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The Strong Ekeland Variational Principle in Quasi-Pseudometric Spaces

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Abstract: We prove a quasi-pseudometric version of the strong Ekeland principle proved by P. Georgiev and T. Suzuki.

Keywords: quasi-metric space; completeness in quasi-metric spaces; variational principles; Ekeland variational principle; strong Ekeland principle

MSC: 58E30; 47H10; 54E25; 54E35; 54E50

Motto: "Je suis très honoré que le CEREMADE m'ait demandé de parler du principe dont je porte le nom." Ivar Ekeland, Paris 2018. [1]

1. Introduction

A variational principle is a proposition asserting that some function, usually bounded below and lower semi-continuous (lsc), attains its minimum. If the original function does not attain its minimum then one looks for an appropriate perturbation such that the perturbed function has a minimum. Variational principles have numerous applications to problems of optimization, in the study of the differentiability properties of mappings, in fixed point theory, etc. Their origins go back to the early stage of development of the calculus of variations and are related to the principle of least action from physics.

Ivar Ekeland announced in 1972, [12] (the proof appeared in 1974 in [13]) a theorem asserting the existence of the minimum of a small perturbation of a lower semicontinuous (lsc) function defined on a complete metric space. This result, known as Ekeland Variational Principle (EkVP), proved to be a very versatile tool in various areas of mathematics and in applications - optimization theory, geometry of Banach spaces, optimal control theory, economics, social sciences, and others. Some of these applications are presented by Ekeland himself in [14].

At the same time, it turned out that this principle is equivalent to a lot of results in fixed point theory (Caristi fixed point theorem), geometry of Banach spaces (drop property), and others (see [24], for instance).

Since then, many extensions of this principle have been published, a good record being given in the book by Meghea [22].

A version of EkVP in $T_!$-quasi-metric spaces was proved in [4]. The result was extended to arbitrary quasi-metric spaces in [19], where it was shown that the validity of this principle actually characterizes the right K-completeness of the underlying quasi-metric space. Other asymmetric versions (meaning quasi-metric, quasi-uniform or in normed or locally convex asymmetric spaces) were proved in [1,2,5,8,10,11], and others.

Strong versions of EkVP were proved by Georgiev [16,17] and Suzuki [33,34]. The aim of this paper is to prove a quasi-metric version of the strong Ekeland Variational principle (see Section 2).

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1 "I am very honored that CEREMADE invited me to speak about the principle whose name I bear." CEREMADE - Centre de Recherche en Mathématiques de la Décision, Paris

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2. Ekeland and the strong Ekeland variational principles in metric and Banach spaces

2.1. Ekeland principle

Ekeland [13] proved the following result, known as Ekeland Variational Principle (EkVP).

Theorem 2.1 (Ekeland Variational Principle). *Let (X,d) be a complete metric space and $f: X \to \mathbb{R}\cup\{+\infty\}$ a lsc bounded below function. Let $\varepsilon > 0$ and $x_0 \in \operatorname{dom} f$.*
Then given $\lambda > 0$ there exists $z = z_{\varepsilon,\lambda} \in X$ such that

$$\begin{aligned} &\text{(a)} \quad f(z) + \frac{\varepsilon}{\lambda} d(z,x_0) \le f(x_0);\\ &\text{(b)} \quad f(z) < f(x) + \frac{\varepsilon}{\lambda} d(z,x) \quad \textit{for all} \;\; x \in X \setminus \{z\}.. \end{aligned} \tag{2.1}$$

If, further, $f(x_0) \le \inf f(X) + \varepsilon$, then

$$\text{(c)} \quad d(z,x_0) \le \lambda.$$

The Ekeland Variational Principle is sometimes written in the following form (see, for instance, [24] or [25, Lemma 3.13]).

Theorem 2.2. *Let (X,d) be a complete metric space and $f: X \to \mathbb{R}\cup\{+\infty\}$ a lsc bounded below function. Let $\varepsilon > 0$ and $x_0 \in \operatorname{dom} f$.*
Then given $\lambda' > 0$ there exists $z = z_{\lambda'} \in X$ such that

$$\begin{aligned} &\text{(a}'\text{)} \quad f(z) + \lambda' d(z,x_0) \le f(x_0);\\ &\text{(b}'\text{)} \quad f(z) < f(x) + \lambda' d(z,x) \quad \textit{for all} \;\; x \in X \setminus \{z\}. \end{aligned} \tag{2.2}$$

If, further, $f(x_0) \le \inf f(X) + \varepsilon$, then

$$\text{(c}'\text{)} \quad d(z,x_0) \le \varepsilon/\lambda'.$$

The equivalence of Theorems 2.1 and 2.2 follows by the substitution

$$\lambda' = \frac{\varepsilon}{\lambda} \iff \lambda = \frac{\varepsilon}{\lambda'}. \tag{2.3}$$

2.2. The strong Ekeland variational principle

Let X be a Banach space and $f: X \to \mathbb{R}\cup\{\infty\}$ a function. A point $x_0 \in \operatorname{dom} f$ is called

- a *minimum point* for f if $f(x_0) \le f(x)$ for all $x \in X$;
- a *strict minimum point* for f if $f(x_0) < f(x)$ for all $x \in X \setminus \{x_0\}$;
- a *strong minimum point* for if $f(x_0) = \inf f(X)$ and every sequence (x_n) in X such that $\lim_n f(x_n) = \inf_X f$ is norm-convergent to x_0.

