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Posted Date: 21 December 2023

doi: 10.20944/preprints202312.1561.v1

Keywords: arrow time; path integral; classical motion; quantum transition doi:10.20944/preprints202312.1561.v1

Communication

Can a Particle Move Zigzag in Time?

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Abstract: Amplitudes of quantum transitions containing time zigzags are considered. The discussion is carried out in the framework of the Minkowski metric and the standard quantum mechanics without adding new postulates. We argue that time zigzags are not suppressed at the quantum level, but their contribution to the amplitude is zero. The result is valid for a single particle and a non-interacting scalar field.

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1. Introduction

The problem of the arrow of time origin has been of great interest for a long time [1–4]. The view that the fundamental rules of physics make no distinction between past and future is widely accepted. The common conclusion is that the presence of the time arrow is related to the macroscopic physics [5,6]. At the same time, there are some arguments that the arrow of time has been incorporated into the quantum theory in the form of the arrow of causality [4].

One of the main points of discussion concerns the origin of the time arrow at the planckian energies. The idea that the notion of time is formed together with the Universe appearance and is closely related to its expansion [7,8] looks reasonable. There are three questions that need to be answered. The first one is "How was the arrow of time formed and supported during the Universe evolution?" The impossibility of reversal motion of complicated systems is usually related to the entropy growth. But the statistical arguments cannot be applied to small systems like several particles.

The second question is "are there any spacetimes in which closed spacetime geodesics exist?" This subject is extensively discussed with a positive answer in papers [9–11] and many others. The coexistence of neighboring regions with opposite arrows of time is discussed in [13].

In this paper we consider the third question, "why can a particle zigzag in space but not in time?". Such processes have not yet been observed in the Minkowski metric, so the probability must be at least strongly suppressed [12].

The situation is quite clear at the classical level but it turns out to be nontrivial at the quantum one. Consider firstly a classical trajectory represented in Figure 1. The time derivative is infinite at point (t_c, q_c) and hence, classical motion near the turning point t_c is impossible. One can imagine that a particle classically approaches the turning point, then turns back being in a specific quantum state and moves away classically. If such a transition is feasible at all, it should be suppressed, as in the case of quantum mechanical tunneling. The study presented in this paper leads to a different conclusion: a time zigzag is allowed at the quantum level, but gives zero contribution to the transition amplitude.

doi:10.20944/preprints202312.1561.v1

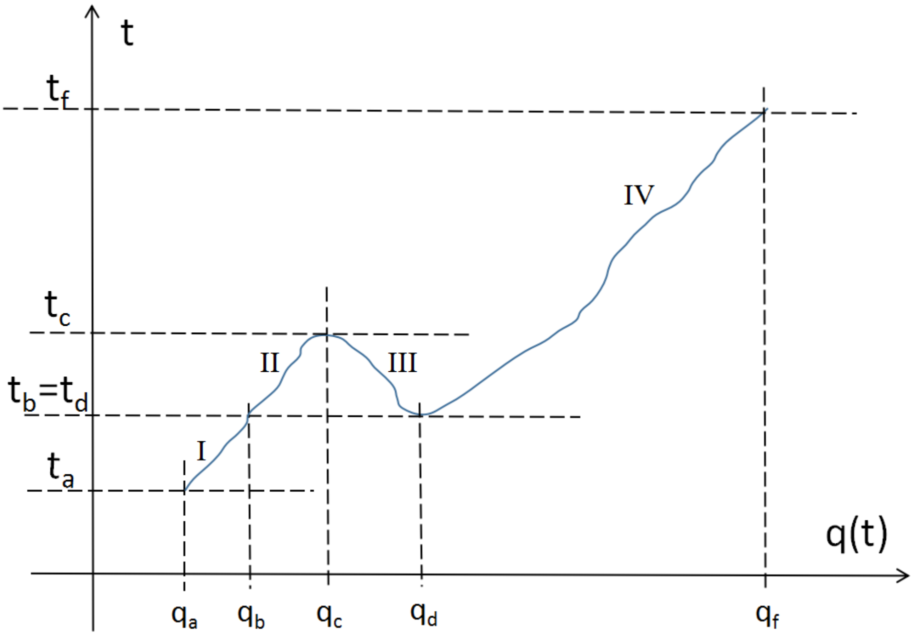

Figure 1. An example of trajectory. A particle moves back in time in the region III. The trajectory is parameterized by a parameter τ such that each point in the trajectory is in one-to-one correspondence with the specific value of the parameter τ. Points b and d are characterized by different parameter values $\tau_b \neq \tau_d$ but equal time $t_b = t_d$.

