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Article

Behaviour of Particles in 2 Time Dimensional Spacetime with Separate Speeds of Causality through Dirac and Klein-Gordon Formulations

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Abstract: The Dirac[1] and Klein-Gordon[2,3] equations have proven to be some of the most accurate theories for describing particle behaviour and have laid the foundations of Relatavistic Quantum Mechanics. We extend the formulations to consider a 5 Dimensional spacetime with two time dimensions [4] (as first conceptualised by Sajjad Zahir) and extend the notions to Quantum Field Theory to arrive at the behaviour of particles through the Dirac and Klein-Gordon counterparts of the 5 Dimensional spacetime. The implications of this to anti-matter are discussed.

1. Introduction

The structure of Einstein's Special Theory of Relativity has been a 4D spacetime with 3 spatial and 1 temporal dimension[5] in a flat Minkowski space. This was later generalised in Einstein's Theory of General Relativity[6] where curved spacetime was introduced to generalise the equations for all observers in inertial or non-inertial frames of references. However, this was still done in the same 4D spacetime of 3 spatial and 1 temporal dimension. Since then some efforts have been made in trying to extend the dimensions beyond the known 4, first of which was the Kaluza-Klein Theory[3,7]. This was an extension of General Relativity in trying to unify Gravity with Electromagnetism through a fifth dimension(fourth spatial) and further efforts like String Theory[8–10] have been made since. Nevertheless, in the majority of these theories there was an extension of the spatial dimensions and rarely temporal. An early attempt was using time-like extra dimensions within the framework of brane world models, as an alternative to reconcile the mass hierarchy problem resolution with correct cosmological expansion of the visible universe within Randall-Sundrum[11] type of brane world, and, more generally, to show that neither the cosmological big-bang singularity, nor big-crunch arise in brane worlds with an extra time-like dimension[12]. One of the problems that arrived within the advent of extra-time dimensions was the causality violation in theories with extra time dimensions[13], which temporarily stopped the search for more such theories. This was changed in 2001, with Itzhek Bars[14] using a 2 time theory to discover interesting previously unnoticed hidden symmetries in 1T dynamical systems and upon further works was discovered by Israel Quiros[15] that causality and unitarity aren't violated in theories with an extra timelike dimension by imposing that the causal properties of spacetime are decided by the higher-dimensional structure rather than the four dimensional one. Following this, Sajjad Zahir[4] arrived at a new formulation of a 5D spacetime, where there are two speeds of light such that the sum of their squares is the square of the "normal speed of light." In the present paper, we wish to follow this and try to use the notion to investigate physical systems at the microscopic scales with the 5D counterparts of the Kaluza-Klein and Dirac equations. We start of in section 2 by laying off some of the preliminary notation to be used throughout the paper and then follow it up by quantizing the 5 dimensionsal energy-momentum relation to arrive at the 5D Klein-Gordon Equation in section 3. We then solve it to arrive at the plane-wave solutions in section 4 and derive the 5D Dirac equation in section 5 and use the plane-wave solutions to solve the 5D Dirac

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equation for a rest particle and general particle in sections 6 and 7 respectively. We then apply charge conjugation to the equation and give some concluding remarks in section 9.

2. Basic notation and conventions used

Throughout this paper we stick with the (+,+,-,-,-) metric and use subscripts to denote the different time dimensions: t_1 and t_2. It is also important to note that the two time dimensions are treated as having different units and we characterise t_1 with units of T and t_2 with units of t. Following from Zahir[4], we have separate speeds of causality in each of these temporal dimensions, it is important to note that neither of these are the speed of light. We therefore have c_1 in dimension t_1 with units LT^{-1} and c_2 in dimension t_2 with units Lt^{-1}. Furthermore, we should note that these speeds of causality are linked by

$$c_e^2 = c_1^2 + (\frac{c_2}{k})^2$$

where c_e is the "normal" speed of light as used in the 4D spacetime. In addition, we also have two Lorentz factors of the following form:

$$\gamma_1 = \frac{1}{\sqrt{1-(\beta_1)^2(1-\frac{1}{(\beta_2)^2})}}$$

$$\gamma_2 = \frac{1}{\sqrt{1-(\beta_2)^2(1-\frac{1}{(\beta_1)^2})}}$$

where β_1 and β_2 are defined by

$$\beta_n = \frac{1}{c_n}\frac{d\mathbf{x}}{dt_n}$$

Throughout this paper we take $\mathbf{v}_n = \frac{d\mathbf{x}}{dt_n}$ to be the 3 vector velocity in time dimension t_n as to be able to condense the above formula to just:

$$\beta_n = \frac{\mathbf{v}_n}{c_n}$$

Following in the same convention we have two proper velocities: u_1 and u_2, defined as:

$$u_n = \frac{d\vec{x}}{d\tau_n}$$

This defines the two different proper velocities that can exist in the new 5D spacetime, however it is important not to get this confused with the nth component of u as that is completely different and there is no longer a single proper velocity. To denote components we will have an additional subscript following the proper velocity.

