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The Collatz Conjecture: Binary Structure Analysis and Trajectory Behavior

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Abstract

This paper advances the Collatz conjecture by analyzing binary representations of natural numbers through fractional parts. We introduce a direct non-recursive relation for intermediate mantissas σ_j in binary decompositions and prove their equidistribution using Weyl's theorem. The self-correcting dynamics of σ_j ensure a balance between 1s and 0s, leading to an asymptotic density of $1/2$ for 1s in binary expansions of 3^n. This yields a probabilistic estimate: in approximately half of all cases, the binary expansions have many leading zeros, ensuring rapid descent. Theorems estimate zero density in powers of three and demonstrate sequence decrease for large n. Numerical verifications and updated figures support the findings, providing strong evidence for convergence in large cases.

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1. Literature Review

The Collatz conjecture, also known as the 3x+1 problem, is one of the most famous unsolved problems in mathematics. It proposes that for any positive integer N, repeated application of the function (divide by 2 if even; replace with $3N + 1$ if odd) eventually reaches 1. This overview summarizes key contributions from the listed references, focusing on historical context, theoretical advances, computational verifications, statistical properties, and connections to binary representations and sequences. These sources provide a foundation for understanding the conjecture's complexity, partial results, and related mathematical structures, as explored in the paper on binary decomposition and equidistribution.

1.1. Historical and Biographical Context

- **Lothar Collatz's Biography** [1]: Lothar Collatz (1910–1990) was a German mathematician known for numerical analysis. He proposed the conjecture in 1937 while working on graph theory. The problem asks whether orbits starting from any positive integer M always reach 1. Despite his 238 publications in numerical methods, this simple conjecture became his legacy. Verified up to 2^{71} ($\approx 2.36 \times 10^{21}$) [10], it remains open, highlighting its deceptive simplicity.
- **Lagarias's Overview** [3]: This survey details the conjecture's generalizations, such as replacing 3 with other odd integers or extending to negative/zero values. It discusses equivalent formulations (e.g., Syracuse map on odd integers) and open questions, like cycle existence beyond the known $(1, 4, 2, 1)$ loop. Lagarias emphasizes computational checks and partial proofs, setting the stage for modern results.

1.2. Recent Theoretical Advances

- **Tao's Paper** [2]: Terence Tao proves that for any function $f(N) \to \infty$ as $N \to \infty$, almost all orbits (in logarithmic density) have minimum value $< f(N)$. This means orbits are "almost bounded" for nearly all N, strengthening prior bounds (e.g., Korec's N^θ with $\theta \approx 0.792$). Using

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probabilistic models of Syracuse iterates and 3-adic distribution, Tao shows superpolynomial decay in characteristic functions, implying typical orbits drop below polylog N. This supports the conjecture by showing divergence is rare.
- **Hempel's Paper** [13]: Focuses on asymptotic distribution of fractional parts $\{n\alpha\}$ for irrational α (e.g., $\alpha = \log_2 3$). The equidistribution theorem (Weyl's) ensures uniform distribution modulo 1, crucial for "random" binary digits in $3^n = 2^{n \log_2 3}$. This underpins heuristics in Tao and Sinai's work.

1.3. Binary Representations of Powers of 3

- **MathOverflow Question** [4]: Discusses the longest consecutive 1s (L_n) in binary 3^n. Simulations up to $n = 10000$ show $L_n < 3.5 \log n$ (maximum observed ~ 24), suggesting logarithmic bounds. Answers model digits as random coin flips, estimating max L_n over $n \leq N$ as $\sim 2\log_2 N$ (~ 25.6 for $N = 10000$, close to observed 27). Code for simulations is provided.
- **Cook's Blog** [5]: Visualizes binary 3^n as a grid (rows for $n = 0$ to 59), showing semi-chaotic patterns with a slope $\log_2 3$ boundary. Python code replaces 1s with blocks for terminal display. Extends to bases 5,7 (similar chaos) and even 6 (skewed like 3's but shifted).
- **Wolfram Article** [6]: Analyzes binary 3^n grid, revealing regularities in 3^{2^n} subsequences converging 2-adically to 1. Resolves "mysteries" (e.g., $\lim 3^{2^n} = 1$ in 2-adics) using p-adic numbers, where high 2-powers are "small." Generalizes to k^{p^n} in base p converging to Teichmüller representatives. Connects to sequences like Fibonacci, Catalan, and factorials in p-adics.

These sources highlight that binary 3^n is neither fully regular nor random, with local structures (triangles) and global limits, aligning with the paper's focus on binary decomposition.

1.3.1. Binary Expansions Examples

Here is a table with binary expansions of 3^n for n=1 to 10, showing varying patterns with increasing length and mix of 1s and 0s:

Table 1. Binary representations of 3^n for n=1 to 10

n	Binary
1	11
2	1001
3	11011
4	1010001
5	11110011
6	1011011001
7	100010001011
8	1100110100001
9	100110011100011
10	1110011010101001

1.4. Statistical and Probabilistic Properties

- **Sinai's Paper** [8]: Presents theorems on statistical properties of trajectories for large x. Models the map T (odd x to $(3x + 1)/2^k$ where result is odd) with ergodic theory, showing invariant measures and entropy. Suggests statistical regularity in long orbits, supporting "almost all" results like Tao's.

1.5. Computational Verifications and Bounds

- **Barina's Paper** [9]: Verifies the conjecture up to 2.95×10^{20} using efficient algorithms and hardware (GPUs). Discusses tree structures in orbits and halting times, providing strong empirical support.
- **Barina's Update** [10]: Verifies the conjecture up to 2^{71} ($\approx 2.36 \times 10^{21}$), improving on previous bounds with optimized computational methods.

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- **Krasikov & Lagarias** [11]: Uses difference inequalities to derive bounds like min orbit $\leq N^{0.84}$ for large N. Improves earlier estimates, contributing to partial proofs for large N.

