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Posted Date: 29 December 2023

doi: 10.20944/preprints202312.2327.v1

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Linear Relations between Numbers of Terms and First Terms of Sums of Consecutive Squared Integers Equal to Squared Integers

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Abstract: Sums of M consecutive squared integers $(a+i)^2$ equaling squared integers (for $a \geq 1$, $0 \leq i \leq M-1$) yield remarkable regular linear features when plotting values of M in function of a. These features correspond to groupings of pairs of a values for successive same values of M around straight lines of equation $\mu M \approx 2a$ and are characterized in this paper for rational values of μ.

Keywords: Sums of consecutive squared integers equal to square integers; Quadratic diophantine equation; Generalized Pell equation; Fundamental solutions; Chebyshev polynomials

MSC: 11E25 ; 11D09 ; 33D45

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1. Introduction

The study of integer squares equal to sums of consecutive squared integers can be dated back to 1873 when Lucas stated [11] that $(1^2 + ... + n^2)$ is an integer square only for $n = 1$ and 24. Lucas proposed later in 1875 [12] the well known cannonball problem, which was proven by several authors [2,10,13,14,19,20,33].
Instead of starting at 1, finding all values of a for which the sum of M consecutive integer squares starting from $a^2 \geq 1$ is itself an integer square s^2 is a more general problem that was addressed by several authors (see e.g. [1,3,8,24]). More recently, this author showed [26] that there are no integer solutions if $M \equiv 3, 5, 6, 7, 8$ or $10 \ (mod\, 12)$ and that there are integer solutions for non squared integer M congruent to $0, 9, 24$ or $33 \ (mod\, 72)$, or to $1, 2$ or $16 \ (mod\, 24)$, or to $11 \ (mod\, 12)$, and for squared integer M congruent to $1 \ (mod\, 24)$.
In this paper, we investigate and characterize the properties of groupings of pairs of a values for a same value of M that are found around inclined straight lines of equation $\mu M \approx 2a$ in the (a, M) plot for rational values of μ.

2. Linear features in the (a, M) plot

For $M > 1, a, i, s \in \mathbb{Z}^*$, the sum of M consecutive squared integers $(a+i)^2$ equaling a squared integer s^2 can be written [28] as

$$\sum_{i=0}^{M-1} (a+i)^2 = M\left[\left(a + \frac{M-1}{2}\right)^2 + \frac{M^2-1}{12}\right] \tag{1}$$

For $1 \leq a \leq 10^5$ and $2 \leq M \leq 10^5$, there are only 4078 couples of values of a and M among the approximately 10^{10} possibilities such that (1) holds. Figure 1 shows the distribution of these 4078 couples in a (a, M) plot where several groupings of interest are seen.

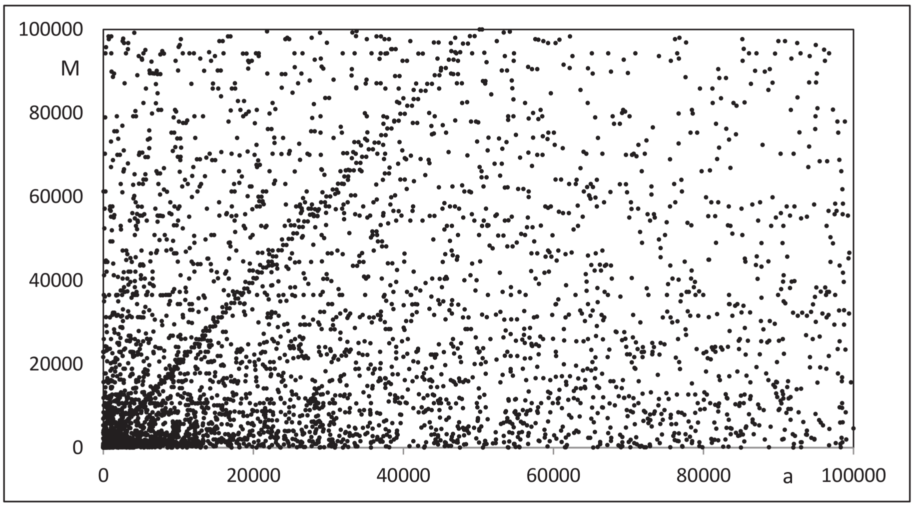

Figure 1. Distribution of M versus a for the 4078 couples (a, M) found such as (1) holds, with $1 \leq a \leq 10^5$ and $2 \leq M \leq 10^5$