This argument also works in the case of field theory. Consider the non-interacting scalar field $\phi(t, x)$. Suppose the field is homogeneous, $\phi(t)$. Then Figure 1 with the substitution $q \to \phi$ remains valid and describes the time dependence of the field amplitude. The turning points are characterised by the infinite field derivatives, which forbids classical motion.

There is another known argument against a classical particle turning back in time. In this case, the particle would have to cross the light cone, which is impossible for a massive particle. This argument is not valid for field dynamics.

In this paper, we discuss the quantum transition amplitude of the time zigzag motion in the Minkowski space. We argue that although zigzag motion in time is unobservable indeed, it is not forbidden by the standard laws of quantum mechanics. The arguments why these two statements do not contradict each other are explained below. It will be proved that the result does not depend on the time interval of particle motion in the opposite time direction.

No complementary postulates are involved, although the path integral measure needs to be upgraded.

2. Transition Amplitude

The transition amplitude kernel has a well-known formal form

$$K(t_f, q_f; t_a, q_a) = \int \mathcal{D}q \exp\left\{ i \int_{t_a}^{t_f} dt \left[\frac{1}{2} \left(\frac{dq}{dt} \right)^2 - V(q) \right] \right\} \tag{1}$$

for the quantum mechanical tasks, see e.g., [14]. It is usually calculated assuming that $t_a < t_f$. The summed trajectories are defined by their values $q(t_i)$ at times t_i, which implicitly means that $q(t)$ is a single-valued function. Trajectories we are interested in, do not satisfy this criterion, as it is evident from Figure 1 . Indeed, for any t_i such that $t_b < t_i < t_c$ there exist three values of the function $q(t)$.

This suggests that if we allow for both forward and backward movement in time, the known measure of integration

$$\mathcal{D}q = \mathcal{N} \prod_i dq(t_i). \tag{2}$$

is invalidated. Here $\mathcal{N}$ is the normalization factor.

The solution to this problem is quite clear, as will be shown using the one-zigzag case as an example. To be more precise, when we consider such a transition amplitude, we sum only those trajectories describing a particle that starts its motion somewhere in the past, then turns backward in time at the moment t_c and turns forward again at t_d. This means that $dt < 0$ only between certain times t_c and t_d. One of such trajectory is presented in Figure 1. The calculation of such a transition amplitude gives a direct answer to the question posed in the title.

As a first step, we need to modernise the measure (2). To do this, we choose the parameter τ, which varies in the interval (τ_a, τ_f), and relate it to the time t in the interval (t_a, t_f). Knowledge of the turning points τ_c and τ_d greatly facilitates parameterization. One of the way to do this is to choose a function $t(\tau)$ in the piecewise form

$$t = \tau \qquad \text{at} \quad \tau < \tau_c \equiv t_c, \tag{3}$$

$$t = 2\tau_c - \tau \qquad \text{at} \quad \tau_c \le \tau \le \tau_d \equiv 2t_c - t_d, \tag{4}$$

$$t = \tau + 2(\tau_c - \tau_d) \qquad \text{at} \quad \tau > \tau_d. \tag{5}$$

The parameter τ grows monotonically along the trajectory so that $\tau_a < \tau_b < \tau_c < \tau_d < \tau_f$. A particle moves backward in physical time t in the interval (τ_c, τ_d). The important parameter value τ_b is defined from the condition

$$t(\tau_b) = t(\tau_d). \tag{6}$$

Now we can assign a unique parameter τ to each point of the trajectory. $q(\tau)$ is a single-valued function of τ for "one zigzag" trajectory, so the appropriate measure in functional integral (1) is

$$\mathcal{D}q = \mathcal{N} \prod_i dq(\tau_i) \tag{7}$$

The next step concerns the transition amplitude, which contains the turning points at instants τ_c and τ_d. It can be divided into four parts

$$K_Z(\tau_f, q_f; \tau_a, q_a) = \int dq_b dq_c dq_d K_{IV}(\tau_f, q_f; \tau_d, q_d) \times$$
$$K_{III}(\tau_d, q_d; \tau_c, q_c) K_{II}(\tau_c, q_c; \tau_b, q_b) K_I(\tau_b, q_b; \tau_a, q_a) \tag{8}$$

due to the principle of superposition. Here the turning instants τ_c and τ_d are fixed and the time parametrization (3)–(5) is taken into account. Remind that τ_b is obtained from the condition $t_b(\tau_b) = t_d(\tau_d)$, see Figure 1. The transition amplitude K_Z defined in (8) contributes to the total amplitude (1).