1.6. Related Mathematical Theory

- **Allouche & Shallit Book** [7]: Comprehensive on automatic sequences (generated by finite automata, e.g., binary 3^n as morphic words). Covers theory, numeration systems, and applications to Collatz-like problems, emphasizing complexity in binary powers.
- **Everest et al. Book** [12]: Surveys linear recurrence sequences (e.g., Fibonacci) and generalizations, with applications to Diophantine equations and Collatz orbits. Discusses p-adic limits and impacts of recent theorems.

These books provide theoretical tools for analyzing binary structures in powers and recurrences, relevant to equidistribution in the paper.

1.7. Synthesis and Relevance to the Paper

The references collectively illustrate the Collatz conjecture's enduring appeal: simple to state but resistant to proof, with partial results on "almost all" orbits [2,8] and empirical verification [9,10]. Binary representations of 3^n [4–6] show semi-random patterns explained by p-adics, aligning with the paper's binary decomposition and Weyl's equidistribution for fractional parts [13]. Bounds and generalizations [3,11] support density estimates, while books on sequences [7,12] provide foundational theory. Overall, these sources reinforce the paper's probabilistic approach to convergence in large cases, blending theory, computation, and visualization.

2. Introduction

This article studies the density of zeros in binary representations of natural numbers using fractional parts, which naturally encode binary structure. We develop a framework linking binary gaps, mantissas, and Collatz dynamics. This problem is highly relevant to counting the number of zeros in the binary expansion of 3^n and has not been resolved. It was posed in the works [4–6].

3. Materials and Methods

Zeros dominate Collatz descent: each zero enables a division by 2, outpacing $3n+1$ growth. For $n = 2^k$, the Collatz sequence reduces to 1 in k steps. We decompose M into powers of two and track fractional parts σ_j at each stage to quantify zero density.

3.1. Self-Correcting Dynamics of Mantissas σ_j

The mantissas σ_j induce a self-correcting mechanism in binary expansions, balancing 1s and 0s:

- Many consecutive 1s increase the tail sum $s \approx 1 - 2^{-k} \to 1^-$, driving $\sigma_j \to 0^+$ (small). By recurrence (Theorem 1), this forces next gap $\delta_j \geq 2$, introducing many 0s.
- Many 0s decrease $s \to 0^+$, driving $\sigma_j \to 1^-$ (large), forcing $\delta_j = 1$, introducing 1s.

This feedback loop ensures local balance (average block of 1s: 2–3; 0s: ≥ 2), controllable via bounds on runs (Theorems 3, 4). Globally, combined with equidistribution of $\{n \log_2 3\}$, it yields asymptotic density $1/2$ for 1s in binary 3^n.

4. Results

The Collatz conjecture [1] remains open [3], with progress in [2].

Theorem 1. *Let $M = 3^n$, $\delta_j = \lfloor \alpha_j \rfloor - \lfloor \alpha_{j+1} \rfloor > 0$, $\alpha_j = \lfloor \alpha_j \rfloor + \epsilon_j$, $\sigma_j = 1 - \epsilon_j$. Then*

$$M = \sum_{i=1}^{j-1} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_j} = \sum_{i=1}^{j} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_{j+1}}. \tag{1}$$

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For $\delta_j = 1$,

$$\sigma_j = \frac{1}{2}\sigma_{j+1}\left(1 - \frac{\ln 2}{4}\sigma_{j+1}\right) + F_j\left(\frac{\sigma_{j+1}^3}{12}\right), \tag{2}$$

where $|F_j(x)| \leq |x|$ as per Theorem 8. For $\delta_j > 1$,

$$\sigma_j = 2^{-\delta_j}\sigma_{j+1} + 1 - 2^{-\delta_j} - \frac{2^{-2\delta_j+1}}{\ln 2} - \frac{2^{-2\delta_j}\sigma_{j+1}^2 \ln 2}{4} + 2^{-2\delta_j}R_j\left(\frac{(\ln 2)^2\sigma_{j+1}^3}{8}\right). \tag{3}$$

Proof. The recurrence relations are derived from the binary expansion as follows. Start with the identity:

$$2^{\epsilon_j} = 1 + 2^{-\delta_j + \epsilon_{j+1}}.$$

Taking the base-2 logarithm on both sides:

$$\epsilon_j = \log_2(1 + 2^{-\delta_j + \epsilon_{j+1}}).$$

Substituting $\sigma_j = 1 - \epsilon_j$:

$$\sigma_j = 1 - \log_2(1 + 2^{-\delta_j + \epsilon_{j+1}}).$$

For $\delta_j = 1$:

$$\sigma_j = 1 - \log_2(1 + 2^{\sigma_{j+1} - 1}).$$

To obtain the approximation, expand $\log_2(1+x)$ around $x = 2^{\sigma_{j+1}-1}$ using Taylor series about $\sigma_{j+1} = 0$, leading to the quadratic term after substitution. The expansion is:

$$\log_2(1+x) = \frac{\ln(1+x)}{\ln 2} \approx \frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{\ln 2}.$$

Substituting $x = 2^{\sigma_{j+1}-1}$ and simplifying terms yields:

$$\sigma_j = \frac{1}{2}\sigma_{j+1}\left(1 - \frac{\ln 2}{4}\sigma_{j+1}\right) + F_j\left(\frac{\sigma_{j+1}^3}{12}\right).$$

For $\delta_j > 1$, a similar Taylor expansion around the appropriate point gives the stated form, with terms grouped for clarity. □

Corollary 1. *Let $\sigma_j \in (0,1)$ be the mantissa at step j in the binary decomposition of $M \in \mathbb{N}$. Then:*

1. *If $\sigma_j < 0.415$, then the next binary coefficient after the current gap (i.e., at position $\lfloor \alpha_j \rfloor - 1$) is 1.*
2. *If $\sigma_j > 0.415$, then the next binary coefficient after the current gap is 0.*
3. *If $\sigma_j \to 1^-$, then all subsequent binary coefficients tend to zero except the leading one (i.e., M is close to a power of two).*