A sequence (x_n) satisfying $\lim_n f(x_n) = \inf f(X)$ is called a *minimizing sequence* for f.

Remark 2.3. A strong minimum point is a strict minimum point, but the converse is not true.

Indeed, if there exist $z \ne z'$ such that $f(z) = m = f(z')$, where $m = \inf f(X)$, then the sequence $x_{2k-1} = z$, $x_{2k} = z', k \in \mathbb{N}$, satisfies $\lim_n f(x_n) = m$, but it is not convergent. Also, the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 e^{-x}$, has a strict minimum at 0, $f(0) = 0$, $f(n) \to 0$, but the sequence $(n)_{n\in\mathbb{N}}$ does not converge to 0.

Condition (b$'$) in Theorem 2.2 asserts that, in fact, z is strict minimum point for the perturbed function $\tilde{f} := f + \lambda' d(z,\cdot)$. Georgiev [16,17] proved a stronger variant of Ekeland variational principle, guaranteeing the existence of a strong minimum point z for $\tilde{f}$.

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Theorem 2.4 (Strong Ekeland Variational Principle). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lsc function bounded from below on X. Then for every $\gamma, \delta > 0$ and $x_0 \in \operatorname{dom} f$ there exists $z \in X$ such that*

$$\begin{aligned}
&\text{(a)}\;\; f(z) + \gamma d(x_0, z) < f(x_0) + \delta;\\
&\text{(b)}\;\; f(z) < f(x) + \lambda d(z,x) \quad \textit{for all} \quad x \in X \setminus \{z\};\\
&\text{(c)}\;\; f(x_n) + \lambda d(z,x_n) \to f(z) \;\;\Rightarrow\;\; x_n \to z, \;\; \textit{for every sequence } (x_n) \textit{ in } X.
\end{aligned} \tag{2.4}$$

Geogiev, *loc. cit.*, also showed the equivalence of this strong form of EkVP with stronger forms of Danes' drop theorem, flower petal theorem, Phelps lemma, and others, extending so the results obtained by Penot [24]. He gave a direct proof to the strong drop theorem, the strong EkVP being a consequence of the equivalence mentioned above. Later Turinici [35] has shown that this strong form can be deduced from Theorem 2.2.

Observe that there is a discrepancy between the conditions (a′) in Theorem 2.2 and condition (a) in Theorem 2.4, condition (a′) being stronger than (a). As was remarked by Suzuki [33,34], a strong version of the Ekeland variational principle with condition (a′) instead of (a) can be proved by imposing supplementary conditions on the underlying metric (or Banach) space X, which are, in some sense, also necessary.

Let $f\colon X \to (-\infty, +\infty]$ be a proper function defined on a metric space (X,ρ). For $x_0 \in \operatorname{dom} f$ and $\lambda > 0$ consider an element $z = z_{x_0,\lambda}$ satisfying the following conditions:

$$\begin{aligned}
&\text{(i)}\;\; f(z) + \lambda\rho(z, x_0) \le f(x_0)\,;\\
&\text{(ii)}\;\; f(z) < f(x) + \lambda\rho(z,x) \quad \text{for all} \quad x \in X \setminus \{z\}\,;\\
&\text{(iii)}\;\; f(x_n) + \lambda\rho(z,x_n) \to f(z) \;\;\Rightarrow\;\; x_n \to z, \;\; \text{for every sequence } (x_n) \text{ in } X.
\end{aligned} \tag{2.5}$$

If $(X, \|\cdot\|)$ is a normed space, then $\rho(x,y)$ is replaced by $\|y - x\|$.

A metric space (X,ρ) is called *boundedly compact* if every bounded closed subset of X is compact, or equivalently, if every bounded sequence in X contains a convergent subsequence.

Remark 2.5. It is obvious that a boundedly compact metric space is complete, and that a normed space is boundedly compact if and only if it is finite dimensional.

Theorem 2.6 ([33]). *Let (X,ρ) be a boundedly compact metric space, $f\colon X \to (-\infty, +\infty]$ a lsc bounded from below function, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$.*

Then there exists a point $z \in X$ satisfying the conditions (2.5).