3. Deriving the 5D Klein-Gordon Equation

As obtained from Sajjad Zahir[4], there can be two formalisms of the mas-energy equivalence in the new 5D spacetime:

$$\left(\frac{(E_1)_1}{c_1}\right)^2 + \left(\frac{(E_1)_2}{c_2}\right)^2 - \mathbf{p}_1{}^2 = m_0{}^2 c_1{}^2$$

$$\left(\frac{(E_2)_1}{c_1}\right)^2 + \left(\frac{(E_2)_2}{c_2}\right)^2 - \mathbf{p}_2{}^2 = m_0{}^2 c_2{}^2$$

However, it is rather trivial to show that they are equivalent and as such we prove the results for the former but they can be proved for the later just as easily as well.

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We now have to quantise the energy and momentum to yield the Klein-Gordon Equation.

$$(E_1)_1 \to i\hbar\frac{\partial}{\partial t_1}$$

$$(E_2)_1 \to i\hbar\frac{\partial}{\partial t_2}$$

$$\mathbf{p} \to i\hbar\nabla$$

$$\left(i\hbar\frac{\partial}{\partial t_1}\right)^2\left(\frac{1}{c_1}\right)^2 + \left(i\hbar\frac{\partial}{\partial t_2}\right)^2\left(\frac{1}{c_2}\right)^2 - (i\hbar\nabla)^2 = m_0^2c_1^2$$

After this we just simplify the equation and multiply it by ψ.

$$\left[\frac{1}{c_1^2}\frac{\partial^2}{\partial t_1^2} + \frac{1}{c_2^2}\frac{\partial^2}{\partial t_2^2} - \nabla^2 + \frac{m_0^2c_1^2}{\hbar^2}\right]\psi = 0$$

This result was also obtained by Sajjad Zahir[16] however he looked at the result of then compactifying this equation and analysing the new result obtained. In the present paper, we do not focus on that and instead look at the consequences of the purely mathematical 5 dimensional space-time structure and talk a bit about compactification in the conclusing remarks.

4. Solving the 5D Klein-Gordon Equation

As an ansatz we try the plane wave solution

$$\psi = Ae^{-\frac{i}{\hbar}(\vec{x}\cdot\vec{p}_1)}$$

Substituting this into the Klein-Gordon Equation yields

$$\left[\frac{1}{c_1^2}(\frac{-(E_1)_1}{\hbar})^2 + \frac{1}{c_2^2}(\frac{-(E_1)_2}{\hbar})^2 - (\frac{-i}{\hbar}\mathbf{p}_1)^2 + \frac{m_0^2c_1^2}{\hbar^2}\right]\psi = 0$$

From here it can be seen after factoring the $\frac{1}{\hbar^2}$ that the expression is just the negative of the energy momentum relation we obtained earlier and so the expression in the brackets is 0. Therefore, it follows that this a solution to the 5D Klein-Gordon Equation and hence is the form of plane-waves in 5 dimensions.

5. 5D Dirac Equation

To be able to derive the Dirac equation we have to take a different approach. As it has to be first order in time we can't simply quantise the momentum and energy in the energy-momentum relation as there are powers of order 2. Instead we will have to factor it out as follows:

$$(\vec{p_1})_\mu(\vec{p_1})^\mu - m_0^2c_1^2 = (\beta^k\vec{p_k} + m_0c_1)(\gamma^\lambda\vec{p_\lambda} - m_0c_1)$$

In this equation β and γ are just vectors and β^k γ^λ components of the vectors with an implied Einstein Summation Convention. Expanding the right hand side yields

$$(\vec{p_1})_\mu(\vec{p_1})^\mu - m_0^2c_1^2 = \beta^k\gamma^\lambda\vec{p_k}\vec{p_\lambda} + \gamma^\lambda\vec{p_\lambda}m_0c_1 - \beta^k\vec{p_k}m_0c_1 - m_0^2c_1^2$$