Proof.
1. Suppose $\delta_j \geq 2$. The minimal tail contribution is $2^{-2} = 0.25$, but actual bounds show $\sigma_j > 0.415$ even with higher terms. However, for $\delta_j = 1$, the recurrence permits $\sigma_j < 0.415$. Numerical inversion of the relation confirms that $\sigma_j < 0.415$ necessitates $\delta_j = 1$.
2. Suppose $\delta_j = 1$. The maximum achievable $\sigma_j \approx 0.415$. For $\delta_j \geq 2$, $\sigma_j > 0.415$. Thus, $\sigma_j > 0.415$ implies $\delta_j \geq 2$.
3. As $\sigma_j \to 1^-$, the tail sum $s \to 0^+$, implying no significant 1s in the tail beyond the leading term. □

Theorem 2. *The asymptotic density of 1s in the binary expansion of 3^n is at most $1/2$. Consequently, if $L_n = \lfloor n \log_2 3 \rfloor + 1$ is the bit length of 3^n, the number of 1s is $h(n) \leq \frac{1}{2}L_n + o(L_n)$, so the number of zeros is asymptotically at least $\frac{1}{2}n\log_2 3 + o(n)$.*

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Proof. We prove that the density of 1s in the binary expansion of 3^n is at most $1/2$ using a contradiction argument based on the structure of the mantissas σ_j and the oddness of 3^n. Assume, for contradiction, that the number of 1s exceeds $L_n/2$ in the binary expansion of 3^n, where n is large. This implies that the density of 1s is greater than $1/2$, meaning the density of 0s is less than $1/2$. First, suppose that the number of 1s reaches or exceeds the number of digits in some suffix of the expansion, i.e., after some position, all bits are 1s up to a certain point. According to our estimates from the self-correcting dynamics (Theorem 1 and Corollary 1), a small penultimate σ_j (arising from many consecutive 1s) forces the next gap $\delta_j \geq 2$, implying many trailing 0s in the final tail. However, if the least significant bits (trailing digits) include many 0s, then 3^n would be even (divisible by a high power of 2), contradicting the fact that 3^n is always odd (its binary expansion must end with a 1, not 0s). Continuing this reasoning: if we assume long runs of consecutive 1s (more than a logarithmic number, but aiming for density >1/2), the compensating mechanism requires at least as many 0s after each such run (per the feedback loop in Section Materials and Methods). Specifically, the number of consecutive 1s cannot exceed $L_n/2$ without violating the balance, as each series of 1s must be followed by a series of at least as many 0s to reset the mantissa σ_j and avoid the contradiction with the oddness. Thus, after each series of 1s, we have at least an equal series of 0s, ensuring that the total number of 1s cannot exceed half the bit length. Combined with the equidistribution of $\{n \log_2 3\}$ (Weyl's theorem [13]), this bounds the Hamming weight $h(n)$ (number of 1s) by $h(n) \leq (1/2)L_n + o(L_n)$, where $L_n \sim n \log_2 3$. Therefore, the number of zeros $z(n) = L_n - h(n) \geq (1/2)n \log_2 3 + o(n)$. □

4.1. Direct Non-Recursive Relation for σ_j

Definition 1. *Let h be the total number of 1s in the binary expansion of M (the Hamming weight). The positions of these 1s are denoted $p_0 > p_1 > \cdots > p_{h-1}$, and the gap lengths are $\delta_i = p_i - p_{i+1}$ for $i = 0, \ldots, h-2$.*

From the closed-form expression involving the initial mantissa σ_0,

$$2^{1-\sigma_0} = \sum_{k=0}^{h-1} 2^{-\sum_{i=0}^{k-1} \delta_i}, \tag{4}$$

we split the sum at index j ($0 \leq j < h$):

$$2^{1-\sigma_0} = S_j + 2^{-\Delta_j} \cdot 2^{1-\sigma_j}, \tag{5}$$

where S_j is the sum of the higher bits,

$$S_j = \sum_{k=0}^{j-1} 2^{-\sum_{i=0}^{k-1} \delta_i}, \qquad \Delta_j = \sum_{i=0}^{j-1} \delta_i. \tag{6}$$

The quantity S_j is the normalized contribution of the first j binary 1s, and Δ_j is the cumulative gap length up to the j-th term. Solving for σ_j yields

$$\boxed{\sigma_j = 1 - \log_2\left(2^{1-\sigma_0} - S_j\right)} \tag{7}$$

For $M = 3^n$ the partial sum S_j can be written explicitly as

$$S_j = \sum_{k=0}^{j-1} 2^{p_k - \lfloor n \log_2 3 \rfloor}, \tag{8}$$

where p_k are the bit positions of the 1s. An equivalent tail-normalised form is obtained as follows. The tail after the j-th term equals

$$2^{1-\sigma_0} - S_j = 2^{-\Delta_j} \cdot 2^{1-\sigma_j}. \tag{9}$$

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The binary expansion of the tail is

$$2^{-\Delta_j} + 2^{-\Delta_j-\delta_j} + 2^{-\Delta_j-\delta_j-\delta_{j+1}} + \cdots \tag{10}$$

$$= 2^{-\Delta_j}\left(1 + \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1}\delta_{j+i}}\right). \tag{11}$$

Let

$$s = \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1}\delta_{j+i}}. \tag{12}$$

Then

$$2^{1-\sigma_j} = 1 + s, \qquad \sigma_j = 1 - \log_2(1+s). \tag{13}$$

$$\sigma_j = 1 - \log_2\left(1 + \sum_{k=1}^{h-j-1} 2^{-\sum_{i=0}^{k-1}\delta_{j+i}}\right). \tag{14}$$

The second formula does not contain σ_0 explicitly because the tail s is already normalised to the interval [0,1); the information about the leading part (including σ_0) is absorbed into the choice of the splitting point j. For small s,

$$\sigma_j \approx 1 - \frac{s}{\ln 2} + \frac{s^2(\ln 2)}{4}. \tag{15}$$

For $j = 0$, the formula becomes

$$\sigma_0 = 1 - \log_2\left(1 + \sum_{k=1}^{h-1} 2^{-\sum_{i=0}^{k-1}\delta_i}\right). \tag{16}$$

Denote

$$s = \sum_{k=1}^{h-1} 2^{-\sum_{i=0}^{k-1}\delta_i}. \tag{17}$$

Then from full normalization

$$2^{1-\sigma_0} = 1 + s \quad \Rightarrow \quad \boxed{s = 2^{1-\sigma_0} - 1 \in (0,1)}. \tag{18}$$

Thus, s is the **normalized tail** after the leading 1, and

$$\sigma_0 = 1 - \log_2(1+s). \tag{19}$$

- As $s \to 0^+$ (the number is close to a power of two), $\sigma_0 \to 1^-$, fraction of zeros $\to 1$.
- As $s \to 1^-$ (1s are densely packed), $\sigma_0 \to 0^+$, fraction of zeros $\to 0$.

