infty} \rho(x_{n(k)+1}, y_{m(k)}). \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$\lim_{k\to\infty} \rho(x_{n(k)+1}, y_{m(k)}) = \epsilon. \tag{2.18}$$

By applying Lemma 1 to (2.13) with (2.12), (2.14) and (2.18), we obtain that

$$F(\epsilon^+) \leq \tau + F(\epsilon^+) \leq F(\epsilon^+)$$

which leads to a contradiction.

Hence, $\{x_n\}$ is a Cauchy sequence.

From the completeness of E, there exists

$$x_* = \lim_{n\to\infty} x_n \in E.$$

It follows from (2.1) that

$$\begin{aligned} F(\rho(x_{n+1}, Tx_*)) &< \tau + F(\rho(x_{n+1}, Tx_*)) \\ &\leq \tau + F(H(Tx_n, Tx_*)) \leq F(m(x_n, x_*)), \end{aligned} \tag{2.19}$$

where $m(x_n, x_*) = \max\{\rho(x_n, x_*), \rho(x_n, x_{n+1}), \rho(x_*, Tx_*),$
$\frac{1}{2}[\rho(x_*, x_{n+1}) + \rho(x_n, Tx_*)]\}$.

Since (F1) is satisfied, from (2.19) we have that

$$\rho(x_{n+1}, Tx_*) < m(x_n, x_*). \tag{2.20}$$

We have that

$$\lim_{n\to\infty} \rho(x_{n+1}, Tx_*) = \lim_{n\to\infty} m(x_n, x_*) = \rho(x_*, Tx_*). \tag{2.21}$$

Assume that $\rho(x_*, Tx_*) > 0$.

By Lemma 1, we have that

$$\begin{aligned} \lim_{n\to\infty} F(\rho(x_{n+1}, Tx_*)) &= \lim_{n\to\infty} F(m(x_n, x_*)) \\ &= \lim_{t\to\rho(x_*,Tx_*)^+} F(t) = F(\rho(x_*, Tx_*)^+). \end{aligned} \tag{2.22}$$

Applying (2.22) to (2.19), we obtain that

$$F(\rho(x_*, Tx_*)^+) \leq \tau + F(\rho(x_*, Tx_*)^+) \leq F(\rho(x_*, Tx_*)^+)$$

which leads to a contradiction.

Hence, $\rho(x_*, Tx_*) = 0$, and hence $x_* \in Tx_*$. □

The following example interprets Theorem 2.

Example 1. *Let* $E = [0, 1]$ *and* $\rho(x, y) = |x - y|$, $\forall x, y \in E$.

Then (E, ρ) *is a complete metric space.*

Define a set-valued map $T : E \to CL(E)$ *by*

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$$Tx = \begin{cases} \{1\}, & (x = 0) \\ \{\frac{2}{5}, \frac{1}{2}\}, & (0 < x \leq 1). \end{cases}$$

Let $\tau = \ln \frac{3}{2}$ *and* $F(t) = \ln t$, $\forall t > 0$.
We show that T *is a generalized* F*-contraction.*
We have that for all $x, y \in E$,

$$H(Tx, Ty) > 0 \Longleftrightarrow (x = 0 \text{ and } 0 < y \leq 1), (0 < x \leq 1 \text{ and } y = 0).$$

$1°$: *Let* $x = 0$ *and* $0 < y \leq 1$.
Then, we obtain that

$$\begin{aligned} &\tau + F(H(Tx, Ty)) - F(\rho(x, Tx)) \\ =&\tau + F(\frac{3}{5}) - F(1) \\ =&\ln \frac{3}{2} + \ln \frac{3}{5} - \ln 1 \\ =&\ln 9 - \ln 10 < 0. \end{aligned}$$