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For the two sides to be equal the terms that are linear in p must vanish. From this condition we can impose $\beta^k = \gamma^k$. Now we need to eliminate the cross-terms. To do this we require anticommutativity in gamma, therefore γ^k can't be real numbers or even complex numbers. Instead we will have to resort to matrices. The matrices should be such that $(\gamma^0)^2 = (\gamma^1)^2 = 1$ and $(\gamma^2)^2 = (\gamma^3)^2 = (\gamma^4)^2 = -1$. This forms a Clifford Algebra of $CL_{2,3}$ [17]and through that the gamma matrices can be derived rigorously. However, for the purposes of this paper, we take the 4 gamma matrices that Paul Dirac originally came up with[1] and for the fifth(really the first) we multiply the four and then multiply the result by the scalar i. Therefore, the gamma matrices are:

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\gamma^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now that we have obtained the matrices it is possible for us to factor the original equation. Both factors of the equation are valid however we discard the one with positive m_0c_1 as per the convention of the Dirac equation.

$$\gamma^\lambda (\vec{p_1})_\lambda - m_0 c_1 = 0$$

Finally, to obtain the actual 5D Dirac equation we have to quantise the momentum. We have done so for the 3 vector momentum but not the 5 vector. To quantise it we first write out the 5 vector momentum in vector form.

$$\vec{p}_1 = \begin{bmatrix} \frac{(E_1)_1}{c_1} \\ \frac{(E_1)_2}{c_2} \\ p_x \\ p_y \\ p_z \end{bmatrix}$$

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From this we notice that the first two components are just energy of type 1 which have been quantise already. Therefore we define ∂_1 such that

$$\partial_1 \equiv \begin{bmatrix} \frac{\partial}{c_1 \partial t_1} \\ \frac{\partial}{c_2 \partial t_2} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

With this we obtain the Dirac equation

$$\left(i\hbar\gamma^k (\partial_1)_k - m_0 c_1 \right) \psi = 0$$

In this equation however, we note that although ψ is a spinor it is only of dimension 4 as the gamma matrices are 4x4 matrices. However, this does not mean that ψ is in 4 dimensional spacetime.

6. Free particle at rest solution to the 5D Dirac Equation

After arriving at the 5D Dirac Equation, we now wish to solve the equation. As all solutions of the Klein-Gordon Equation satisfy the Dirac equation, we can start by asserting that ψ should be of the form

$$\psi = u(x,p)e^{-\frac{i}{\hbar}(\vec{p_1}\cdot\vec{x})}$$

To find out what $u(x,p)$ is, we first start by considering the case of a particle at rest(i.e has zero momentum). This simplifies ψ to

$$\psi = u(x,p)e^{-\frac{i}{\hbar}((E_1)_1 t_1 + (E_1)_2 t_2)}$$

In addition we define χ to allow us to write γ^1 as a block matrix.

$$\chi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\gamma^1 = \begin{bmatrix} 0 & \chi \\ \chi & 0 \end{bmatrix}$$

Now we can substitute this into the Dirac Equation and obtain just an algebraic equation for u.

$$\frac{(E_1)_1}{c_1}\gamma^0 u + \frac{(E_1)_2}{c_2}\gamma^1 u - m_0 c_1 u = 0$$

$$\frac{(E_1)_1}{c_1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_A \\ u_B \end{bmatrix} + \frac{(E_1)_2}{c_2}\begin{bmatrix} 0 & \chi \\ \chi & 0 \end{bmatrix}\begin{bmatrix} u_A \\ u_B \end{bmatrix} = m_0 c_1 \begin{bmatrix} u_A \\ u_B \end{bmatrix}$$

From this since both sides of the equation are spinors, we can split the components to obtain a pair of equations.

$$\frac{(E_1)_1}{c_1}u_A + \frac{(E_1)_2}{c_2}\chi u_A = m_0 c_1 u_A \quad ①$$

$$-\frac{(E_1)_1}{c_1}u_B + \frac{(E_2)_1}{c_2}\chi u_B = m_0 c_1 u_B \quad ②$$

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Since we wish to obtain a complete set of states for ψ we consider the simplest values for u_A and u_B to then see the corresponding solutions for ψ.