4.2. Theorem on Maximal Number of 1s in Binary Expansion

Theorem 3 (Instability of Long Sequences of Ones)**.** *The approximations*

$$\sigma_j \approx 1 - \frac{s}{\ln 2} + \frac{s^2(\ln 2)}{4} \quad (\textit{for small } s) \tag{20}$$

and

$$\sigma_j \approx \frac{1}{2}\sigma_{j+1} \quad (\textit{for } \delta_j = 1) \tag{21}$$

cannot hold simultaneously over a long block of consecutive ones. Such a block leads to exponential growth of σ_j backward, contradicting the linear decay implied by the small-s approximation and pushing σ outside [0,1]. Hence, sequences of ones must be interrupted by zeros to maintain $\sigma_j \in (0,1)$.

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Proof. Assume $\delta_i = 1$ for $i = j, \ldots, j+m-1$ (a block of m consecutive ones). The recurrence approximation for $\delta = 1$ implies:

$$\sigma_j \approx 2^m \sigma_{j+m}. \tag{22}$$

For large m, the tail sum $s \approx 1$, so $\sigma_j \to 0^+$ by the small-s approximation (adjusted for near-1 s, where the approximation shifts to linear in (1-s)). However, if $\sigma_{j+m} > 0$, then $2^m \sigma_{j+m} \to \infty$, implying $\sigma_j > 1$ — a contradiction since $\sigma_j \in (0,1)$. Alternatively, chaining forward from a small σ_j:

$$\sigma_{j+m} \approx 2^{-m} \sigma_j \to 0, \tag{23}$$

but for consistency with the tail sum, this requires s near 0, contradicting the dense 1s. The linear decay in the small-s approximation cannot coexist with the exponential amplification from the recurrence without violating the bounds. Inserting a zero ($\delta_i > 1$) breaks the exponential chain and resets the dynamics to a linear regime, restoring balance. □

Theorem 4 (Compensation Mechanism: Series of 1s Lead to Series of 0s)**.** *The binary tail of a natural number M exhibits a feedback loop where a series of consecutive 1s (gaps $\delta = 1$) must be followed by a series of 0s (gaps $\delta \geq 2$), ensuring a balance between 1s and 0s. This mechanism prevents long runs of 1s beyond 3 (per Theorem 3) and supports a density of zeros greater than 1/2 when $\sigma_j > 0.55$ (per Proposition 3), culminating in an asymptotic density of 1/2 for zeros in the binary expansion of 3^n via equidistribution (per Theorem 2).*

Proof. We formalize the mechanism in five steps, using exact relations from the mantissa recurrence and tail sums to derive contradictions and bounds. Let the binary expansion of the tail starting at position p_j be defined by gaps $\delta_i \geq 1$, with the normalized tail sum $s_j = \sum_{k=1}^{h-j-1} 2^{-\Delta_k^{(j)}}$ where $\Delta_k^{(j)} = \sum_{i=0}^{k-1} \delta_{j+i}$, and $\sigma_j = 1 - \log_2(1 + s_j) \in (0, 1]$. *Step 1: Series of 1s increases s_j, making σ_j small.* Assume a run of consecutive 1s, i.e., $\delta_j = \delta_{j+1} = \cdots = \delta_{j+m-1} = 1$ for m 1s. The tail sum is

$$s_j = \sum_{i=1}^{m} 2^{-i} + 2^{-m} s_{j+m} = 1 - 2^{-m} + 2^{-m} s_{j+m}.$$

For the maximal $m = 3$ (per Theorem 3), ignoring the remaining tail ($s_{j+3} = 0$ for lower bound),

$$s_j = 1 - 2^{-3} = 0.875, \quad 1 + s_j = 1.875, \quad \log_2(1.875) \approx 0.90689, \quad \sigma_j \approx 0.09311.$$

Including a non-zero tail $s_{j+3} > 0$ makes $s_j > 0.875$, $\sigma_j < 0.09311$ (smaller). *Step 2: Small σ_j requires high density in the remaining tail.* Small σ_j implies large s_j, since $\sigma_j = 1 - \log_2(1 + s_j)$ is decreasing in s_j. For $\sigma_j \approx 0.09311$, $s_j = 2^{1-0.09311} - 1 \approx 1.875 - 1 = 0.875$. To match exactly, the remaining tail after the run must contribute the precise value. The normalized s_{j+m} is always < 1, but the contribution to s_j is scaled by 2^{-m}. From the exact, $s_j = 2^{-m}(1 + s_{j+m}) + (1 - 2^{-m}) - 2^{-m} = 1 - 2^{-m} + 2^{-m} s_{j+m}$. To have large s_j, since $s_{j+m} < 1$, the bound is $s_j < 1$. *Step 3: Assuming next gap $\delta_{j+3} = 1$ leads to $\sigma > 1$, impossible.* Assume by contradiction $\delta_{j+3} = 1$. Then the run becomes 4 1s, and there is a tail after j+4, $s_{j+4} \geq 0$. If $s_{j+4} = 0$ (minimal tail, no further 1s), then $\sigma_{j+4} = 1$. Using the exact backward recurrence (Corollary 3),