Thus,

$$\tau + F(H(Tx, Ty)) < F(\rho(x, Tx))$$

which implies

$$\tau + F(H(Tx, Ty)) < F(m(x, y)).$$

$2°$: *Let* $0 \leq x < 1$ *and* $y = 1$. *Then, we have that*

$$\begin{aligned} &\tau + F(H(Tx, Ty)) - F(\rho(y, Ty)) \\ =&\tau + F(\frac{3}{5}) - F(1) \\ =&\ln \frac{3}{2} + \ln \frac{3}{5} - \ln 1 \\ =&\ln 9 - \ln 10 < 0. \end{aligned}$$

Thus,

$$\tau + F(H(Tx, Ty)) < F(\rho(y, Ty))$$

which leads to

$$\tau + F(H(Tx, Ty)) < F(m(x, y)).$$

Hence, F *is generalized* F*-contraction. The assumptions of Theorem 2 are satisfied. By Theorem 2,* T *possesses two fixed points,* $\frac{2}{5}$ *and* $\frac{1}{2}$.

Remark 1. *Theorem 2 is a positive answer to Question 4.3 of [9].*

By Theorem 2, we have the following result.

Corollary 1. *Let* (E, ρ) *be a complete metric space. Suppose that* $T : E \to CL(E)$ *is a set-valued map such that for all* $x, y \in E$ *with* $H(Tx, Ty) > 0$

$$\tau + F(H(Tx, Ty)) \leq F(l(x, y))$$

where $\tau > 0$ *and* $F : (0, \infty) \to (-\infty, \infty)$ *is a function, and*

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$$l(x,y) = \max\{\rho(x,y), \frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)], \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}.$$

If $(F1)$ is satisfied, then T possesses a fixed point.

Corollary 2. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CB(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx,Ty) > 0$*

$$\tau + F(H(Tx,Ty)) \leq F(\rho(x,y))$$

where $\tau > 0$ and $F : (0,\infty) \to (-\infty,\infty)$ is a function. If $(F1)$ is satisfied, then T possesses a fixed point.

Corollary 3. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx,Ty) > 0$*

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty) + e[\rho(x,Ty) + \rho(y,Tx)]) \end{aligned} \tag{2.23}$$

where $\tau > 0$ and $F : (0,\infty) \to (-\infty,\infty)$ is a function, and $a,b,c,e \geq 0$ and $a + b + c + 2e = 1$. If $(F1)$ is satisfied, then T possesses a fixed point.

Proof. It follows from (2.23) that

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty) + e[\rho(x,Ty) + \rho(y,Tx)]) \\ = &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty)] + 2e\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]) \\ \leq &F((a+b+c+2e)\max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}) \\ = &F(m(x,y)). \end{aligned}$$

By Theorem 2, T possesses a fixed point. □

Corollary 4. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx,Ty) > 0$*

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b[\rho(x,Tx) + \rho(y,Ty)] + c[\rho(x,Ty) + \rho(y,Tx)]) \end{aligned} \tag{2.24}$$

where $\tau > 0$ and $F : (0,\infty) \to (-\infty,\infty)$ is a function, and $a,b,c \geq 0$ and $a + 2b + 2c = 1$. If $(F1)$ is satisfied, then T possesses a fixed point.

Proof. It follows from (2.24) that

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b[\rho(x,Tx) + \rho(y,Ty)] + c[\rho(x,Ty) + \rho(y,Tx)]) \\ = &F(a\rho(x,y) + 2b\frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)] + 2c\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]) \\ \leq &F((a+2b+2c)\max\{\rho(x,y), \frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)], \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}) \\ = &F(l(x,y)). \end{aligned}$$

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By Corollary 2.2, T possesses a fixed point. □

Corollary 5. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$*

$$\tau + F(H(Tx,Ty)) \le F(\frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)])$$

where $\tau > 0$ and $F : (0,\infty) \to (-\infty,\infty)$ is a function. If $(F1)$ is satisfied, then T possesses a fixed point.

Corollary 6. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$*

$$\tau + F(H(Tx,Ty)) \le F(\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)])$$

where $\tau > 0$ and $F : (0,\infty) \to (-\infty,\infty)$ is a function. If $(F1)$ is satisfied, then T possesses a fixed point.