The most visible is the grouping around a straight line of equation $M = 2a + c$ where c is a constant, corresponding to a double infinite family of a values that starts with the identity and the Pythagorean relations

$$0^2 + 1^2 = 1^2 \tag{2}$$
$$3^2 + 4^2 = 5^2 \tag{3}$$

for a same value of $M = 2$ and respectively for $a = 0$ and 3. This double infinite family has the characteristics that couples of a values correspond to a same value of M. There are other similar groupings and double infinite families around straight lines of general equation $\mu M = 2a + c_\mu$ for certain rational values of $\mu > 0$ and where c_μ is a constant different for each value of μ. Only groupings around inclined lines are considered as the limit cases of $\mu = 0$ and $\mu \to \infty$ corresponding to groupings around respectively vertical and horizontal lines are not treated here. The "horizontal" case for which one or several solutions in a exist for each values of M was investigated in [26–28].

Definition 1. *For $1 \leq j \leq 2$, $M_{\mu,k} > 1$, $a_{j,\mu,k} \in \mathbb{Z}^*$, for a given value of $\mu \in \mathbb{Q}^+$, two values of $a_{j,\mu,k}$ are called a pair $\left(a_{1,\mu,k}, a_{2,\mu,k}\right)$ if for a same value of $M_{\mu,k}$ and $\forall k \in \mathbb{Z}$,*

$$a_{1,\mu,k} + a_{2,\mu,k} = \mu M_{\mu,k} + 1 \tag{4}$$
$$a_{2,\mu,k} - a_{1,\mu,k} = f_{\mu,k} \tag{5}$$

hold, where $f_{\mu,k} = f_\mu(k)$ is a linear integer function of k for each value of μ, yielding

$$\mu M_{\mu,k} = 2a_{j,\mu,k} \pm f_{\mu,k} - 1 \tag{6}$$

where the upper or lower sign is taken for $j = 1$ or 2.

The two families of $a_{1,\mu,k}$ and $a_{2,\mu,k}$ are characterized for each values of μ around the straight line of equation $\mu M = 2a + c_\mu$ in the following theorem. However, relations (4) to (6) hold only for certain values, called allowed values, of $M_{\mu,k}$ and of μ that are determined further.

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Theorem 1. *For $1 \leq j \leq 2, i, \eta, \delta, M_{\mu,k} > 1, a_{j,\mu,k} \in \mathbb{Z}^*$, $k, s_{j,\mu,k} \in \mathbb{Z}$, for allowed values of $\mu \in \mathbb{Q}^+$, let $\mu = (\eta/\delta)$ be an irreducible fraction; if $\left(a_{1,\mu,k}, a_{2,\mu,k}\right)$ is a pair of $a_{j,\mu,k}$ values for a same value of $M_{\mu,k}$ and if*

$$M_{\mu,k} = \frac{\left(3f_{\mu,k}^2 - 1\right)}{3\left(\mu+1\right)^2 + 1} = \frac{\delta^2\left(3f_{\mu,k}^2 - 1\right)}{3\left(\eta+\delta\right)^2 + \delta^2} \tag{7}$$

holds $\forall k \in \mathbb{Z}$, then the sums of squares of $M_{\mu,k}$ consecutive integers $\left(a_{j,\mu,k} + i\right)$ for $i = 0$ to $M_{\mu,k} - 1$ are always equal to squared integers $s_{j,\mu,k}^2$, with

$$a_{j,\mu,k} = \frac{1}{2\delta}\left(\eta M_{\mu,k} + \delta \mp \sqrt{\frac{M_{\mu,k}\left(3\left(\eta+\delta\right)^2 + \delta^2\right) + \delta^2}{3}}\right) \tag{8}$$

$$s_{j,\mu,k} = \frac{M_{\mu,k}}{2\delta}\left(\sqrt{\frac{M_{\mu,k}\left(3\left(\eta+\delta\right)^2 + \delta^2\right) + \delta^2}{3}} \mp \left(\eta+\delta\right)\right) \tag{9}$$

where the upper (respectively lower) sign is taken for $j = 1$ (resp. 2).