Remark 2.7. 1. Let X be a vector space. A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called *quasi-convex* if

$$f((1-t)x + ty) \le \max\{f(x), f(y)\}\,,$$

for all $x, y \in X$ and $t \in [0,1]$. This is equivalent to the fact that the sublevel sets $\{x \in X : f(x) \le \alpha\}$ are convex for all $\alpha \in \mathbb{R}$ (see [23]).

2. One says that a Banach space X is a *dual Banach space* if there exists a Banach space Y such that $Y^* = X$. Obviously, a reflexive Banach space is a dual Banach space with $X = (X^*)^*$ and, in this case, the weak (i.e. $\sigma(X^*, X^{**})$) and the weak* (i.e. $\sigma(X^*, X)$) topologies on X agree.

In the Banach space case the following results can be proved.

Theorem 2.8 ([33]). *Let X be a Banach space, $f\colon X \to (-\infty, +\infty]$ a bounded from below function, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$.*

1. *If X is a dual Banach space and f is w^*-lsc, then there exists a point $z \in X$ satisfying (2.5) with $x_n \xrightarrow{w^*} x$ in the condition (iii).*

2. *Suppose that the Banach space X is reflexive. If f is weakly lsc, then there exists a point $z \in X$ satisfying the conditions (2.5). The same is true if f is quasi-convex and norm-lsc.*

As it was shown by Suzuki [34], in some sense, the results from Theorems 2.6 and 2.8 are the best that can be expected.

Theorem 2.9. *For a metric space (X, ρ) the following are equivalent.*

1. *The metric space X is boundedly compact.*
2. *For every proper lsc bounded from below function $f\colon X \to (-\infty, +\infty]$, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$ there exists a point $z \in X$ satisfying the conditions (2.5).*
3. *For every Lipschitz function $f\colon X \to [0, +\infty)$, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$ there exists a point $z \in X$ satisfying the conditions (2.5).*

A similar result holds in the case of normed spaces.

Theorem 2.10. *For a normed space $(X, \|\cdot\|)$ the following are equivalent.*

1. *X is a reflexive Banach space.*
2. *For every proper lsc bounded from below quasi-convex function $f\colon X \to (-\infty, +\infty]$, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$ there exists a point $z \in X$ satisfying the conditions (2.5).*
3. *For every Lipschitz convex function $f\colon X \to [0, +\infty)$, $x_0 \in \operatorname{dom} f$ and $\lambda > 0$ there exists a point $z \in X$ satisfying the conditions (2.5).*

3. The case of quasi-pseudometric spaces

We present in this section some versions of Ekeland and strong Ekeland principles in quasi-pseudometric spaces.

3.1. Quasi-pseudometric spaces

A *quasi-pseudometric* on an arbitrary set X is a mapping $d : X \times X \to [0, \infty)$ satisfying the following conditions:

$$\begin{aligned} &(\mathrm{QM1}) \qquad d(x,y) \geq 0, \quad \text{and} \quad d(x,x) = 0;\\ &(\mathrm{QM2}) \qquad d(x,z) \leq d(x,y) + d(y,z), \end{aligned}$$

for all $x, y, z \in X$. If further

$$(\mathrm{QM3}) \qquad d(x,y) = d(y,x) = 0 \Rightarrow x = y,$$

for all $x, y \in X$, then d is called a *quasi-metric*. The pair (X, d) is called a *quasi-pseudometric space*, respectively a *quasi-metric space*[2] The conjugate of the quasi-pseudometric d is the quasi-pseudometric $\bar{d}(x,y) = d(y,x)$, $x, y \in X$. The mapping $d^s(x,y) = \max\{d(x,y), \bar{d}(x,y)\}$, $x, y \in X$, is a pseudometric on X which is a metric if and only if d is a quasi-metric.

If (X, d) is a quasi-pseudometric space, then for $x \in X$ and $r > 0$ we define the balls in X by the formulae

$$\begin{aligned} B_d(x,r) &= \{y \in X : d(x,y) < r\} \text{ - the open ball, and}\\ B_d[x,r] &= \{y \in X : d(x,y) \leq r\} \text{ - the closed ball.} \end{aligned}$$

Topological properties

[2] In [6] the term "quasi-semimetric" is used instead of "quasi-pseudometric", while in [18] it is called hemi-metric.

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The topology τ_d (or $\tau(d)$) of a quasi-pseudometric space (X, d) can be defined starting from the family $\mathcal{V}_d(x)$ of neighborhoods of an arbitrary point $x \in X$:

$$\begin{aligned} V \in \mathcal{V}_d(x) &\iff \exists r > 0 \text{ such that } B_d(x, r) \subseteq V \\ &\iff \exists r' > 0 \text{ such that } B_d[x, r'] \subseteq V. \end{aligned}$$