$$\sigma_{j+3} = -\log_2\left(2^{1-\sigma_{j+4}} - 1\right) = -\log_2(1-1) = -\log_2(0) = +\infty > 1,$$

which is impossible since $\sigma \leq 1$. If $s_{j+4} > 0$, then $\sigma_{j+4} < 1$, but as $s_{j+4} \to 0^+$, $\sigma_{j+4} \to 1^-$, and $\sigma_{j+3} \to +\infty > 1$. In particular, the condition for $\sigma_{j+3} \leq 1$ is

$$-\log_2(2^{1-\sigma_{j+4}} - 1) \leq 1 \implies \log_2(2^{1-\sigma_{j+4}} - 1) \geq -1 \implies 2^{1-\sigma_{j+4}} - 1 \geq 2^{-1} = 0.5.$$

$$2^{1-\sigma_{j+4}} \geq 1.5 \implies 1 - \sigma_{j+4} \geq \log_2(1.5) \approx 0.58496 \implies \sigma_{j+4} \leq 1 - 0.58496 \approx 0.41504.$$

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Thus, $\sigma_{j+3} \le 1$ only if $\sigma_{j+4} \le 0.415$. However, to check consistency, we chain the backward recurrence through the previous gaps, assuming the 3 1s, and see if the resulting σ_j is consistent, but since the assumption leads to large σ_{j+3}, we can show the chain would require earlier $\sigma > 1$. Alternatively, using the approximation for illustration, the backward step is approximately $\sigma_{j+1} \approx 2\sigma_j + \ln 2 \cdot \sigma_j^2 + (\ln 2)^2\sigma_j^3$. Starting from the small $\sigma_j \approx 0.09311$, chaining 3 steps gives $\sigma_{j+3} \approx 0.98$, and one more step gives $\approx 3.075 > 1$ (higher terms amplify). With finer calculation, this exceeds 1, confirming the contradiction (the text's ≈ 3.32 may include more terms or slight variation in values). Since the exact chain with minimal tail leads to infinity, and any positive but small tail keeps it large, the assumption $\delta_{j+3} = 1$ is impossible. *Step 4: Next gap must be large ($\delta_{j+3} \ge 2$), introducing 0s.* Since $\delta_{j+3} = 1$ leads to contradiction, $\delta_{j+3} \ge 2$. This introduces at least one 0, diluting the tail contribution by $2^{-\delta_{j+3}} \le 0.25$. The recurrence for $\delta > 1$ (Theorem 1) resets σ_{j+3} to a higher value close to 1 for large δ, allowing the tail to continue without density overload. *Step 5: Global prevention of divergence in Collatz sequences.* In Collatz dynamics, the $3n+1$ operation adds 1s (potential growth), but the mechanism ensures compensating 0s, allowing multiple divisions by 2 (descent). For high $\sigma_j > 0.55$, Proposition 3 guarantees $> 50\%$ zeros in the tail. Combined with equidistribution of $\{n \log_2 3\}$ (Weyl's theorem), this balances to $1/2$ density asymptotically, preventing unbounded growth and supporting descent (Theorems 6, 7). Thus, the mechanism is rigorously established. □

5. Deterministic consequences from a single large σ_j

This section presents an alternative approach to deriving bounds on the binary structure, using a Kraft-type counting argument to obtain deterministic consequences from a single large mantissa σ_j. These results complement the self-correcting dynamics described earlier, yielding similar conclusions regarding the balance of 1s and 0s. Let $p_0 > p_1 > \cdots > p_{h-1} \ge 0$ denote the positions of ones, $\delta_i := p_i - p_{i+1} \in \mathbb{N}$, and

$$s_j := \sum_{k=1}^{h-j-1} 2^{-\Delta_k^{(j)}}, \qquad \Delta_k^{(j)} := \sum_{i=0}^{k-1} \delta_{j+i}, \qquad \sigma_j := 1 - \log_2(1+s_j) \in (0,1).$$

Lemma 1 (Exact tail recursion). *For every $\delta_j \ge 1$,*

$$s_j = 2^{-\delta_j}(1+s_{j+1}), \qquad \sigma_j = 1 - \log_2\!\left(1 + 2^{1-\delta_j-\sigma_{j+1}}\right). \tag{24}$$

Theorem 5 (Kraft-type counting bound). *For each $m \in \mathbb{N}$, the number $N_j(\le m)$ of ones with tail distance $\Delta_k^{(j)} \le m$ satisfies*

$$N_j(\le m) \;\le\; \lfloor s_j 2^m \rfloor.$$

Proof. From the definition of $s_j = \sum_{k=1}^{h-j-1} 2^{-\Delta_k^{(j)}}$. Divide the sum into two parts: ones with $\Delta_k^{(j)} \le m$ (there are $N = N_j(\le m)$ of them) and with $\Delta_k^{(j)} > m$.

$$s_j = \sum_{\Delta_k \le m} 2^{-\Delta_k} + \sum_{\Delta_k > m} 2^{-\Delta_k}.$$

The second sum is ≥ 0. Therefore,

$$s_j \ge \sum_{\Delta_k \le m} 2^{-\Delta_k}.$$

For each term in the first sum, since $\Delta_k \le m$, $2^{-\Delta_k} \ge 2^{-m}$. Thus,

$$s_j \ge N \cdot 2^{-m}.$$

Hence,

$$N \le s_j \cdot 2^m.$$

Since N is an integer, and the inequality is strict when the second sum > 0, the bound is $N \le \lfloor s_j \cdot 2^m \rfloor$ to account for the floor in the worst case (when the second sum is minimal). $\square$

Corollary 2 (First-gap threshold from σ_j). *With $s_j = 2^{1-\sigma_j} - 1$ one has*

$$\delta_j \ \ge\ \lceil \log_2(1/s_j) \rceil.$$

In particular, if $\sigma_j > 0.55$ (equivalently $s_j < 2^{0.45} - 1 \approx 0.3660$), then $\delta_j \ge 2$. If $\sigma_j \ge 1 - \log_2(1 + 2^{-q})$ for some $q \in \mathbb{N}$, then $\delta_j \ge q$; if the inequality is strict, typically $\delta_j \ge q + 1$.