Proof. Let $1 \leq j \leq 2, i, \eta, \delta, M_{\mu,k} > 1, a_{j,\mu,k} \in \mathbb{Z}^*, k, s_{j,\mu,k} \in \mathbb{Z}$, $\mu = (\eta/\delta) \in \mathbb{Q}^+$ forming an irreducible fraction, i.e. $\gcd(\eta,\delta) = 1$. Let further $f_{\mu,k}$ be a yet unknown integer function of k for each value of μ. Replacing in the second equality of (1) M by $M_{\mu,k}$ and a by $a_{j,\mu,k}$ from (6) yields successively

$$\begin{aligned}
\sum_{i=0}^{M_{\mu,k}-1}\left(a_{j,\mu,k}+i\right)^2 &= \frac{M_{\mu,k}}{4}\left[\left(2a_{j,\mu,k} + M_{\mu,k} - 1\right)^2 + \frac{M_{\mu,k}^2 - 1}{3}\right] \\
&= \frac{M_{\mu,k}}{4}\left[\left(\left(\mu+1\right)M_{\mu,k} \mp f_{\mu,k}\right)^2 + \frac{M_{\mu,k}^2 - 1}{3}\right] \\
&= \frac{M_{\mu,k}^2}{4}\left[\frac{\left(3\left(\eta+\delta\right)^2 + \delta^2\right)M_{\mu,k}}{3\delta^2} + \frac{3f_{\mu,k}^2 - 1}{3M_{\mu,k}}\right. \\
&\quad \left.\mp 2\left(\frac{\eta+\delta}{\delta}\right)f_{\mu,k}\right]
\end{aligned} \tag{10}$$

where the upper (respectively lower) sign in (10) is taken for $j = 1$ (resp. 2). For the expression between brackets in (10) to be a square, replace in (10) $f_{\mu,k}$ by

$$f_{\mu,k} = \sqrt{\frac{M_{\mu,k}\left(3\left(\eta+\delta\right)^2 + \delta^2\right) + \delta^2}{3\delta^2}} \tag{11}$$

from (7), yielding immediately (9). Replacing $f_{\mu,k}$ (11) in (6) yields then (8). □

In addition, from Theorem 2, the following relations hold $\forall k \in \mathbb{Z}$

$$a_{2,\mu,k} - a_{1,\mu,k} = \sqrt{\frac{M_{\mu,k}\left(3(\eta+\delta)^2+\delta^2\right)+\delta^2}{3\delta^2}} \quad (12)$$

$$s_{2,\mu,k} + s_{1,\mu,k} = M_{\mu,k}\sqrt{\frac{M_{\mu,k}\left(3(\eta+\delta)^2+\delta^2\right)+\delta^2}{3\delta^2}} \quad (13)$$

$$= M_{\mu,k}\left(a_{2,\mu,k} - a_{1,\mu,k}\right) \quad (14)$$

$$s_{2,\mu,k} - s_{1,\mu,k} = M_{\mu,k}\left(\frac{\eta+\delta}{\delta}\right) \quad (15)$$

$$= a_{2,\mu,k} + a_{1,\mu,k} + M - 1 \quad (16)$$

3. Parametric expressions of $f_{\mu,k}$, $M_{\mu,k}$, $a_{j,\mu,k}$, $s_{j,\mu,k}$

Above results hold only for certain allowed values of $M_{\mu,k}$ and of $\mu \in \mathbb{Q}^+$, that can be determined as follows. Relation (7) reads also

$$\left(\delta f_{\mu,k}\right)^2 - M_{\mu,k}(\eta+\delta)^2 = \delta^2\left(\frac{M_{\mu,k}+1}{3}\right) \quad (17)$$

It was shown [30] that for (17) to hold:
- $\delta \equiv 0\,(mod\,6)$, and $\eta \equiv 1$ or $5\,(mod\,6)$, $M_{\mu,k} \equiv 0$ or $24\,(mod\,72)$, and $M_{\mu,k}\left(mod\ \left(\delta^2/3\right)\right) \equiv 0$;
- $\delta \equiv 1$ or $5\,(mod\,6)$, and $\eta \equiv 1, 3$ or $5\,(mod\,6)$ and either $f_{\mu,k} \equiv 1\,(mod\,2)$ and $M_{\mu,k} \equiv 2\,(mod\,24)$, or $f_{\mu,k} \equiv 0\,(mod\,2)$ and $M_{\mu,k} \equiv 11\,(mod\,12)$, and $M_{\mu,k}\left(mod\,\delta^2\right) \equiv 0$.

Parametric expressions of $f_{\mu,k}$, $M_{\mu,k}$, $a_{j,\mu,k}$ and $s_{j,\mu,k}$ in function of $k \in \mathbb{Z}$, $\mu = (\eta/\delta)$ and initial values are found as follows.