The convergence of a sequence (x_n) to x with respect to τ_d, called d-convergence and denoted by $x_n \xrightarrow{d} x$, can be characterized in the following way

$$x_n \xrightarrow{d} x \iff d(x, x_n) \to 0. \tag{3.1}$$

Also

$$x_n \xrightarrow{\bar{d}} x \iff \bar{d}(x, x_n) \to 0 \iff d(x_n, x) \to 0\,, \tag{3.2}$$

and

$$\begin{aligned} x_n \xrightarrow{d^s} x &\iff d^s(x, x_n) \to 0 \\ &\iff d(x, x_n) \to 0 \;\text{ and }\; \underline{d}(x_n, x) \to 0 \\ &\iff x_n \xrightarrow{d} x \;\text{ and }\; x_n \xrightarrow{\bar{d}} x\,, \end{aligned} \tag{3.3}$$

As a space equipped with two topologies, τ_d and $\tau_{\bar{d}}$, a quasi-pseudometric space can be viewed as a bitopological space in the sense of Kelly [20]. In fact, this is the main example of such a space considered in [20] and, later on, the quasi-uniform spaces were considered as well.

The following topological properties are true for quasi-pseudometric spaces.

Proposition 3.1 (see [6]). *If (X, d) is a quasi-pseudometric space, then the following hold.*

1. *The ball $B_d(x, r)$ is τ_d-open and the ball $B_d[x, r]$ is $\tau_{\bar{d}}$-closed. The ball $B_d[x, r]$ need not be τ_d-closed.*
2. *The topology τ_d is T_0 if and only if d is a quasi-metric.*
   The topology τ_d is T_1 if and only if $d(x, y) > 0$ for all $x \neq y$ in X.
3. *For every fixed $x \in X$, the mapping $d(x, \cdot) : X \to (\mathbb{R}, |\cdot|)$ is τ_d-usc and $\tau_{\bar{d}}$-lsc.*
   For every fixed $y \in X$, the mapping $d(\cdot, y) : X \to (\mathbb{R}, |\cdot|)$ is τ_d-lsc and $\tau_{\bar{d}}$-usc.

The following remarks show that imposing too many conditions on a quasi-pseudometric space it becomes pseudometrizable.

Remark 3.2 ([20]). Let (X, d) be a quasi-metric space. Then

(a) if the mapping $d(x, \cdot) : X \to (\mathbb{R}, |\cdot|)$ is τ_d-continuous for every $x \in X$, then the topology τ_d is regular;
(b) if $\tau_d \subseteq \tau_{\bar{d}}$, then the topology $\tau_{\bar{d}}$ is pseudometrizable;
(c) if $d(x, \cdot) : X \to (\mathbb{R}, |\cdot|)$ is $\tau_{\bar{d}}$-continuous for every $x \in X$, then the topology $\tau_{\bar{d}}$ is pseudometrizable.

Remark 3.3. The characterization of Hausdorff property (or T_2) of quasi-metric spaces can be given in terms of uniqueness of the limits, as in the metric case. The topology of a quasi-pseudometric space (X, d) is Hasudorff if and only if every sequence in X has at most one d-limit if and only if every sequence in X has at most one $\bar{d}$-limit (see [36]).

In the case of an asymmetric normed space there exists a characterization in terms of the quasi-norm (see [6], Proposition 1.1.40).

Recall that a topological space (X, τ) is called:

- T_0 if for every pair of distinct points in X, at least one of them has a neighborhood not containing the other;

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- T_1 if for every pair of distinct points in X, each of them has a neighborhood not containing the other;
- T_2 (or *Hausdorff*) if every two distinct points in X admit disjoint neighborhoods;
- *regular* if for every point $x \in X$ and closed set A not containing x there exist the disjoint open sets U, V such that $x \in U$ and $A \subseteq V$.

Completeness in quasi-pseudometric spaces

The lack of symmetry in the definition of quasi-metric spaces causes a lot of troubles, mainly concerning completeness, compactness and total boundedness in such spaces. There are a lot of completeness notions in quasi-metric spaces, all agreeing with the usual notion of completeness in the metric case, each of them having its advantages and weaknesses (see [26], or [6]).

As in what follows we shall work only with two of these notions, we shall present only them, referring to [6] for others.

We use the notation

$$\mathbb{N} = \{1, 2, \dots\} \text{ – the set of natural numbers,}$$
$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ – the set of non-negative integers.}$$

Definition 3.4. Let (X, d) be a quasi-pseudometric space. A sequence (x_n) in (X, d) is called:

- *left d-K-Cauchy* if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} &\forall n, m, \ \text{ with } \ n_\varepsilon \le n < m, \quad d(x_n, x_m) < \varepsilon \\ \iff &\forall n \ge n_\varepsilon, \ \forall k \in \mathbb{N}, \quad d(x_n, x_{n+k}) < \varepsilon; \end{aligned} \tag{3.4}$$

- *right d-K-Cauchy* if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} &\forall n, m, \ \text{ with } \ n_\varepsilon \le n < m, \quad d(x_m, x_n) < \varepsilon \\ \iff &\forall n \ge n_\varepsilon, \ \forall k \in \mathbb{N}, \quad d(x_{n+k}, x_n) < \varepsilon. \end{aligned} \tag{3.5}$$

The quasi-pseudometric space (X, d) is called:

- *sequentially left d-K-complete* if every left d-K-Cauchy sequence is d-convergent;
- *sequentially right d-K-complete* if every right d-K-Cauchy sequence is d-convergent;
- *sequentially left (right) Smyth complete* if every left (right) d-K-Cauchy sequence is d^s-convergent.