The amplitudes $K_I, K_{II}, K_{III}, K_{IV}$ do not have time zigzags, so they have the well-known form in terms of the new time/parameter τ due to the linear character of parametrization (3)–(5). The only difference in the K_{III} part of amplitude is the sign "minus" in the exponent

$$K_{III}(\tau_d, q_d; \tau_c, q_c) = \int \prod_i dq(\tau_i) \exp\{-iS_{III}\} \tag{9}$$
$$S_{III} = \int_{\tau_c}^{\tau_d} d\tau \left[\frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 - V(q) \right]$$

which is the result of motion in the opposite time direction, $dt = -d\tau$ according to (4). Sign "$-i\ldots$" is important in the following deliberation. As was discussed in [4], it relates to the arrow of causality.

The central point of this study is the transition amplitude

$$K_{II,III}(\tau_d, q_d; \tau_b, q_b) = \int dq_c K_{III}(\tau_d, q_d; \tau_c, q_c) K_{II}(\tau_c, q_c; \tau_b, q_b) \tag{10}$$

or in the path integral representation

$$K_{II,III} = \int dq_c \int Dq_{II}(\tau) Dq_{III}(\tau) e^{iS[q_{II}(\tau)]} e^{-iS[q_{III}(\tau)]}. \tag{11}$$

This amplitude consists of two pieces of trajectory - number II with time t goes in forward direction, "clockwise" and number III with time t going back in time, "anticlockwise", see Figure 1. Note that the $d\tau > 0$ for both pieces of trajectory. There is common point q_c at $\tau = \tau_c$ where the trajectories II are finished and the trajectories III are started. Next section is devoted to study a particle motion in an arbitrary potential.

3. Particle in an Arbitrary Potential

Consider a particle motion in a potential $V(q)$ taking into account the II and III parts of the trajectory. The time intervals $\tau_c - \tau_b$ and $\tau_d - \tau_c$ satisfy the conditions $\tau_c - \tau_b = \tau_d - \tau_c > 0$ and can be parted in N intervals in the standard manner

$$\tau_c - \tau_b = N\epsilon, \quad \tau_d - \tau_c = N\epsilon, \quad \epsilon \to 0. \tag{12}$$

Object defined as

$$\begin{aligned} K_0(\tau_c, q_1', q_1, n) \equiv & \int dq_c K_{III}(\tau_c + (n+1)\epsilon, q_1'; \tau_c + n\epsilon, q_c) \\ & \times K_{II}(\tau_c - n\epsilon, q_c; \tau_c - (n+1)\epsilon, q_1) \end{aligned} \tag{13}$$

is important for the following discussion. Note that two time instants $\tau_c + n\epsilon$ and $\tau_c - n\epsilon$ refer to the same physical time t because τ_c is the turning point. The object K_0 defined in (13) appears to be proportional to the δ function. Indeed, substituting the standard representation of the transition amplitude as in (9) into the definition (13) gives

$$K_0(\tau_c, q_1', q_1, n) \tag{14}$$

$$\begin{aligned} &= \int dq_c \, (2\pi i\epsilon)^{-1/2} \exp\left[-\frac{i}{2} \frac{(q_1' - q_c)^2}{\epsilon} + i\epsilon V(q_c) + o(\epsilon^2) \right] \times \\ &\times (2\pi i\epsilon)^{-1/2} \exp\left[\frac{i}{2} \frac{(q_1 - q_c)^2}{\epsilon} - i\epsilon V(q_c) + o(\epsilon^2) \right] \\ &= \exp\left[\frac{i}{2} ({q_1}^2 - {q_1'}^2) \right] \int dq_c \, (2\pi i\epsilon)^{-1} \exp\left[i q_c \frac{q_1' - q_1}{\epsilon} \right] = \delta(q_1 - q_1'). \end{aligned} \tag{15}$$

We also put the particle mass $m = 1$ and $\hbar = 1$ for not to overburden the calculations. In contrast to the typical sign in the third line, the second line contains the sign "minus" in the exponent, see (9), so the integral in the last line strongly differs from the usual form, see [15], which is the key point.