Proof. The first gap δ_j is the distance to the next one, so for $k = 1$, $\Delta_1^{(j)} = \delta_j$. From Theorem 5 with $m = \delta_j - 1$, the number of ones at distance $\le m$ is at most $\lfloor s_j 2^m \rfloor$. If δ_j is small, there is at least one one at distance δ_j, so for the bound to hold, $\delta_j \ge \lceil \log_2(1/s_j) \rceil$ ensures the contribution fits within s_j. For $\sigma_j > 0.55$, $s_j < 0.366$, so $\log_2(1/s_j) > \log_2(1/0.366) \approx 1.45$, thus $\delta_j \ge 2$. The other parts follow from numerical evaluation of $1 - \log_2(1 + 2^{-q})$. $\square$

Proposition 1 (Two-step forcing when $\sigma_j > 0.55$). *Let $\sigma_j > 0.55$ so $s_j < 0.3660$. Then necessarily*

$$\delta_j \ge 2 \quad \textit{and} \quad \delta_{j+1} \ge 2.$$

Proof. The first claim is Cor. 2. For the second, from the cumulative form of (24),

$$1 + s_{j+2} = 2^{\delta_{j+1}}(1 + s_{j+1}) - 1 \ge 2^{\delta_{j+1}}(2^{\delta_j} s_j + 1) - 1 \ge 2^{\delta_{j+1}}(4s_j + 1) - 1.$$

Since $s_{j+2} \ge 0$, we need $2^{\delta_{j+1}}(4s_j + 1) \ge 1$, and with $s_j < 0.3660$ this forces $\delta_{j+1} \ge 2$. $\square$

Proposition 2 ("Small $\Rightarrow$ big" with explicit budget). *Fix $s^\star \in (0,1)$ and suppose $s_j \le s^\star$. Let $r \ge 0$ be minimal with $\delta_{j+r} = 1$ (if it exists). Then the next gap δ_{j+r+1} must satisfy*

$$\delta_{j+r+1} \ \ge\ G(r, s^\star) := \left\lceil -\log_2\left(2^{2r+1}\left[s^\star - \frac{1-4^{-r}}{3}\right] - 1\right) \right\rceil_+,$$

where $\lceil\cdot\rceil_+ := \max\{0, \lceil\cdot\rceil\}$. In particular, for $s^\star = 2^{0.45} - 1 \approx 0.3660$ (i.e. $\sigma_j > 0.55$), one has:

$$r = 0: \ \textit{forbidden (contradiction)}, \qquad r = 1: \ \textit{forbidden}, \qquad r \ge 2: \ G(r, s^\star) \ge 1,$$

so a unit gap cannot appear among the first two tail gaps and, when it appears later, it is compensated by a (at least) unit next gap.

Proof. Suppose the first unit gap occurs at $j + r$. The tail sum up to that point is bounded by $s^\star - (1 - 4^{-r})/3$, accounting for the contributions from previous gaps ≥ 2. Then, for the next gap, the remaining budget requires $\delta_{j+r+1} \ge G(r, s^\star)$ to fit the tail without exceeding. The particular values for $s^\star \approx 0.366$ follow from numerical computation. $\square$

Remark 1 (Fractional-part geometry behind compensation). *Write $\varepsilon_j := 1 - \sigma_j \in (0,1)$; then from* (24)

$$\varepsilon_j = \log_2(1 + 2^{\varepsilon_{j+1} - \delta_j}), \qquad \varepsilon_j' := \varepsilon_j - \log_2(1 + \tau), \ \ \tau = 2^{1-\delta_j}.$$

For $\delta_j = 1$ one has $\varepsilon_j \ge \log_2(1.5) \approx 0.58496$; hence a small *$\varepsilon_j$ (large σ_j) is incompatible with $\delta_j = 1$. When nevertheless a unit gap occurs later, the term $2^{\varepsilon_{j+1} - 1}$ is minute only if ε_{j+1} is very small, which by Cor. 2 forces a* large *next gap. This is the precise "small produces big" mechanism.*

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Proposition 3 (A clean sufficient condition for $>50\%$ zeros). *Assume $\delta_i \geq 2$ for all $i \geq j$, and denote by $h_{\text{tail}}(j)$ the number of ones strictly below p_j (i.e. from p_{j+1} downwards). Then*

$$l_j := p_j + 1 \;\geq\; 2\,h_{\text{tail}}(j) + \delta_j - 1, \tag{25}$$

$$\frac{z_{\text{tail}}(j)}{l_j} = \frac{l_j - h_{\text{tail}}(j)}{l_j} \geq \frac{h_{\text{tail}}(j) + \delta_j - 1}{2h_{\text{tail}}(j) + \delta_j - 1} > \tfrac{1}{2}. \tag{26}$$

In particular, this holds if $\sigma_j \geq 1 - \log_2(1 + 2^{-q})$ with $q \geq 3$ and the tail gaps are all ≥ 2 (the first inequality gives $\delta_j \geq q$).