Theorem 3. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is an Işik type set-valued contraction, i.e. for each $x, y \in E$ and each $u \in Tx$, there exists $v \in Ty$ such that*

$$\rho(u,v) \le \phi(\rho(x,y)) - \phi(\rho(u,v)) \tag{2.25}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a function such that

$$\lim_{t\to 0^+} \phi(t) = 0. \tag{2.26}$$

Then, T possesses a fixed point.

Proof. Let $x_0 \in E$, and let $x_1 \in Tx_0$. Then there exits $x_2 \in Tx_1$ such that

$$\rho(x_1,x_2) \le \phi(\rho(x_0,x_1)) - \phi(\rho(x_1,x_2)).$$

Again, there exists $x_3 \in Tx_2$ such that

$$\rho(x_2,x_3) \le \phi(\rho(x_1,x_2)) - \phi(\rho(x_2,x_3)).$$

Inductively, we have a sequence $\{x_n\} \subset E$ such that for all $n = 1, 2, 3, \cdots,$

$$x_n \in Tx_{n-1} \text{ and } \rho(x_n,x_{n+1}) \le \phi(\rho(x_{n-1},x_n)) - \phi(\rho(x_n,x_{n+1})). \tag{2.27}$$

From (2.27) $\{\phi(\rho(x_{n-1},x_n))\}$ is a non-increasing sequence and bounded below by 0. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \phi(\rho(x_{n-1},x_n)) = r.$$

We show that $\{x_n\}$ is a Cauchy sequence.
Let m, n be any positive integers such that $m > n$.
Then, we have that

$$\begin{aligned}
&\rho(x_n,x_m)\\
\le&\rho(x_n,x_{n+1}) + \rho(x_{n+1},x_{n+2}) + \cdots + \rho(x_{m-1},x_m)\\
\le&\phi(\rho(x_{n-1},x_n)) - \phi(\rho(x_{m-1},x_m))\\
\le&\phi(\rho(x_{n-1},x_n)) - r.
\end{aligned} \tag{2.28}$$

 doi:10.20944/preprints202401.0244.v1

Letting $m, n \to \infty$ in (2.28), we obtain that

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of E that

$$x_* = \lim_{n\to\infty} x_n \text{ exists.} \tag{2.29}$$

Now, we show that x_* is a fixed point for T.
It follows from (2.25) that for $x_n \in Tx_{n-1}$, there exists $v \in Tx_*$ such that

$$\rho(x_n, v) \le \phi(\rho(x_{n-1}, x_*)) - \phi(\rho(x_n, v)) \le \phi(\rho(x_{n-1}, x_*)). \tag{2.30}$$

Taking limit $n \to \infty$ in equation (2.30) and using (2.26), we infer that

$$\lim_{n\to\infty} \rho(x_n, v) = 0$$

which implies

$$x_* = v \in Tx_*.$$

□

Example 2. *Let $E = \{x_n : x_n = 1 + 2 + 3 + \cdots + n, n = 1, 2, 3, \cdots\}$ and $\rho(x, y) = |x - y|, \forall x, y \in E$.*
Then, (E, ρ) is a complete metric space.
Define a map $T : E \to CL(E)$ by

$$Tx = \begin{cases} \{x_1\}, & (x = x_1) \\ \{x_1, x_2, x_3, \cdots x_{n-1}\}, & (x = x_n). \end{cases}$$

Let $\phi(t) = \frac{1}{2}t, \forall t \ge 0$.
We show that condition (2.25) is satisfied.
Consider the following two cases.
1°: Let $x = x_1$ and $y = x_n, n = 2, 3, 4, \cdots$.
Then, for $u = x_1 \in Tx$, there exists $v = x_1 \in Ty$ such that

$$\rho(u, v) = 0 < \frac{1}{2}\rho(x_1, x_n) = \phi(\rho(x_1, x_n)) = \phi(\rho(x_1, x_n)) - \phi(\rho(u, v)).$$