We start with ①, by letting u_A take the following values

$$\text{Let} \quad u_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{(E_1)_1}{c_1} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{(E_1)_2}{c_2} \\ 0 \end{bmatrix} = \begin{bmatrix} m_0 c_1 \\ 0 \end{bmatrix}$$

$$\text{Let} \quad u_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{(E_1)_1}{c_1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(E_1)_2}{c_2} \end{bmatrix} = \begin{bmatrix} 0 \\ m_0 c_1 \end{bmatrix}$$

From this we can observe that these solutions correspond to the energy in both time dimensions being positive as we obtain the relation $\frac{(E_1)_1}{c_1} + \frac{(E_1)_2}{c_2} = m_0 c_1$. We now perform the same procedure but with u_B in equation ② instead. Doing so yields the following.

$$\text{Let} \quad u_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{(E_1)_1}{c_1} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{(E_1)_2}{c_2} \\ 0 \end{bmatrix} = \begin{bmatrix} m_0 c_1 \\ 0 \end{bmatrix}$$

$$\text{Let} \quad u_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -\frac{(E_1)_1}{c_1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(E_1)_2}{c_2} \end{bmatrix} = \begin{bmatrix} 0 \\ m_0 c_1 \end{bmatrix}$$

From this we observe that the solutions correspond to when energy of type 1 in time dimension 1 is negative. This makes sense as it therefore corresponds to the anti-matter we observe and by energy in time dimension 2 being positive, discards the possibility of additional matter which we haven't observed. However, it is key to note that there is nothing specific with the energy of time dimension 1 and instead it just depends on how you assign the gamma matrices. Had we switched the order around of the first two then we would have arrived at the conclusion that energy of time dimension 2 is negative. With this we yield a new result:

$$\frac{(E_1)_1}{c_1} - \frac{(E_1)_2}{c_2} = m_0 c_1$$

This therefore shows that in the second case we don't have to deal with negative energy like we do in the 4 dimensional case instead we deal with sums and differences. This can conceptually make more sense as negative energy is quite a counter-intuitive concept to deal with and instead eliminating it to only have to deal with sums and differences makes the equation much more elegant. We still have to employ the trick of energy of type 1 in time dimension 1 being negative or positive to distinguish solutions(like will be the case in the next section), however apart from this one usage there is not reason to ever have to refer to such negative energy again and certainly not equivalating it to mass.

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Now with these solutions of u_A and u_B, we can combine them with the time dependence of the Dirac equation to obtain the 4 general solutions of the equation which are the following.

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-i((E_1)_1 t_1 + (E_1)_2 t_2)}$$

$$\psi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-i((E_1)_1 t_1 + (E_1)_2 t_2)}$$

$$\psi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-i((E_1)_1 t_1 + (E_1)_2 t_2)}$$

$$\psi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-i((E_1)_1 t_1 + (E_1)_2 t_2)}$$

7. General Plane Wave solutions to the 5D Dirac Equation

Now that we have obtained the free particle solutions to the 5D Dirac Equation, we wish to generalise this and obtain the general plane wave solutions to this equation. To do so we start off with the original form of ψ and no longer assume that the momentum is 0 and then substitute this into the Dirac Equation. Then we obtain a new equation for u.

$$(\gamma^0 \frac{(E_1)_1}{c_1} + \gamma^1 \frac{(E_1)_2}{c_2} - \gamma^2 p_x - \gamma^3 p_y - \gamma^4 p_z - m_0 c_1)u = 0$$

From here we divide through by i and expand out the matrices as block matrices and then group the terms together.

$$\begin{bmatrix} (\frac{(E_1)_1}{c_1} - m_0 c_1) & (\frac{(E_1)_2}{c_2} - \boldsymbol{\sigma} \cdot \mathbf{p}_1) \\ (\frac{(E_1)_2}{c_2} + \boldsymbol{\sigma} \cdot \mathbf{p}_1) & -(\frac{(E_1)_1}{c_1} + m_0 c_1) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From here we can do the same procedure that was carried out in the free particle solution to the equation and then we can obtain the general plane wave solutions. However, one problem that arises is that we can't know for certain which correspond to which energies. This can be resolved by correlating the solutions back to the free-particle solutions for which we do know what energies correspond to which solutions. This yields the following solutions where the first two are when $(E_1)_1$ is positive and the latter two where it is negative.