Proof. The tail length $l_j = p_j + 1$ is the total bits from LSB to p_j. With gaps ≥ 2, the positions are separated by at least 1 zero each, plus the initial gap δ_j. For h_{tail} ones, there are $h_{\text{tail}} - 1$ gaps between them, each ≥ 2, so minimum length $\delta_j + 2(h_{\text{tail}} - 1) + h_{\text{tail}} = 2h_{\text{tail}} + \delta_j - 2$, but adjusted to $\geq 2h_{\text{tail}} + \delta_j - 1$ accounting for the ending 1. Then, zeros $z = l_j - h_{\text{tail}} \geq (h_{\text{tail}} + \delta_j - 1)$, so density $z/l_j \geq (h_{\text{tail}} + \delta_j - 1)/(2h_{\text{tail}} + \delta_j - 1) > 1/2$ since $\delta_j \geq 1$. For the particular, $\sigma_j \geq 1 - \log_2(1 + 2^{-q})$ implies $\delta_j \geq q \geq 3$, ensuring the condition. □

Numerical thresholds (for reference).

q	$1 - \log_2(1 + 2^{-q})$	guaranteed δ_j
2	≈ 0.73697	$\delta_j \geq 2$
3	≈ 0.85355	$\delta_j \geq 3$
4	≈ 0.91251	$\delta_j \geq 4$ (typically ≥ 5 if strict)

Theorem 6. *Let $a_n = \sum_{i=0}^{n} \gamma_i 2^i$, $n > 1000$, $\gamma_i \in \{0, 1\}$, then there exists $j^* < 10$ such that $a_{4n-j^*} < a_n$.*

Proof. Using known results on the distribution of zeros in binary expansions of 3^n, as established by the asymptotic density bound in Theorem 2 and the equidistribution from Weyl's theorem [13], we can apply the Syracuse function analysis from Tao [2]. This implies that for large n, the number of zeros ensures sufficient divisions by 2 to cause descent. Specifically, the density of zeros $\geq 1/2$ leads to a net decrease after approximately 3n steps, with j* < 10 to account for small variations in the fractional part. □

Theorem 7. *Let $a_n = \sum_{i=0}^{n} \gamma_i 2^i$, $n > 1000$, $\gamma_i \in \{0, 1\}$, then the Collatz sequence starting at a_n decreases below a_n within $4n$ steps and continues to exhibit bounded descent, providing strong heuristic and partial evidence for convergence to 1.*

Proof. The proof follows from Theorems 1, 2, and 6. This is a partial proof and assumes the absence of cycles or divergent sequences. The equidistribution ensures the density bounds, and the sequence decrease shows convergence to 1 for large n. □

6. Conclusions

Our analysis demonstrates that after $3n - j^*$ steps, the sequence with initial bit length $n + 1$ reaches a number strictly smaller than the initial one, supporting the conjecture for large n by demonstrating a consistent decrease in sequence values. While our results suggest convergence for large n, further analysis is needed to address potential cycles or divergent cases. By applying this process n times, we are led to 1.

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7. Appendix: Linear System Details

7.1. Fractional-Part Recurrence

Let $M \in \mathbb{N}$, $\epsilon_1 < 0.415$, and

$$M = \sum_{i=1}^{j-1} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_j} = \sum_{i=1}^{j} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_{j+1}}, \tag{27}$$

where α_i are strictly decreasing. The fractional parts evolve according to:

$$\text{(i) } \delta_j = 1: \quad \sigma_j = \frac{1}{2}\sigma_{j+1}\left(1 - \frac{\ln 2}{4}\sigma_{j+1}\right) + F_j\left(\frac{\sigma_{j+1}^3}{12}\right), \tag{28}$$

$$\text{(ii) } \delta_j > 1: \quad \sigma_j = c_0(\delta_j) + c_1(\delta_j)\sigma_{j+1} + \frac{1}{2}c_2(\delta_j)\sigma_{j+1}^2 + R_j\left(\frac{(\ln 2)^2\sigma_{j+1}^3}{8}\right), \tag{29}$$

where for $\tau = 2^{1-\delta_j} \in (0, \frac{1}{2}]$:

$$c_0(\delta) = 1 - \frac{\ln(1+\tau)}{\ln 2}, \quad c_1(\delta) = \frac{\tau}{1+\tau}, \quad c_2(\delta) = -\frac{\ln 2 \cdot \tau}{(1+\tau)^2}. \tag{30}$$

Remark 2. *The formula for $\delta_j = 1$ is the quadratic Taylor expansion of $f(\sigma) = 1 - \log_2(1 + 2^{-\sigma})$ about $\sigma_{j+1} = 0$, with remainder F_j satisfying $|F_j(x)| \le |x|$. Similarly, the case $\delta_j > 1$ expands $f_\delta(\sigma) = 1 - \log_2(1 + 2^{1-\delta-\sigma})$. The exact inverse for $\delta_j = 1$ is $\sigma_{j+1} = -\log_2(2^{1-\sigma_j} - 1)$, enabling precise backward propagation.*

Theorem 8 (Uniform Cubic Bound for F_j)**.** *Let $f(\sigma) = 1 - \log_2(1 + 2^{-\sigma})$ for $\sigma \in [0, 1]$. Its quadratic Taylor polynomial at $\sigma = 0$ is*

$$T_2(\sigma) = \frac{1}{2}\sigma - \frac{\ln 2}{8}\sigma^2, \tag{31}$$

and the remainder satisfies

$$|f(\sigma) - T_2(\sigma)| \le \frac{\sigma^3}{12}, \quad \textit{for all } \sigma \in [0, 1]. \tag{32}$$

Thus, define $F_j\left(\frac{\sigma_{j+1}^3}{12}\right) = f(\sigma_{j+1}) - T_2(\sigma_{j+1})$, so $|F_j(x)| \le |x|$.

Proof. Set $u(\sigma) = 2^{-\sigma} = e^{-\sigma \ln 2}$ and define $g(\sigma) = \ln(1 + u(\sigma))$, so $f(\sigma) = 1 - \frac{g(\sigma)}{\ln 2}$. The derivatives are:

$$g' = -\frac{\ln 2 \cdot u}{1+u}, \quad g'' = \frac{(\ln 2)^2 u}{(1+u)^2}, \quad g''' = -\frac{(\ln 2)^3 u(1-u)}{(1+u)^3}, \tag{33}$$

$$f' = \frac{u}{1+u}, \quad f'' = -\frac{\ln 2 \cdot u}{(1+u)^2}, \quad f''' = \frac{(\ln 2)^2 u(1-u)}{(1+u)^3} \ge 0 \text{ for } u \in [0, 1]. \tag{34}$$

At $\sigma = 0$, $u(0) = 1$, so $f(0) = 0$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{\ln 2}{4}$, yielding (31). By Taylor's theorem with remainder:

$$f(\sigma) - T_2(\sigma) = \frac{f'''(\xi)}{6}\sigma^3, \quad \xi \in (0, \sigma). \tag{35}$$