2°: Let $x = x_n$ and $y = x_m, m > n, n = 2, 3, 4, \cdots$.
For $u = x_k \in Tx$ $(k = 1, 2, 3, \cdots, n-1)$, there exists $v = x_k \in Ty$ such that

$$\rho(u, v) = 0 < \frac{1}{2}\rho(x_n, x_m) = \phi(\rho(x_n, x_m)) = \phi(\rho(x_n, x_m)) - \phi(\rho(u, v)).$$

This show that T satisfies condition (2.25). Thus, all conditions of Theorem 3 hold. From Theorem 3 T possesses a fixed point, $x_* = x_1$.

Corollary 7. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for each $x, y \in E$,*

$$H(Tx, Ty) < \phi(\rho(x, y)) - \phi(H(Tx, Ty)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function such that

$$\lim_{t\to 0^+} \phi(t) = 0.$$

Then, T possesses a fixed point.

Proof. Let $x, y \in E$ and let $u \in Tx$.

As ϕ is strictly increasing,

$$\rho(u, Ty) + \phi(\rho(u, Ty)) < \phi(\rho(x, y)).$$

Applying Lemma 4, there exists $v \in Ty$ such that

$$\rho(u, v) + \phi(\rho(u, v)) < \phi(\rho(x, y)).$$

By Theorem 3, T possesses a fixed point. □

From Theorem 3 we have the following result.

Corollary 8. *[14] Let (E, ρ) be a complete metric space. Suppose that $f : E \to E$ is a map such that for each $x, y \in E$,*

$$\rho(fx, fy) \leq \phi(\rho(x, y)) - \phi(\rho(fx, fy))$$

where $\phi : [0, \infty) \to [0, \infty)$ is a function such that

$$\lim_{t \to 0^+} \phi(t) = 0.$$

Then, f possesses a fixed point.

3. Application

In this section, we give an application of our result to integral inclusion.

Let $[a, b] \subset \mathbb{R}$ be a closed interval, and let $C([a, b], \mathbb{R})$ be the family of continuous mapping from $[a, b]$ into $\mathbb{R}$.

Let $E = C([a, b], \mathbb{R})$ and $\rho(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$ for all $x, y \in E$.

Then, (E, ρ) is a complete metric space.

Consider the Fredholm type integral inclusion:

$$x(t) \in \int_a^b K(t, s, x(s))ds + f(t), t \in [a, b] \tag{3.1}$$

where $f \in E$, $K : [a, b] \times [a, b] \times \mathbb{R} \to CB(\mathbb{R})$, and $x \in E$ is the unknown function.

Suppose that the following conditions are satisfied:

(1°) for each $x \in E, K(\cdot, \cdot, x(s)) = K_x(\cdot, \cdot)$ is continuous;

(2°) there exists a continuous function $Z : [a, b] \times [a, b] \to [0, \infty)$ such that for all $t, s \in [a, b]$ and all $u, v \in E$,

$$|k_u(t, s) - k_v(t, s)| \leq Z(t, s)\rho(u(s), v(s))$$

where $k_u(t, s) \in K_u(t, s), k_v(t, s) \in K_v(t, s)$;

(3°) there exists $\alpha > 1$ such that

$$\sup_{t \in [a,b]} \int_a^b Z(t, s)ds \leq \frac{1}{2 + \alpha}.$$

Theorem 4. *Let (E, ρ) be a complete metric space. If conditions (1°),(2°) and (3°) are satisfied, then the integral inclusion (3.1) has a solution.*

Proof. Define a set-valued map $T : E \to CB(E)$ by

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$$Tx = \{y \in E : y(t) \in \int_a^b K(t,s,x(s))ds + f(t), t \in [a,b]\}.$$

Let $x \in E$ be given.