Theorem 2. *For $1 \le j \le 2, \eta, \delta, M_{\mu,k} > 1 \in \mathbb{Z}^+$, $a_{j,\mu,k} \in \mathbb{Z}^*$, $k, s_{j,\mu,k} \in \mathbb{Z}$, $\mu, \nu \in \mathbb{Q}^+$, for allowed values of $\mu = (\eta/\delta)$ and for pairs $\left(a_{1,\mu,k}, a_{2,\mu,k}\right)$, $f_{\mu,k}$ is a linear function of k, $M_{\mu,k}$ and $a_{j,\mu,k}$ are quadratic functions of k, and $s_{j,\mu,k}$ is a cubic function of k, as follows*

$$f_{\mu,k} = \left(3(\eta+\delta)^2+\delta^2\right)\nu k + f_{\mu,0} \quad (18)$$

$$M_{\mu,k} = 3\delta^2\left(3(\eta+\delta)^2+\delta^2\right)\nu^2k^2 + 6\delta^2 f_{\mu,0}\nu k + M_{\mu,0} \quad (19)$$

$$a_{j,\mu,k} = \frac{1}{2}\left[3\eta\delta\left(3(\eta+\delta)^2+\delta^2\right)\nu^2k^2 + \left(6\eta\delta f_{\mu,0} \mp \left(3(\eta+\delta)^2+\delta^2\right)\right)\nu k\right] + a_{j,\mu,0} \quad (20)$$

$$s_{j,\mu,k} = \frac{1}{2}\left[3\delta^2\left(3(\eta+\delta)^2+\delta^2\right)^2\nu^3k^3 + 3\delta\left(3(\eta+\delta)^2+\delta^2\right)\left(3\delta f_{\mu,0} \mp (\eta+\delta)\right)\nu^2k^2 + \left(\left(3\delta f_{\mu,0} \mp (\eta+\delta)\right)^2 - \left((\eta+\delta)^2+\delta^2\right)\right)\nu k\right] + s_{j,\mu,0} \quad (21)$$

where $\nu = 1$ for $\delta \equiv 1$ or $5\,(mod\,6)$ and $\nu = (2/3)$ for $\delta \equiv 0\,(mod\,6)$ and where the upper (respectively lower) sign is taken for $j = 1$ (resp. 2).

Proof. For $1 \le j \le 2, \eta, \delta, x, M_{\mu,k} > 1 \in \mathbb{Z}^+$, $a_{j,\mu,k} \in \mathbb{Z}^*$, $k, s_{j,\mu,k} \in \mathbb{Z}$, $\mu, \nu \in \mathbb{Q}^+$, for allowed values of $\mu = (\eta/\delta)$ and for pairs $\left(a_{1,\mu,k}, a_{2,\mu,k}\right)$, let $f_{\mu,k}$ (5) be a linear function of k, $f_{\mu,k} = xk + f_{\mu,0}$ where

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$f_{\mu,0} = (a_{2,\mu,0} - a_{1,\mu,0})$ is the initial value for $k = 0$ of the difference (5) and x an integer function to be defined for some parameters. Then (7) yields

$$M_{\mu,k} = \frac{\delta^2\left(3\left(xk + f_{\mu,0}\right)^2 - 1\right)}{3\left(\eta+\delta\right)^2 + \delta^2} = \frac{3\delta^2 xk\left(xk + 2f_{\mu,0}\right)}{3\left(\eta+\delta\right)^2 + \delta^2} + \frac{\delta^2\left(3f_{\mu,0}^2 - 1\right)}{3\left(\eta+\delta\right)^2 + \delta^2} \tag{22}$$

The second term on the right of (22) is $M_{\mu,0}$ by (7).
(i) If $\delta \equiv 1$ or $5\,(mod\,6)$, as $M_{\mu,k} \in \mathbb{Z}^+$, $\left(3\left(\eta+\delta\right)^2 + \delta^2\right)$ must divide x in the first term of (22), yielding then (18) and (19) with $\nu = 1$.
(ii) If $\delta \equiv 0\,(mod\,6)$, simplifying the first term by 3, (22) reads

$$M_{\mu,k} = \frac{\delta^2 xk\left(xk + 2f_{\mu,0}\right)}{\left(\eta+\delta\right)^2 + \left(\delta^2/3\right)} + M_{\mu,0} \tag{23}$$