Let us take a closer look at the amplitude in the vicinity of the turning point τ_c.

$$\begin{aligned} K_{II,III}(\tau_d, q_d; \tau_b, q_b) &\equiv \int dq_c K_{III}(\tau_d, q_d; \tau_c, q_c) K_{II}(\tau_c, q_c; \tau_b, q_b) \\ &= \int K_{III}(\tau_d, q_d; \tau_c + \epsilon, q_c'') dq_c'' K_0(\tau_c, q_c'', q_c', n = 0) dq_c' K_{II}(\tau_c - \epsilon, q_c'; \tau_b, q_b) \\ &= \int dq_c' K_{III}(\tau_d, q_d; \tau_c + \epsilon, q_c') K_{II}(\tau_c - \epsilon, q_c'; \tau_b, q_b). \end{aligned} \tag{16}$$

Here the first line is the standard decomposition of the transition amplitude, the second line contains definition (13). In the third line we use the fact that K_0 is equal to the δ function according to (13). As a result, the transition amplitude describing the motion in the time interval $(\tau-\epsilon,\tau+\epsilon)$ disappears.

The procedure described above is repeated at the second iteration with $n=1$:

$$\begin{aligned}
K_{II,III}(\tau_d,q_d;\tau_b,q_b) &= \int dq_1 K_{III}(\tau_d,q_d;\tau_c+\epsilon,q_1)K_{II}(\tau_c-\epsilon,q_1;\tau_b,q_b)\\
&= \int dq_2 dq_2' K_{III}(\tau_d,q_d;\tau_c+2\epsilon,q_2)K_0(\tau_c,q_2,q_2',n=1)K_{II}(\tau_c-2\epsilon,q_2';\tau_b,q_b)\\
&= \int dq_2 K_{III}(\tau_d,q_d;\tau_c+2\epsilon,q_2)K_{II}(\tau_c-2\epsilon,q_2;\tau_b,q_b). \qquad (17)
\end{aligned}$$

The second and third terms in the middle line are the transition amplitudes acting in small time interval ϵ.

Repeating such procedure N times one obtains

$$\begin{aligned}
&K_{II,III}(\tau_d,q_d;\tau_b,q_b)\\
&= \int dq_N K_{III}(\tau_d,q_d;\tau_c+N\epsilon,q_N)K_{II}(\tau_c-N\epsilon,q_N;\tau_b,q_b) \qquad (18)
\end{aligned}$$

According to (12), $\tau_c-N\epsilon=\tau_b$ and $\tau_c+N\epsilon=\tau_d$. Both amplitudes under integral (18) do not contain inverse time motion. Therefore, we can use their standard normalization

$$\begin{aligned}
K_{II}(\tau_c-N\epsilon,q_N;\tau_b,q_b) &= K_{II}(\tau_b,q_N;\tau_b,q_b) = \delta(q_N-q_b),\\
K_{III}(\tau_c+N\epsilon,q_N;\tau_d,q_d) &= K_{III}(\tau_d,q_N;\tau_d,q_d) = \delta(q_N-q_d)
\end{aligned}$$

to substitute it into (18) that leads to the following result

$$K_{II,III}(\tau_d,q_d;\tau_b,q_b) = \delta(q_b-q_d), \quad \tau_b\neq\tau_d. \qquad (19)$$

This result is valid if a time reverse motion is assumed in the interval (τ_b,τ_d).

Let us finally substitute (19) into (8) to obtain the transition amplitude $\tau_a,q_a\to\tau_f,q_f$.

$$K(\tau_f,q_f;\tau_a,q_a) = i\int dq_b K_{IV}(\tau_f,q_f;\tau_b,q_b)K_I(\tau_b,q_b;\tau_a,q_a) \qquad (20)$$

The expression (20) is the standard form for the quantum transition amplitude for a particle moving ”clockwise” in the ordinary time regime. The part containing the reverse motion completely disappears, regardless of the potential shape and the duration of the time interval.