For the set-valued map $K_x(t,s) : [a,b] \times [a,b] \to CB(\mathbb{R})$, by applying Michael's selection theorem, there exists a continuous map $k_x(t,s) : [a,b] \times [a,b] \to \mathbb{R}$ such that

$$k_x(t,s) \in K_x(t,s), \forall t,s \in [a,b].$$

Thus,

$$\int_a^b k_x(t,s)ds + f(t) \in Tx,$$

and so $Tx \neq \emptyset$.

Since f and k_x are continuous, $Tx \in CB(E)$ for each $x \in E$.

Let $y_1 \in Tx_1$.

Then,

$$y_1(t) \in \int_a^b K(t,s,x_1(s))ds + f(t), t \in [a,b].$$

Hence, there exists $k_{x_1}(t,s) \in K_{x_1}(t,s), \forall t,s \in [a,b]$ such that

$$y_1(t) = \int_a^b k_{x_1}(t,s)ds + f(t), \forall t,s \in [a,b].$$

It follows from (2°) that there exists $z(t,s) \in K_{x_2}(t,s)$ such that

$$|k_{x_1}(t,s) - z(t,s)| \leq Z(t,s)\rho(x_1(s), x_2(s)), \forall t,s \in [a,b].$$

Let $U : [a,b] \times [a,b] \to CB(\mathbb{R})$ be defined by

$$U(t,s) = K_{x_2}(t,s) \cap \{u \in \mathbb{R} : \rho(k_{x_1}(t,s), u) \leq \rho(x_1(s), x_2(s))\}.$$

From (1°) U is continuous. Hence, it follows that there exists a continuous map $k_{x_2} : [a,b] \times [a,b] \to \mathbb{R}$ such that

$$k_{x_2}(t,s) \in U(t,s), \forall t,s \in [a,b].$$

Let

$$y_2(t) = \int_a^b k_{x_2}(t,s)ds + f(t), \forall t,s \in [a,b].$$

Then,

$$y_2(t) \in \int_a^b K_{x_2}(t,s)ds + f(t) = \int_a^b K(t,s,x_2(s))ds + f(t), \forall t,s \in [a,b],$$

and so $y_2 \in Tx_2$.

Thus, we obtain that

$$\begin{aligned}
&\rho(y_1, y_2) = |\int_a^b k_{x_1}(t,s) - k_{x_2}(t,s)ds| \\
&\leq \sup_{t \in [a,b]} \int_a^b |k_{x_1}(t,s) - k_{x_2}(t,s)|ds \\
&\leq \sup_{t \in [a,b]} \int_a^b Z(t,s)ds\rho(x_1(s), x_2(s)) \\
&\leq \frac{1}{2+\alpha}\rho(x_1(s), x_2(s)).
\end{aligned}$$

Thus, we have that

$$(1 + \frac{1}{2}\alpha)\delta(Tx_1, Tx_2) \leq \frac{1}{2}\rho(x_1, x_2)$$

which implies

$$(1 + \frac{1}{2}\alpha)H(Tx_1, Tx_2) \leq \frac{1}{2}\rho(x_1, x_2).$$

Hence, we obtain that

$$H(Tx_1, Tx_2)) \leq \phi(\rho(x_1, x_2)) - \phi(\alpha H(Tx_1, Tx_2))$$
$$<\phi(\rho(x_1, x_2)) - \phi(H(Tx_1, Tx_2)) \text{ where } \phi(t) = \frac{1}{2}t, \forall t \geq 0.$$

By Corollary 7, T possesses a fixed point, and hence the integral inclusion (3.1) has a solution. □

4. Conclusions

Our results are generalizations and extensions of F-contractions and Işik contractions to set-valued maps on metric spaces. We give a positive answer to the Question 4.3 of [9] and an application to integral inclusion.

Data Availability Statement: Not applicable.

Acknowledgments: The author express his gratitude to the referees for careful reading and giving variable comments. This research was supported by Hanseo University.

Conflicts of Interest: The author declares no conflict of interest.