As $M_{\mu,k} \in \mathbb{Z}^+$, $\left(\left(\eta+\delta\right)^2 + \left(\delta^2/3\right)\right)$ is a factor of x. However, as $\eta \equiv 1$ or $5\,(mod\,6)$ for $\delta \equiv 0\,(mod\,6)$, $\left(\left(\eta+\delta\right)^2 + \left(\delta^2/3\right)\right) \equiv 1\,(mod\,2)$ and as $f_{\mu,k}\,(mod\,2) \equiv f_{\mu,0}\,(mod\,2)\ \forall k \in \mathbb{Z}$, x must be replaced by $2\left(\left(\eta+\delta\right)^2 + \left(\delta^2/3\right)\right)$, yielding then (18) and (19) with $\nu = (2/3)$.
(iii) Further, replacing $f_{\mu,k}$ and $M_{\mu,k}$ by (18) and (19) in $a_{j,\mu,k}$ from (6) and in $s_{j,\mu,k}$ (9) with (11), yield directly (20) and (21) with the upper (or lower) sign for $j = 1$ (or 2). □

4. Finding allowed values of $\mu = (\eta/\delta)$ and $M_{\mu,k}$

Finding the allowed values of $\mu = (\eta/\delta)$, $M_{\mu,0}$ and $f_{\mu,0}$ requires solving the generalized Pell equation (17) for $k = 0$ in variables $(\delta f_{\mu,0})$ and $(\eta + \delta)$.
In general, for $X, Y, D, N, x_f, y_f, n \in \mathbb{Z}^+$ and D square free (i.e. $\sqrt{D} \notin \mathbb{Z}$), a generalized Pell equation $X^2 - DY^2 = N$ admits either no solution, or one or several fundamental solution(s) (X_1, Y_1) and also one or several infinite branches of solutions (X_n, Y_n). Several methods exist to find the fundamental solutions of the generalized Pell equation (see [15,19,31]). Two methods are used further: first a brute force search method, i.e. trying several values of Y until the smallest $X_1 = \sqrt{N + DY_1^2} \in \mathbb{Z}^+$ is found; second, Matthews' method [16] based on an algorithm by Frattini [4–6] using Nagell's bounds [18,21]. Once fundamental solution(s) (X_1, Y_1) have been found one way or another, noting $\left(x_f, y_f\right)$ the fundamental solutions of the related simple Pell equation $X^2 - DY^2 = 1$, the other solutions (X_n, Y_n) can be found by

$$X_n + \sqrt{D}Y_n = \pm\left(X_1 + \sqrt{D}Y_1\right)\left(x_f + \sqrt{D}y_f\right)^n \tag{24}$$

for a proper choice of sign $\pm$ [17], which can be written also in function of Chebyshev's polynomials [28]

$$X_n = X_1 T_{n-1}\left(x_f\right) + DY_1 y_f U_{n-2}\left(x_f\right) \tag{25}$$

$$Y_n = X_1 y_f U_{n-2}\left(x_f\right) + Y_1 T_{n-1}\left(x_f\right) \tag{26}$$

where $T_{n-1}\left(x_f\right)$ and $U_{n-2}\left(x_f\right)$ are Chebyshev polynomials of the first and second kinds evaluated at x_f.

The generalized Pell equation (17) can be written as

$$\left(\lambda f_{\mu,0}\right)^2 - \left(\frac{\lambda^2 M_{\mu,0}}{\delta^2}\right)(\eta+\delta)^2 = \frac{\lambda^2\left(M_{\mu,0}+1\right)}{3} \tag{27}$$

with $X = \lambda f_{\mu,0}$, $Y = (\eta+\delta)$, $D = \left(\lambda^2 M_{\mu,0}/\delta^2\right)$ and $N = \lambda^2\left(M_{\mu,0}+1\right)/3$, and where $\lambda = 1$ if $\delta \equiv 1$ or $5\,(mod\,6)$ and $\lambda = 3$ if $\delta \equiv 0\,(mod\,6)$.
To use Matthews' method [16], the parameters D and N must be fixed with values of δ and $M_{\mu,0}$ that can be chosen from the allowed congruent values (see Section 3) and be tried one by one until fundamental solutions are found. Alternatively, fixing the values of η and δ, a brute force search method can be used to find $f_{\mu,0} \in \mathbb{Z}^+$ for the smallest value of $M_{\mu,0} \in \mathbb{Z}^+$, with from (11)

$$f_{\mu,0} = \sqrt{\frac{M_{\mu,0}\left(3(\eta+\delta)^2+\delta^2\right)+\delta^2}{3\delta^2}} \tag{28}$$