The discussion above shows that the reverse motion is feasible, but utterly unobservable. This conclusion is the result of calculations based on the standard quantum mechanics. The application of this method to Lagrangians containing higher derivatives is not so obvious and deserves further discussion.

4. Particular Cases

4.1. Oscillator

Let us consider the transition amplitude for the oscillator. Its classical motion is ruled by equation

$$\ddot{q}(t)+\omega^2 q(t)=0. \qquad (21)$$

Classical trajectories like that in Figure1 are forbidden as was discussed in the Introduction. Below, we discuss only quantum motion.

Consider the part of the transition amplitude (8) where the integration goes over trajectories with turning points q_c and q_d like in Figure 1. The amplitude of the transition for the oscillator is well known, so we do not need to divide the time intervals into infinitely small segments. Simple calculations confirm the previous result (19):

$$
\begin{aligned}
&K_{II,III}(\tau_d, q_d; \tau_b, q_b) = \int dq_c K_{III}(\tau_d, q_d; \tau_c, q_c) K_{II}(\tau_c, q_c; \tau_b, q_b) \\
&= \left[\frac{-\omega}{2\pi i \sin(\omega T)}\right]^{1/2} \exp\left[\frac{-i\omega}{2\sin(\omega T)}\left((q_d^2 + q_c^2)\cos(\omega T) - 2q_d q_c\right)\right] \times \\
&\left[\frac{\omega}{2\pi i \sin(\omega T)}\right]^{1/2} \exp\left[\frac{i\omega}{2\sin(\omega T)}\left((q_c^2 + q_b^2)\cos(\omega T) - 2q_b q_c\right)\right] = \\
&= i\left[\frac{\omega}{2\pi i \sin(\omega T)}\right] \int dq_c \exp\left[\frac{-i\omega}{\sin(\omega T)} q_c (q_b - q_d)\right] = i\delta(q_d - q_b).
\end{aligned}
\tag{22}
$$

Here $T = \Delta T = \tau_c - \tau_b = \tau_d - \tau_c > 0$. The third line is the textbook result for the oscillator transition amplitude, [14]. The same is written in the second line with the reversed time direction. One can see that these parts annihilate each other after integration over q_c and we come back to Formula (20).

A remark is necessary. The standard, time-ordered amplitude satisfies the condition

$$
K(t_d, q_d; t_b, q_b) = \delta(q_d - q_b) \quad \text{at} \quad t_b = t_d \tag{23}
$$

The physical meaning is that the particle velocity must be infinite if $q_d \neq q_b$ and the time interval equals zero. If a trajectory contains a time zigzag, then the velocity of particle can be arbitrary, but the ”zigzag part” of the amplitude (22) or (19) remains proportional to the δ function.

The result obtained above can be easily applied to the quantum transition of free particle in the limit $\omega \to 0$. Therefore, the classical motion in the backward time direction of a free particle is impossible, whereas quantum time zigzags leave no trace in the transition amplitude.

4.2. Scalar Field

The result of the previous section can be easily applied to the scalar field with action

$$
S = \frac{1}{2}\int d^4x \left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] \tag{24}
$$

Suppose that a space region $\mathcal{V}$ of finite size contains a certain field configuration $\phi(t_{in}, x)$, $x \in \mathcal{V}$ at time t_{in}. Can this field turn backwards in time?

In the momentum representation, action (24) describes the set $\{\phi_p\}$ of the harmonic oscillators

$$
S = \sum_{\mathbf{p}} \frac{1}{2}\int dt \left[\dot{\phi}_p(t)^2 - (\mathbf{p}^2 + m^2)\phi_p^2\right]. \tag{25}
$$

Each mode $\phi_p(t)$ evolves independently and hence any field configuration can be considered as a set of independent oscillators with frequencies $\omega_p^2 = \mathbf{p}^2 + m^2$. Therefore, we can apply the results of the previous subsection. The time zigzags of the scalar field are possible, but they are unobservable since they do not affect the transition amplitude.

5. Conclusions

We studied the possibility of a zigzag motion in time for a particle obeying the standard laws of quantum mechanics. The transition amplitudes, which include the reverse (zigzag) motion in time, were calculated using the standard path integral approach. Correct determination of the path integral measure is required to account for time zigzags.