Remark 3.5.

1. It is obvious that a sequence is left d-K-Cauchy if and only if it is right $\bar{d}$-K-Cauchy. Also a left (right) Smyth complete quasi-pseudometric space is left (right) K-complete and the space (X, d) is right Smyth complete if and only if $(X, \bar{d})$ is left Smyth complete. For this reason, some authors call a Smyth complete space a left Smyth complete.
2. The notion of Smyth completeness, introduced by Smyth in [30] (see also [31]), is an important notion in quasi-metric and quasi-uniform spaces as well as for the applications to theoretical computer science (see, for instance, [27,29]). A good presentation of this notion is given in Section 7.1 of the book [18].
3. There are examples showing that a d-convergent sequence need not be left d-K-Cauchy, showing that in the asymmetric case the situation is far more complicated than in the symmetric one (see [26]).
4. If each convergent sequence in a regular quasi-metric space (X, d) admits a left K-Cauchy subsequence, then X is metrizable ([21]).

Remark 3.6.

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1. One can define more general notions of completeness by replacing in Definition 3.4 the sequences with nets. Stoltenberg [32, Example 2.4] gave an example of a sequentially right K-complete T_1 quasi-metric space which is not right K-complete (i.e., not right K-complete by nets). See [9] for some further specifications.
2. In the case of Smyth completeness, the completeness by nets is equivalent to the completeness by sequences (see [28]). Also, the left (or right) Smyth completeness implies the completeness of the pseudometric space (X, d^s). In this case one says that the quasi-pseudometric space (X, d) is bicomplete.

The following result is the quasi-pseudometric analog of a well-known property in metric spaces.

Proposition 3.7 (see [6], Section 1.2)**.** *Let (X, d) be a quasi-pseudometric space. If a right K-Cauchy sequence (x_n) contains a subsequence d-convergent ($\bar{d}$-convergent, d^s-convergent) to some $x \in X$, then the sequence (x_n) is d-convergent ($\bar{d}$-convergent, d^s-convergent) to x.*

3.2. Ekeland principle in quasi-pseudometric spaces

The following version of Ekeland variational principle in quasi-pseudometric spaces was proved in [8]. For a quasi-pseudometric space X, a function $f : X \to \mathbb{R} \cup \{\infty\}$, $\alpha > 0$ and $x \in X$ put

$$S_\alpha(x) = \{y \in X : f(y) + \alpha d(y, x) \le f(x)\} . \tag{3.6}$$

Theorem 3.8. *Let (X, d) be a sequentially right K-complete quasi-pseudometric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below lsc function. Given $\varepsilon, \lambda > 0$ and $x_0 \in \operatorname{dom} f$ there exists $z \in X$ such that*

$$\begin{aligned} &\text{(i)} \quad f(z) + \frac{\varepsilon}{\lambda} d(z, x_0) \le f(x_0);\\ &\text{(ii)} \quad f(y) = f(z) \ \textit{for all} \ y \in S_\gamma(z);\\ &\text{(iii)} \quad f(z) < f(x) + \frac{\varepsilon}{\lambda} d(x, z) \quad \textit{for all} \ x \in X \setminus S_\gamma(z), \end{aligned} \tag{3.7}$$

where $\gamma = \varepsilon/\lambda$.

If, further, $f(x_0) \le \varepsilon + \inf f(X)$, then

$$\text{(iv)} \quad d(z, x_0) \le \lambda.$$

Obviously, an analog of Theorem 2.2 holds in this case too.

Theorem 3.9. *Let (X, d) be a sequentially right K-complete quasi-pseudometric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below lsc function. Given $\varepsilon, \lambda' > 0$ and $x_0 \in \operatorname{dom} f$ there exists $z \in X$ such that*

$$\begin{aligned} &\text{(i')} \quad f(z) + \lambda' d(z, x_0) \le f(x_0);\\ &\text{(ii')} \quad f(y) = f(z) \ \textit{for all} \ y \in S_{\lambda'}(z);\\ &\text{(iii')} \quad f(z) < f(x) + \lambda' d(x, z) \quad \textit{for all} \ x \in X \setminus S_{\lambda'}(z). \end{aligned} \tag{3.8}$$

If, further, $f(x_0) \le \varepsilon + \inf f(X)$, then

$$\text{(iv')} \quad d(z, x_0) \le \varepsilon/\lambda'.$$

The proof of Theorem 3.8 is based on the properties of Picard sequences corresponding to the set-valued map $S_\alpha : X \rightrightarrows X$. A sequence $(x_n)_{n=0}^\infty$ in X is called a *Picard sequence* for S_α is $x_{n+1} \in S_\alpha(x_n)$ for all $n \in \mathbb{N}_0$, for a given $x_0 \in X$. We mention some of the properties of these sets $S_\alpha(x)$ which will be used in what follows.