Relation (28) yields then the allowed values of μ, $M_{\mu,0}$ and $f_{\mu,0}$ given in Table 2 for $\delta = 1$, $\mu = \eta \in \mathbb{Z}^+$, for $0 \le \mu \le 100$ ([1])

Table 1. Values of $M_{\mu,0}$ and $f_{\mu,0}$ for $0 \le \mu \le 100$

μ	$M_{\mu,0}$	$f_{\mu,0}$	μ	$M_{\mu,0}$	$f_{\mu,0}$	μ	$M_{\mu,0}$	$f_{\mu,0}$
1	2	3	29	26	153	63	3263	3656
5	11	20	33	299	588	67	9563	6650
7	74	69	35	479	788	69	2	99
11	2	17	39	1391	1492	77	74	671
15	194	223	43	59	338	81	1202	2843
19	122	221	49	491	1108	83	146	1015
21	983	690	53	1739	2252	85	1874	3723
25	506	585	55	383	1096	97	23	470
27	47	192	57	2327	2798			

and in [29] for $\delta \neq 1$, $\mu = (\eta/\delta) \in \mathbb{Q}^+$, for $0 < \eta, \delta \le 100$.
Once a set of values has been found for $\eta, \delta, M_{\mu,0}$ and $f_{\mu,0}$ as fundamental solution(s) of the generalized Pell equation (27), other allowed values for η and $f_{\mu,0}$ can be found from the other solutions of (27) using the values of δ and $M_{\mu,0}$ by (25) and (26) written as

$$f_{\mu_n,0} = f_{\mu_1,0}T_{n-1}\left(x_f\right) + \left(\frac{\lambda M_{\mu,0}}{\delta^2}\right)(\eta_1+\delta)\, y_f U_{n-2}\left(x_f\right) \tag{29}$$

$$\eta_n = \lambda f_{\mu_1,0} y_f U_{n-2}\left(x_f\right) + (\eta_1+\delta)\, T_{n-1}\left(x_f\right) - \delta \tag{30}$$

Example 1. *For $\delta = 1$ and $\mu = \eta = 1$, $M_{1,0} = 2$ and $f_{1,0} = 3$ from Table 1. Using $M_{1,0}$ and δ as constants in (27) with $\lambda = 1$, it reduces to a simple Pell equation $f_{\mu,0}^2 - 2(\mu+1)^2 = 1$ (see e.g. [7,9,23,25]) which admits*

[1] [22] gives all values of $(\mu+1)$ such that $\left(3(\mu+1)^2+1\right)$ is prime for which (28) holds.

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the single fundamental solution $(X_1, Y_1) = (f_{\mu_1,0}, (\mu_1+1)) = (3,2)$ *or* $(f_{\mu_1,0}, \mu_1) = (3,1)$ *and an infinity of other solutions that can be found* $\forall n \in \mathbb{Z}^+$ *by*

$$f_{\mu_n,0} = \frac{\left(3+2\sqrt{2}\right)^n + \left(3-2\sqrt{2}\right)^n}{2} = 3, 17, 99, 577, 3363, \ldots \tag{31}$$

$$\mu_n = \frac{\left(3+2\sqrt{2}\right)^n - \left(3-2\sqrt{2}\right)^n}{2\sqrt{2}} - 1 = 1, 11, 69, 407, 2377, \ldots \tag{32}$$

where μ_n *are the Pell numbers [32] of even indices minus one. These new values of* $(f_{\mu_n,0}, \mu_n)$ *for* $n > 1$ *define new groupings around straight lines of general equation* $\mu_n M = 2a + c_{\mu_n}$*, with the initial value* $M_{\mu_n,0} = 2$.

For $\delta = 1$, $\eta = 1$, $M_{1,0} = 2$, $f_{1,0} = 3$, $\nu = 1$, (19) to (21) yield
$M_{1,k} = 39k^2 + 18k + 2$,
$a_{1,1,k} = \left(39k^2 + 5k\right)/2$, $a_{2,1,k} = \left(39k^2 + 31k + 6\right)/2$,
$s_{1,1,k} = \left(507k^3 + 273k^2 + 44k + 2\right)/2$, $s_{2,1,k} = \left(507k^3 + 429k^2 + 116k + 10\right)/2$,
and values of $M_{\mu,k}$, $a_{1,\mu,k}$ and $a_{2,\mu,k}$ for $-10 \le k \le 10$ are given in Table 2.