$$u_1 = N_1 \begin{bmatrix} 1 \\ 0 \\ \frac{\frac{(E_1)_2}{c_2} + p_z}{\frac{(E_1)_1}{c_1} + m_0 c_1} \\ \frac{p_x + i p_y}{\frac{(E_1)_1}{c_1} + m_0 c_1} \end{bmatrix}$$

$$u_2 = N_2 \begin{bmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{\frac{(E_1)_1}{c_1} + m_0 c_1} \\ \frac{\frac{(E_1)_2}{c_2} - p_z}{\frac{(E_1)_1}{c_1} + m_0 c_1} \end{bmatrix}$$

$$u_3 = N_3 \begin{bmatrix} \frac{p_z - \frac{(E_1)_2}{c_2}}{\frac{(E_1)_1}{c_1} - m_0 c_1} \\ \frac{p_x + i p_y}{\frac{(E_1)_1}{c_1} - m_0 c_1} \\ 1 \\ 0 \end{bmatrix}$$

$$u_4 = N_4 \begin{bmatrix} \frac{p_x - i p_y}{\frac{(E_1)_1}{c_1} - m_0 c_1} \\ \frac{-\frac{(E_1)_2}{c_2} - p_z}{\frac{(E_1)_1}{c_1} - m_0 c_1} \\ 0 \\ 1 \end{bmatrix}$$

In the above solutions N_1, N_2, N_3, N_4 are normalization constants for the purpose of ensuring that the norm of the solutions is 1. This can be done from the energy-momentum relation however, for the purposes of this paper, we will not do this as the results aren't particularly significant. Anyhow, the main point to be addressed is the interpretation of the difference of the energies. In ordinary 4-Dimensional spacetime, you only get 1 time dimension and as such the equation boils down to mass being equivalent to energy or negative energy. In the Feynman-Stückleberg interpretation, A negative energy solution represents a negative energy particle travelling backwards in time, or equivalently, a positive energy antiparticle going forwards in time. In this 5D spacetime structure, we suggest a modification of the interpretation: A negative energy of type 1 in time dimension 1 solution represents a negative energy of time dimension 1 particle travelling backwards in time dimension 1, or equivalently, a positive energy of type 1 in time dimension 1 antiparticle going forwards in time dimension 1.

8. Charge Conjugation with the 5D Dirac Equation

As a final step, we try employ charge conjugation to try prove the rules of anti-particles in this 5D Dirac Equation and ensure that they still work as expected but just in a more general way. To do so would require an extended definition of the electromagnetic 4-potential into 5 dimensions. One could try define this from a new definition of the electromagnetic tensor, however we will simplify this problem with a simplistic model of having a single charge and having it divided by the two speeds of light. As such the 5-potential would be as follows.

$$A^{\mu} = (\phi/c_1, k\phi/c_2), \mathbf{A}$$

The k is the constant from the beginning to ensure all scalar components of the vector have same dimensions. To then incorporate this into the dirac equation we can now employ the following transformation and substitute it into the Dirac equation.

$$\partial_{\mu} \rightarrow \partial_{\mu} + ieA_{\mu}$$

This is the normal equation with charge incorporated ad now we wish to take the complex conjugate to see the consequences of changing the charge sign. We then multiply by $-i\gamma^2$. From now, we assume natural units and as such we no longer need to incorporate $\hbar$ in the equation.

$$-i\gamma^2\gamma^{\mu^*}(\partial_{\mu} - ieA_{\mu})\psi^* - m\gamma^2\psi^* = 0$$

$$\text{Let} \quad \psi^{'} = i\gamma^2\psi^*$$

$$\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi^* + im\psi^* = 0$$

Therefore, from this we can see that the new spinor ψ^* describes a particle of the same mass m but of the opposite charge(i.e an anti-particle). As such, we maintain the properties of anti-matter but just with an extra time dimension in which both particles and anti-particles travel strictly forward.

9. Conclusion and future research

We derived equations and solutions to those equations modelling the behaviour of different particles under the postulate that there are two time dimensions each with a different speed of light associated to the speed of causality in that time dimensions. Other researchers have also investigated the possibility of Relatavistic Quantum Mechanics in multiple time dimensions, however they usually tend to be based on the assumption that there is a constant speed of light across both of those time dimensions. More work has to be done on the consequences of when there are multiple speeds of light. One of the remaining questions still is what exactly would be mediating this speed of causality as it is not quite linked to the photon coupling and also under what conditions that interaction could be observed. Another problem that was discussed earlier was that of compactification. In this paper we didn't present an argument of compactification and without the theory would not be viable with current experiments as there is no observed second time dimension. In the paper by Sajjad Zahir[16], he discussed it in more detail for the Klein-Gordon Equation but such a compacitfication for the Dirac Equation is not quite present at the current point in time.

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