Let (X, d) be a quasi-pseudometric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper function, i.e.,

$$\operatorname{dom} f := \{x \in X : f(x) < \infty\} \neq \emptyset.$$

It is obvious that $S_\alpha(x) = X$ if $f(x) = \infty$ and

$$S_\beta(x) \subseteq S_\alpha(x)$$

for $0 < \alpha < \beta$.

Proposition 3.10. *Let (X, d) be a quasi-pseudometric space, $f : X \to \mathbb{R} \cup \{\infty\}$ a proper function, $\alpha > 0$ and $x \in \operatorname{dom} f$. The set $S_\alpha(x)$ has the following properties:*

$$\begin{aligned}
&\text{(i)} \quad x \in S_\alpha(x) \quad \textit{and} \quad S_\alpha(x) \subseteq \operatorname{dom} f;\\
&\text{(ii)} \quad y \in S_\alpha(x) \ \Rightarrow\ f(y) \le f(x) \ \textit{and}\ S_\alpha(y) \subseteq S_\alpha(x);\\
&\text{(iii)} \quad y \in S_\alpha(x) \setminus \overline{\{x\}} \ \Rightarrow\ f(y) < f(x);\\
&\text{(iv)} \quad \textit{if } f \textit{ is bounded below, then}\\
&\qquad S_\alpha(x) \setminus \overline{\{x\}} \neq \emptyset \ \Rightarrow\ f(x) > \inf f(S_\alpha(x));\\
&\text{(v)} \quad \textit{if } f \textit{ is lsc, then } S_\alpha(x) \textit{ is closed.}
\end{aligned} \tag{3.9}$$

The key result used in the proofs of various variational principles in [8] is the following.

Proposition 3.11 ([8], Prop. 2.14)**.** *If the space (X, d) is sequentially right K-complete and the function f is bounded below and lsc, then there exists a point $z \in X$ such that*

$$\begin{aligned}
&\text{(i)} \quad f(y) = f(z) = \inf f(S_\alpha(z)) \ \textit{and}\\
&\text{(ii)} \quad S_\alpha(y) \subseteq \overline{\{y\}},
\end{aligned} \tag{3.10}$$

for all $y \in S_\alpha(z)$.

Remark 3.12. In fact, in [8], Proposition 3.11 is proved in a slightly more general context, namely for a nearly lsc function f, meaning that

$$f(x) \le \liminf_{n\to\infty} f(x_n),$$

for every sequence (x_n) in X with pairwise distinct terms in X such that $x_n \xrightarrow{d} x$.

3.3. The strong Ekeland principle – Georgiev's version

We show that Turinici proof [35] of the strong EkVP (Theorem 2.4) can be adapted to obtain a proof of a quasi-pseudometric version of the strong Ekeland Variational Principle.

Theorem 3.13. *Let (X, d) be a sequentially right K-complete quasi-pseudometric space and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below lsc function. Given $\gamma, \delta > 0$ and $x_0 \in \operatorname{dom} f$ there exists $z \in X$ such that*

$$\begin{aligned}
&\text{(a)} \quad f(z) + \gamma d(z, x_0) \le f(x_0) + \delta;\\
&\text{(b)} \quad f(y) = f(z) \ \textit{for all}\ y \in S_\gamma(z);\\
&\text{(c)} \quad f(z) < f(x) + \gamma d(x, z) \quad \textit{for all}\ x \in X \setminus S_\gamma(z);\\
&\text{(d)} \quad f(x_n) + \gamma d(z, x_n) \to f(z) \ \Rightarrow\ d(x_n, z) \to 0,\\
&\qquad \textit{for every sequence } (x_n) \textit{ in } X.
\end{aligned} \tag{3.11}$$

Proof. Let
$$X_0 = \{y \in X : f(x) \le f(x_0) + \delta\}\,.$$
Then $x_0 \in X_0$ and X_0 is closed (because f is lsc) and so sequentially right K-complete. Also
$$\inf f(X_0) = \inf f(X)\,. \tag{3.12}$$

Indeed, if $m := \inf f(X)$ and $M := \inf f(X_0)$, then $m \le M$. Let (x_n) be a sequence in X such that $f(x_n) \to m$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $f(x_n) \le m + \delta \le f(x_0) + \delta$, that is, $x_n \in X_0$, for all $n \ge n_0$. But then $M \le f(x_n)$, $\forall n \ge n_0$, which for $n \to \infty$ yields $M \le m$, and so $m = M$.