The main result is that time zigzags are not forbidden at the quantum level. Moreover, the time interval between the turning points can be arbitrarily long, without affecting the transition amplitude.

This conclusion is also applicable to free scalar fields considered as a set of non-interacting oscillators. Theories with higher derivatives need further analysis.

Acknowledgments: The author is grateful to O. Zaslavskii for his interest in this research. The work was funded by the Ministry of Science and Higher Education of the Russian Federation, Project "New Phenomena in Particle Physics and the Early Universe" FSWU-2023-0073 and the Kazan Federal University Strategic Academic Leadership Program.

References

1. Claus Kiefer. Does time exist in quantum gravity? *Einstein Stud.*, 13:287–295, 2017. arXiv:0909.3767, https://doi.org/10.1007/978-1-4939-3210-8_10
2. Sean M. Carroll and Jennifer Chen. Spontaneous inflation and the origin of the arrow of time. 10 2004. arXiv:hep-th/0410270.
3. Matias Aiello, Mario Castagnino, and Olimpia Lombardi. The Arrow of time: From universe time-asymmetry to local irreversible processes. *Found. Phys.*, 38:257–292, 2008. arXiv:gr-qc/0608099, https://doi.org/10.1007/s10701-007-9202-0.
4. John F. Donoghue and Gabriel Menezes. Quantum causality and the arrows of time and thermodynamics. *Prog. Part. Nucl. Phys.*, 115:103812, 2020. arXiv:2003.09047, https://doi.org/10.1016/j.ppnp.2020.103812.
5. Andrea Di Biagio, Pietro Donà, and Carlo Rovelli. The arrow of time in operational formulations of quantum theory. *Quantum*, 5:520, 2021. arXiv:2010.05734, https://doi.org/10.22331/q-2021-08-09-520.
6. P. M. Harrington, D. Tan, M. Naghiloo, and K. W. Murch. Characterizing a statistical arrow of time in quantum measurement dynamics. *Phys. Rev. Lett.*, 123:020502, Jul 2019. URL: https://link.aps.org/doi/10.1103/PhysRevLett.123.020502, https://doi.org/10.1103/PhysRevLett.123.020502.
7. James B. Hartle, S. W. Hawking, and Thomas Hertog. The Classical Universes of the No-Boundary Quantum State. *Phys. Rev. D*, 77:123537, 2008. arXiv:0803.1663, https://doi.org/10.1103/PhysRevD.77.123537.
8. George F. R. Ellis and Barbara Drossel. Emergence of time. *Found. Phys.*, 50(3):161–190, 2020. arXiv:1911.04772, https://doi.org/10.1007/s10701-020-00331-x.
9. Dalia S. Goldwirth, Malcolm J. Perry, Tsvi Piran, and Kip S. Thorne. The Quantum propagator for a nonrelativistic particle in the vicinity of a time machine. *Phys. Rev. D*, 49:3951–3957, 1994. arXiv:gr-qc/9308009, https://doi.org/10.1103/PhysRevD.49.3951.
10. A. Carlini and I. D. Novikov. Time machines and the principle of selfconsistency as a consequence of the principle of stationary action. 2. The Cauchy problem for a selfinteracting relativistic particle. *Int. J. Mod. Phys. D*, 5:445–480, 1996. arXiv:gr-qc/9607063, https://doi.org/10.1142/S021827189600028X.
11. S. Krasnikov. The Time travel paradox. *Phys. Rev. D*, 65:064013, 2002. arXiv:gr-qc/0109029, https://doi.org/10.1103/PhysRevD.65.064013.
12. L. Ya. Kobelev. Why We Can Not Walk To and Fro in Time as Do it in Space? (Why the Arrow of Time is Exists?). *arXiv e-prints*, page physics/0011036, November 2000. arXiv:physics/0011036.
13. L. S. Schulman. Resolution of causal paradoxes arising from opposing thermodynamic arrows of time. *Physics Letters A*, 280(5-6):239–245, March 2001. arXiv:cond-mat/0102014, https://doi.org/10.1016/S0375-9601(01)00093-7.
14. Ashok Das. *Field Theory: A Path Integral Approach*. WSP, 2019. https://doi.org/10.1142/11339.
15. Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. 1995. URL: http://www.slac.stanford.edu/spires/find/books/www?cl=QC174.45%3AP4.