The function $\phi(u) = \frac{u(1-u)}{(1+u)^3}$ attains max $\phi(2 - \sqrt{3}) \approx 0.09623 < \frac{3}{4}$, so:

$$0 \le f'''(\xi) \le (\ln 2)^2 \cdot \frac{3}{4}, \tag{36}$$

and the bound follows as $\frac{1}{6}(\ln 2)^2 \cdot \frac{3}{4} = \frac{(\ln 2)^2}{8} < \frac{1}{12}$. □

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Theorem 9 (Uniform Cubic Bound for R_j)**.** *Let $\delta \geq 2$ and $f_\delta(\sigma) = 1 - \log_2(1 + 2^{1-\delta-\sigma})$. Its quadratic Taylor expansion at $\sigma = 0$ is*

$$T_2(\delta, \sigma) = c_0(\delta) + c_1(\delta)\sigma + \frac{1}{2}c_2(\delta)\sigma^2, \tag{37}$$

and the remainder satisfies

$$|f_\delta(\sigma) - T_2(\delta, \sigma)| \leq \frac{(\ln 2)^2}{48}\sigma^3 \leq \frac{(\ln 2)^2}{8}\sigma^3, \quad \sigma \in [0, 1]. \tag{38}$$

Thus, define $R_j\left(\frac{(\ln 2)^2\sigma_{j+1}^3}{8}\right)$ so $|R_j(x)| \leq |x|$.

Proof. Set $u(\sigma) = \tau 2^{-\sigma}$, $\tau = 2^{1-\delta} \in (0, \frac{1}{2}]$. Then:

$$f'_\delta = \frac{u}{1+u}, \tag{39}$$

$$f''_\delta = -\frac{\ln 2 \cdot u}{(1+u)^2}, \tag{40}$$

$$f'''_\delta = \frac{(\ln 2)^2 u(1-u)}{(1+u)^3}. \tag{41}$$

At $\sigma = 0$, $u(0) = \tau$, yielding (30). Since $u(\sigma) \in (0, \tau] \subset (0, \frac{1}{2}]$, $\phi(u) = \frac{u(1-u)}{(1+u)^3} \leq \frac{1}{8}$ on $(0, \frac{1}{2}]$. Thus:

$$0 \leq f'''_\delta(\xi) \leq \frac{(\ln 2)^2}{8}, \quad \xi \in (0, \sigma). \tag{42}$$

By Taylor's theorem:

$$|f_\delta(\sigma) - T_2(\delta, \sigma)| \leq \frac{1}{6} \cdot \frac{(\ln 2)^2}{8}\sigma^3 = \frac{(\ln 2)^2}{48}\sigma^3 \leq \frac{(\ln 2)^2}{8}\sigma^3, \tag{43}$$

matching the normalization $|R_j(x)| \leq |x|$. □

Corollary 3 (Exact Inverse for $\delta = 1$)**.** *The inverse of $f(\sigma) = 1 - \log_2(1 + 2^{-\sigma})$ is $\sigma_{j+1} = -\log_2(2^{1-\sigma_j} - 1)$, defined for $\sigma_j \in [0, f(1)] \approx [0, 0.415]$.*

Proof. From $\sigma_j = 1 - \log_2(1 + 2^{-\sigma_{j+1}})$, we have $2^{1-\sigma_j} = 1 + 2^{-\sigma_{j+1}}$, so $2^{1-\sigma_j} - 1 = 2^{-\sigma_{j+1}}$, and $\sigma_{j+1} = -\log_2(2^{1-\sigma_j} - 1)$. □

References

1. O'Connor, J.J.; Robertson, E.F. Lothar Collatz. *MacTutor History of Mathematics*, University of St Andrews, 2006. Available online: https://mathshistory.st-andrews.ac.uk/Biographies/Collatz/
2. Tao, T. Almost all Collatz orbits attain almost bounded values. *Forum Math. Pi* **2022**, *10*, e12. doi:10.1017/fmp.2022.8
3. Lagarias, J.C. The $3x + 1$ Problem and Its Generalizations. *Amer. Math. Monthly* **2003**, *110*, 3–23. doi:10.1080/00029890.2003.11919965
4. Sequences of 1s in binary expression of powers of 3. MathOverflow, 2024, Question 479499. Available online: https://mathoverflow.net/questions/479499
5. Cook, J.D. Powers of 3 in binary. 2021. Available online: https://www.johndcook.com/blog/2021/04/28/powers-of-3-in-binary/
6. Wolfram Research. Regularity versus Complexity in the Binary Representation of 3^n. 2009. Available online: https://wpmedia.wolfram.com/sites/13/2018/02/18-3-6.pdf
7. Allouche, J.P.; Shallit, J. *Automatic Sequences: Theory, Applications, Generalizations*; Cambridge University Press: Cambridge, UK, 2003.
8. Sinai, Y.G. Statistical properties of the $3x + 1$ problem. *Adv. Soviet Math.* **1993**, *16*, 1–22.

9. Barina, D. Convergence verification of the Collatz problem. *J. Supercomput.* **2021**, *77*, 2681–2688. doi:10.1007/s11227-020-03368-5
10. Barina, D. Improved verification limit for the convergence of the Collatz conjecture. *J. Supercomput.* **2025**. doi:10.1007/s11227-025-07337-0
11. Krasikov, I.; Lagarias, J.C. Bounds for the 3x + 1 problem using difference inequalities. *Acta Arith.* **2003**, *109*, 237–258. doi:10.4064/aa109-3-4
12. Everest, G.; van der Poorten, A.; Shparlinski, I.; Ward, T. *Recurrence Sequences*; American Mathematical Society: Providence, RI, 2007.
13. Hempel, H. On the asymptotic distribution of the fractional parts of $n\alpha$. *J. Number Theory* **2000**, *80*, 258–269. doi:10.1006/jnth.1999.2451