Let $0 < \lambda < 1$ be such that
$$\frac{\lambda}{1-\lambda}\,(f(x_0) - \inf f(X)) \le \delta\,. \tag{3.13}$$

By Theorem 3.9 applied to X_0, $f|_{X_0}$ and $\lambda' := (1-\lambda)\gamma$, there exists $z \in X_0$ such that
$$\begin{aligned} &\text{(i)} \quad f(z) + \lambda' d(z, x_0) \le f(x_0);\\ &\text{(ii)} \quad f(y) = f(z) \;\; \text{for all} \;\; y \in X_0 \cap S_{\lambda'}(z) = S_{\lambda'}(z);\\ &\text{(iii)} \quad f(z) < f(x) + \lambda' d(x,z) \quad \text{for all} \;\; x \in X_0 \setminus S_{\lambda'}(z)\,. \end{aligned} \tag{3.14}$$

To justify the equality $X_0 \cap S_{\lambda'}(z) = S_{\lambda'}(z)$ in (ii) above, observe that
$$z \in X_0 \quad \Rightarrow \quad S_{\lambda'}(z) \subseteq X_0\,.$$

Indeed, the existence of an element $x \in (X \setminus X_0) \cap S_{\lambda'}(z)$ would yield the contradiction:
$$f(x_0) + \delta < f(x) \le f(x) + \lambda' d(x,z) \le f(z) \le f(x_0) + \delta\,.$$

By (3.14).(i), the definition of λ' and (3.13),
$$\begin{aligned} \gamma d(x_0, z) &\le \frac{1}{1-\lambda}\,[f(x_0) - f(z)]\\ &= f(x_0) - f(z) + \frac{\lambda}{1-\lambda}\,[f(x_0) - f(z)]\\ &\le f(x_0) - f(z) + \frac{\lambda}{1-\lambda}\,[f(x_0) - \inf f(X)]\\ &\le f(x_0) - f(z) + \delta\,, \end{aligned}$$

showing that condition (3.11).(a) holds.

The inequality $\lambda' = (1-\lambda)\gamma < \gamma$ implies
$$S_\gamma(z) \subseteq S_{\lambda'}(z)\,,$$

so that, by (3.14).(ii), $f(y) = f(z)$ for all $y \in S_\gamma(z)$, i.e., (3.11).(b) holds too.

The inequality (3.11).(c) follows from the definition of the set $S_\gamma(z)$.

Observe now that, by the definition of the set $S_{\lambda'}(z)$,
$$f(z) < f(x) + (1-\lambda)\gamma d(x,z) \quad \text{for all} \quad x \in X \setminus S_{\lambda'}(z)\,. \tag{3.15}$$

To prove (3.11).(d), let (x_n) be a sequence in X such that
$$\lim_{n\to\infty}\,[f(x_n) + \gamma d(z, x_n)] = f(z)\,.$$

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If $x_n \in S_{\lambda'}(z)$, then, by (3.14).(ii) $f(x_n) = f(z)$ and the inequality $f(x_n) + \lambda' d(x_n, z) \le f(z)$ implies $d(x_n, z) = 0$.

For all n such that $x_n \in X \setminus S_{\lambda'}(z)$ the inequality (3.15) yields

$$\lambda\gamma d(x_n, z) < f(x_n) + \gamma d(x_n, z) - f(z) \longrightarrow 0 \text{ as } n \to \infty .$$

Consequently,

$$\lim_{n\to\infty} d(x_n, z) = 0 .$$

□

Remark 3.14. Actually, condition (3.11).(d) says that the minimizing sequence (x_n) is $\bar{d}$-convergent to z.

3.4. The strong Ekeland principle – Suzuki's versions

As we have seen in Subsection 3.1 completeness in quasi-pseudometric spaces has totally different features than that in metric spaces. The situation is the same with compactness, see [6].

In order to extend Theorem 2.6 to quasi-pseudometric spaces we consider the following notion. A subset Y of a quasi-pseudometric space (X, d) is called *d-bounded* if there exist $x \in X$ and $r > 0$ such that

$$Y \subseteq B_d[x, r] ,$$

or, equivalently,

$$\sup\{d(x, y) : y \in Y\} < \infty \quad \text{for every} \quad x \in X.$$

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is d-bounded if the set $\{x_n : n \in \mathbb{N}\}$ is d-bounded.

Similar definitions are given for $\bar{d}$-boundedness.

We have seen (Remark 2.5) that a boundedly compact metric space is complete. In the case of quasi-pseudometric spaces we have.