Table 2. Values of $M_{\mu,k}$, $a_{1,\mu,k}$, $a_{2,\mu,k}$ for $\mu = 1, 5, (1/6)$ and $-10 \le k \le 10$

	$\mu = 1$			$\mu = 5$			$\mu = 1/6$		
k	$M_{1,k}$	$a_{1,1,k}$	$a_{2,1,k}$	$M_{5,k}$	$a_{1,5,k}$	$a_{2,5,k}$	$M_{1/6,k}$	$a_{1,1/6,k}$	$a_{2,1/6,k}$
0	2	0	3	11	18	38	312	15	38
-1	23	17	7	218	590	501	5784	532	433
1	59	22	38	458	1081	1210	12408	962	1107
-2	122	73	50	1079	2797	2599	28824	2513	2292
2	194	83	112	1559	3779	4017	42072	3373	3640
-3	299	168	132	2594	6639	6332	69432	5958	5615
3	407	183	225	3314	8112	8459	89304	7248	7637
-4	554	302	253	4763	12116	11700	127608	10867	10402
4	698	322	377	5723	14080	14536	154104	12587	13098
-5	887	475	413	7586	19228	18703	203352	17240	16653
5	1067	500	568	8786	21683	22248	236472	19390	20023
-6	1298	687	612	11063	27975	27341	296664	25077	24368
6	1514	717	798	12503	30921	31595	336408	27657	28412
-7	1787	938	850	15194	38357	37614	407544	34378	33547
7	2039	973	1067	16874	41794	42577	453912	37388	38265
-8	2354	1228	1127	19979	50374	49522	535992	45143	44190
8	2642	1268	1375	21899	54302	55194	588984	48583	49582
-9	2999	1557	1443	25418	64026	63065	682008	57372	56297
9	3323	1602	1722	27578	68445	69446	741624	61242	62363
-10	3722	1925	1798	31511	79313	78243	845592	71065	69868
10	4082	1975	2108	33911	84223	85333	911832	75365	76608

Example 2. *For* $\delta = 1$ *and* $\mu = \eta = 5$, $M_{5,0} = 11$ *and* $f_{5,0} = 20$ *from Table 1. Using* $M_{5,0}$ *and* δ *as constants in (27) with* $\lambda = 1$ *yield the generalized Pell equation* $f_{\mu,0}^2 - 11(\mu+1)^2 = 4$*. Using Matthews' method [16] yields the single fundamental solution* $(f_{\mu_1,0}, (\mu_1+1)) = (2,0)$ *which is of no use. However, as the right hand term is a squared integer, the equation can be rewritten as a simple Pell equation* $(f_{\mu,0}/2)^2 - 11((\mu+1)/2)^2 = 1$*, which admits the fundamental solution* $((f_{\mu_1,0}/2), ((\mu_1+1)/2)) = (10,3)$ *or* $(f_{\mu_1,0}, \mu_1) = (20,5)$ *and an infinity of other solutions* $\forall n \in \mathbb{Z}^+$

$$f_{\mu_n,0} = \left(10+3\sqrt{11}\right)^n + \left(10-3\sqrt{11}\right)^n = 20, 398, 7940, 158402, \ldots \tag{33}$$

$$\mu_n = \frac{\left(10+3\sqrt{11}\right)^n - \left(10-3\sqrt{11}\right)^n}{\sqrt{11}} - 1 = 5, 119, 2393, 47759, \ldots \tag{34}$$

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For $\delta = 1$, $\eta = 5$, $M_{5,0} = 11$, $f_{5,0} = 20$, $\nu = 1$, (19) to (21) yield (see Table 2)
$M_{5,k} = 327k^2 + 120k + 11$,
$a_{1,5,k} = \left(1635k^2 + 491k + 36\right)/2$, $a_{2,5,k} = \left(1635k^2 + 709k + 76\right)/2$,
$s_{1,5,k} = \left(35643k^3 + 17658k^2 + 2879k + 154\right)/2$,
$s_{2,5,k} = \left(35643k^3 + 21582k^2 + 4319k + 286\right)/2$.