References

1. J. Ahmad, A. Al-Rawashdeh, A. Azam, *Some new fixed point theorems for generalized F-contractions in complete metric spaces*, Fixed Point Theory Appl. (2015), 2015:80
2. W.M. Alfaqih, M. Imdad, R. Gubran, *An observation on F-weak contractions and discontinuity at the fixed point with an applications*, J. Fixed Point Theory Appl. (2020) 22:66
3. M. Arshad, S. Khan and J. Ahmad, *Fixed point results for F-contractions involving some new rational expressions*, JP Journal of Fixed Point Theory and Applications, 11(1) (2016), 79-97
4. S.V. Bedre, *Remarks on F-weak contractions and discontinuity at the fixed point*, Advances in theTheory of Nonlinear Analysis and its Applications 4(4)(2020), 260-265.
5. V. Berinde, M. Păcurar, *The role of the Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform. 22(2)(2013), 143-150.
6. S.H. Cho, J.S. Bae, *Fixed points of weak α-contraction type maps*, Fixed Point Theory and Appl. 2014, 175(2014)
7. M. Cosentino, P. Vetro, *Fixed point results for F-contractive mappings of Hardy Rogers-Type*, Filomat 28(4) (2014), 715-722
8. N.V. Dung, V.L. Hang, *A fixed point theorem for generalized F-contractions on complete metric spaces*, Vietnam J. Math. **43**(2015), 743-753
9. N. Fabiano, Z. Kadelburg, N. Mirkov, V. Š. Čavić, S. Radenović, *On F-contractions: A Survey*, Contemporary Mathematics (2022) 3:3, 327
10. H. Huang, Z.D. Mitrović, K. Zoto, S. Radenović, *On convex F-contraction in b-metric spaces*, Axioms **2021**, 10,71. https://doi.org/10.3390/axioms10020071
11. A. Hussain, *Ćirić type α-ψ F-contraction involving fixed point on a closed ball*, Honam Math. J. **41**(2019), 19-34
12. A. Hussain and M. Arshad, *New type of multivalued F-Contraction involving fixed Point on Closed Ball*, J. Math. Comp. Sci. 10 (2017), 246-254
13. N. Hussain and P. Salimi, *Suzuki-wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math., 20 (2014), doi: 10.11650/tjm.18.2014.4462.
14. H. Işik, B. Mohammadi, M.R. Haddadi, V. Parvaneh, *On a new generalization of Banach contraction principle with application*, Mathematics **2019**,7,862; doi/;10.3390/math7090862

15. N. Konwar, P. Debnath, *Fixed point results for a family of Interpolative F-contractions in b-metric spaces*, Axioms 2022, 11, 621. https://doi.org/10.3390/axioms11110621
16. G. Minak, A. Halvaci and I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat, 28(6) (2014), 1143-1151
17. H. Piri, P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory and Appl **2014**, Article ID 210(2014), doi:10.1186/1687-1812-2014-210
18. H. Piri, P. Kumam, *Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces*, Fixed Point Theory and Applications (2016) **2016:90** doi:10.1186/s13663-016-0577-5
19. N.A. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory Appl. (2013), Article ID 277 (2013). doi:10.1186/1687-1812-2013-277
20. T. Stephen, Y. Rohen, M.K. Singh, K.S. Devi, *Some rational F-contractions in b-metric spaces and fixed points*, Nonlinear Functional Analysis and Applications Vol. 27, No. 2 (2022), pp. 309-322
21. D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spacs*, Fixed Point Theory and Appl **2012**, Article ID 94(2012), doi:10.1186/1687-1812-2013-277
22. D. Wardowski, N.V. Dung, *Fixed points f-weak contractions on complete metric spaces*, Demonstr. Math. **1**(2014), 146-155
23. M. Younis, N. Mirkov, A. Savić, M. Pantović, S. Radenović, *Some critical remarks on recent results concerning F-contractions in b-metric spaces*, CUBO, A mathematical Journal, Accepted: 27 December 2022