Proposition 3.15. *Let (X, d) be a quasi-pseudometric space. If every $\bar{d}$-bounded sequence in X contains a d^s-convergent subsequence, then the space X is right Smyth complete.*

Proof. Let (x_n) be a right K-Cauchy sequence in X. Then (x_n) is $\bar{d}$-bounded. Indeed, for $\varepsilon = 1$ there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n_1}) \le 1 \quad \text{fro all} \quad n \ge n_1 ,$$

which implies the $\bar{d}$-boundedness of (x_n). It follows that (x_n) contains a subsequence d^s-convergent to some $x \in X$. By Proposition 3.7 the sequence (x_n) is d^s-convergent to x. □

The analogs of the conditions (2.5) in the quasi-pseudometric case are:

$$\begin{aligned}
&\text{(i)}\ \ f(z) + \lambda\rho(z, x_0) \le f(x_0) ;\\
&\text{(ii)}\ \ f(y) = f(z) \quad \text{for all} \quad y \in S_\lambda(z);\\
&\text{(iii)}\ \ f(z) < f(x) + \lambda\rho(x, z) \quad \text{for all} \quad x \in X \setminus S_\lambda(z) ;\\
&\text{(iv)}\ \ f(x_n) + \lambda\rho(z, x_n) \to f(z) \ \ \Rightarrow \ \lim_{n\to\infty} d(x_n, z) = 0\\
&\qquad \text{for every sequence } (x_n) \text{ in } X.
\end{aligned} \tag{3.16}$$

The quasi-pseudometric analog of Theorem 2.6 is the following.

Theorem 3.16. *Let (X, d) be a quasi-pseudometric space such that every $\bar{d}$-bounded sequence in X contains a d^s-convergent subsequence and $f : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below d-lsc function. Then for every $x_0 \in X$ and $\lambda > 0$ there exists a point $z \in X$ satisfying (3.16).*

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Proof. By Theorem 3.9 there exists $z \in X$ such that

$$\begin{aligned} &\text{(a)} \quad f(z) + \lambda d(z, x_0) \le f(x_0);\\ &\text{(b)} \quad f(y) = f(z) \ \text{ for all } \ y \in S_\lambda(z);\\ &\text{(c)} \quad f(z) < f(x) + \lambda d(x,z) \quad \text{for all } \ x \in X \setminus S_\lambda(z)\,. \end{aligned} \tag{3.17}$$

Let (x_n) be a sequence in X such that

$$\lim_{n\to\infty} [f(x_n) + \lambda d(x_n, z)] = f(z)\,. \tag{3.18}$$

Suppose that $(d(x_n, z))_{n\in\mathbb{N}}$ does not converge to 0. Then there exist $\gamma > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of (x_n) such that $d(x_{n_k}, z) \ge \gamma$ for all $k \in \mathbb{N}$. Passing to this sequence we can suppose, without restricting the generality, that the sequence (x_n) satisfies (3.18) and that

$$d(x_n, z) \ge \gamma\,, \tag{3.19}$$

for all $n \in \mathbb{N}$.

Let $n_1 \in \mathbb{N}$ be such that

$$f(x_n) + \lambda d(x_n, z) \le f(z) + 1$$

for all $n \ge n_1$. Then

$$\begin{aligned} \lambda d(x_n, z) &= f(x_n) + \lambda d(x_n, z) - f(x_n)\\ &\le f(z) + 1 - \inf f(X)\,, \end{aligned}$$

for all $n \ge n_1$, which shows that the sequence (x_n) is $\bar{d}$-bounded. By hypothesis, it contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ d^s-convergent to some $y \in X$.

Observe that

$$y \notin \overline{\{z\}}^d. \tag{3.20}$$

Indeed,

$$y \in \overline{\{z\}}^d \iff d(y, z) = 0\,,$$

which would imply

$$d(x_{n_k}, z) \le d(x_{n_k}, y) + d(y, z) = d(x_{n_k}, y) \to 0 \ \text{ as } \ k \to \infty\,,$$

in contradiction to (3.19).

Since $S_\lambda(z) \subseteq \overline{\{z\}}^d$ (see Proposition 3.11), (3.20) implies $y \notin S_\lambda(z)$. Taking into account (3.17).(c) and the d-lsc of f and $d(\cdot, z)$, one obtains the contradiction

$$f(z) < f(y) + \lambda d(y, z) \le \lim_{n\to\infty} [f(x_n) + \lambda d(x_n, z)] = f(z)\,.$$

Consequently, we must have $\lim_{n\to\infty} d(x_n, z) = 0$. □

4. Conclusions

We have proved (Theorem 3.16) a version of strong Ekeland in a quasi-pseudometric space (X, d) having d^s-compact $\bar{d}$-bounded sets. As it was shown by Suzuki [34], in a metric space X the validity of strong EkVP is equivalent to the fact all closed bounded subsets of X are compact. I do not know whether a similar result holds in the asymmetric case - a question that deserves further investigation.

A notion of reflexivity of normed spaces was also considered in the asymmetric case (see [15] or [6, Section 2.5.6]), but in a more complicated way than in the classical one. The extension of Theorems 2.8 and 2.10 to the asymmetric case could be another theme of reflection.

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