Example 3. *For $\eta = 1$ and $\delta = 6$, $M_{1/6,0} = 312$ and $f_{1/6,0} = 23$ [29]. Using $M_{1/6,0}$ and δ as constants in (27) with $\lambda = 3$ yields $\left(3f_{\mu,0}\right)^2 - 78\left(\eta + 6\right)^2 = 939$, which by [16] has two fundamental solutions $\left(\left(3f_{\mu_1,0}\right), \left(\eta_1 + 6\right)\right) = (69,7)$ and $(381,43)$, yielding $\left(f_{\mu_1,0}, \eta_1\right) = (23,1)$ and $(127,37)$. The fundamental solutions of the related simple Pell equation $X^2 - 78Y^2 = 1$ are $\left(x_f, y_f\right) = (53,6)$. Other values of $\left(f_{\mu_n,0}, \eta_n\right)$ can be found on the two infinite branches corresponding to these two fundamental solutions by (29) and (30) as*

$$f_{\mu_n,0} = 23T_{n-1}(53) + 1092U_{n-2}(53) \quad (35)$$
$$= 23, 2311, 244943, 25961647, 2751689639, ... \quad (36)$$
$$\eta_n = 414U_{n-2}(53) + 7T_{n-1}(53) - 6 \quad (37)$$
$$= 1, 779, 83197, 8818727, 934702489, ... \quad (38)$$

for the first fundamental solution, and

$$f_{\mu_n,0} = 127T_{n-1}(53) + 6708U_{n-2}(53) \quad (39)$$
$$= 127, 13439, 1424407, 150973703, 16001788111, ... \quad (40)$$
$$\eta_n = 2286U_{n-2}(53) + 43T_{n-1}(53) - 6 \quad (41)$$
$$= 37, 4559, 483841, 51283211, 5435537149, ... \quad (42)$$

for the second fundamental solution. For $\eta = 1$, $\delta = 6$, $M_{1/6,0} = 312$, $f_{1/6,0} = 23$, $\nu = (2/3)$, (19) to (21) yield (see Table 2)

$M_{1/6,k} = 8784k^2 + 3312k + 312$,
$a_{1,1/6,k} = 732k^2 + 215k + 15$, $a_{2,1/6,k} = 732k^2 + 337k + 38$,
$s_{1,1/6,k} = 535824k^3 + 297924k^2 + 55188k + 3406$,
$s_{2,1/6,k} = 535824k^3 + 308172k^2 + 59052k + 3770$.

5. Conclusions

It is shown that regular linear features exist in the distribution of couples of values a and M in the (a, M) plot, where a and M are the first term and the number of terms in sums of consecutive squared integers equal to integer squares. These regular features correspond to groupings of pairs of a values for successive same values of M around straight lines of equation $\mu M \approx 2a$ for positive rational values of $\mu = (\eta/\delta)$.
For allowed values of η and δ such as $\eta \equiv 1 \,(mod\, 2)$ and $\delta \equiv 0, 1$ or $5 \,(mod\, 6)$, if $M_{\mu,k} = \delta^2\left(3\left(a_{2,\mu,k} - a_{1,\mu,k}\right)^2 - 1\right) / \left(3\left(\eta + \delta\right)^2 + \delta^2\right)$ holds $\forall k \in \mathbb{Z}$ and for pairs $\left(a_{1,\mu,k}, a_{2,\mu,k}\right)$, then the sums of $M_{\mu,k}$ consecutive squared integers starting with $a_{1,\mu,k}$ or $a_{2,\mu,k}$ are always equal to squared integers $s^2_{1,\mu,k}$ or $s^2_{2,\mu,k}$ $\forall k \in \mathbb{Z}$. Parametric equations are found in function of $k \in \mathbb{Z}$: linear for $\left(a_{2,\mu,k} - a_{1,\mu,k}\right)$, quadratic for $M_{\mu,k}$, $a_{1,\mu,k}$ and $a_{2,\mu,k}$, and cubic for $s_{1,\mu,k}$ and $s_{2,\mu,k}$.
The allowed values of $\eta, \delta, M_{\mu,0}$ and of the difference $f_{\mu,0} = a_{2,\mu,0} - a_{1,\mu,0}$ are found by solving the generalized Pell equation $\left(\delta f_{\mu,0}\right)^2 - M_{\mu,0}\left(\eta + \delta\right)^2 = \delta^2\left(M_{\mu,0} + 1\right)/3$ and further allowed values of η_n and $f_{\mu_n,0}$ can be calculated for fixed values of δ and $M_{\mu,0}$ using Chebyshev polynomials.

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