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[Ibtisam Alshammari](#) , [Mesfer H. Alqahtani](#) , [Islam M. Taha](#) *

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Article

On r -Fuzzy Soft δ -Open Sets and Applications via Fuzzy Soft Topologies

Ibtesam Alshammari ¹, Mesfer H. Alqahtani ² and Islam M. Taha ^{3,*}

¹ Department of Mathematics, Faculty of Science, University of Hafr Al Batin, Saudi Arabia

² Department of Mathematics, University College of Umluj, University of Tabuk, Saudi Arabia

³ Department of Mathematics, Faculty of Science, Sohag University, Egypt

* Correspondence: imtaha2010@yahoo.com

Abstract: In this work, we introduce the concepts of r -fuzzy soft δ -open (resp. semi-open and α -open) sets on fuzzy soft topological space, the relations of these sets with each other are established. Next, we introduce the concepts of fuzzy soft δ -continuous (resp. β -continuous, semi-continuous, pre-continuous and α -continuous) functions, and some properties of these functions along with their mutual relationships are investigated. Also, a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α -continuity are given. Finally, as a weaker form of fuzzy soft continuity by Aygünoğlu et al. (2014), the concepts of fuzzy soft almost (resp. weakly) continuous functions are introduced, and some properties are specified. Moreover, we explore the notion of continuity in a very general setting namely fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuous functions, and a historical justification are obtained.

Keywords: Fuzzy soft set; fuzzy soft topological space; r -fuzzy soft δ -open set; fuzzy soft δ -continuity; fuzzy soft almost (weakly) continuity and fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuity

1. Introduction

In [1], the author initiated a novel concept of soft set theory, which is a completely new approach for modeling uncertainty and vagueness. He showed many applications of this theory in solving several practical problems in engineering, economics, medical science, social science, etc. Maji et al. [2] introduced the concept of fuzzy soft sets which combines soft sets [1] and fuzzy sets [3]. After that, many authors have contributed to soft set (fuzzy soft set) theory in the different fields such as topology, algebra and etc., see [4–14].

The concept of fuzzy soft topology is introduced and some of its properties such as fuzzy soft continuity, interior fuzzy soft set, closure fuzzy soft set and fuzzy soft subspace topology are obtained in [15,16] based on fuzzy topologies in Šostaks sense [17]. The concept of r -fuzzy soft regularly open sets was introduced by Çetkin and Aygün [16] based on the paper Aygünoğlu et al. [15]. Also, the concepts of r -fuzzy soft β -open (resp. pre-open) sets was also considered by Taha [18].

The organization of this paper is as follows:

(i) In Section 2, we define new types of fuzzy soft sets on fuzzy soft topological space based on the paper Aygünoğlu et al. [15]. Also, the relations of these sets with each other are established with the help of some examples.

(ii) In Section 3, we introduce the concepts of fuzzy soft δ -continuous (resp. β -continuous, semi-continuous, pre-continuous and α -continuous) functions, and the relations of these functions with each other are investigated. Moreover, a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α -continuity are given.

(iii) In Section 4, as a weaker form of fuzzy soft continuity [15], the concepts of fuzzy soft almost (resp. weakly) continuous functions are introduced, and some properties are obtained. Also, we show that fuzzy soft continuity \Rightarrow fuzzy soft almost continuity \Rightarrow fuzzy soft weakly continuity, but the converse need not be true in general. In the end, we explore the notion of continuity in a very general setting namely fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuous functions, and a historical justification are given.

(iv) Finally, we close this paper with some conclusions and make a plan for suggest some future works in Section 5.

Throughout this article, nonempty sets will be denoted by U, V etc., E is the set of all parameters for U and $A \subseteq E$, the family of all fuzzy sets on U is denoted by I^U (where $I_o = (0, 1]$, $I = [0, 1]$), and for $t \in I$, $\underline{t}(u) = t$, for all $u \in U$. The following definitions will be used in the next sections:

Definition 1.1. [2, 4, 15] A fuzzy soft set f_A on U is a function from E to I^U such that $f_A(e)$ is a fuzzy set on U , for each $e \in A$ and $f_A(e) = \underline{0}$, if $e \notin A$. The family of all fuzzy soft sets on U is denoted by $\widetilde{(U, E)}$.

Definition 1.2. [19] A fuzzy soft point e_{u_t} on U is a fuzzy soft set defined as follows:

$$e_{u_t}(k) = \begin{cases} u_t, & \text{if } k = e, \\ \underline{0}, & \text{if } k \in E - \{e\}, \end{cases}$$

where u_t is a fuzzy point on U . e_{u_t} is said to belong to a fuzzy soft set f_A , denoted by $e_{u_t} \tilde{\in} f_A$, if $t \leq f_A(e)(u)$. The family of all fuzzy soft points on U is denoted by $\widetilde{P_t(U)}$.

Definition 1.3. [15] A function $\tau : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ is called a fuzzy soft topology on U if it satisfies the following conditions for every $e \in E$,

- (i) $\tau_e(\Phi) = \tau_e(\tilde{E}) = 1$,
- (ii) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$, for every $f_A, g_B \in \widetilde{(U, E)}$,
- (iii) $\tau_e(\bigsqcup_{i \in \Delta} (f_A)_i) \geq \bigwedge_{i \in \Delta} \tau_e((f_A)_i)$, for every $(f_A)_i \in \widetilde{(U, E)}$, $i \in \Delta$.

Then, (U, τ_E) is called a fuzzy soft topological space (briefly, FSTS) in Šostaks sense [17].

Definition 1.4. [15] Let (U, τ_E) and (V, τ_F^*) be a FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U, E)} \rightarrow \widetilde{(V, F)}$ is said to be fuzzy soft continuous if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau_k^*(g_B)$ for every $g_B \in \widetilde{(V, F)}$, $e \in E$ and $(k = \psi(e)) \in F$.

Definition 1.5. [16, 18] Let (U, τ_E) be a FSTS. A fuzzy soft set $f_A \in \widetilde{(U, E)}$ is said to be r -fuzzy soft regularly open (resp. pre-open and β -open) if $f_A = I_\tau(e, C_\tau(e, f_A, r), r)$ (resp. $f_A \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r)$) and $f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r)$ for every $e \in E$ and $r \in I_0$.

Lemma 1.1. [18] Every r -fuzzy soft regularly open set [16] is r -fuzzy soft pre-open [18].

The converse of Lemma 1.1 is not true, in general, as shown by Example 1.1.

Example 1.1. [18] Let $U = \{u_1, u_2\}$, $E = \{e, k\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (k, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$. Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = g_E, \\ \frac{1}{3}, & \text{if } m_E = f_E, \\ \frac{1}{3}, & \text{if } m_E = g_E \sqcap f_E, \\ \frac{1}{4}, & \text{if } m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ \frac{1}{2}, & \text{if } m_E = g_E \sqcap f_E, \\ \frac{1}{4}, & \text{if } m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, f_E is $\frac{1}{4}$ -fuzzy soft pre-open set, but it is not $\frac{1}{4}$ -fuzzy soft regularly open set.

In a fuzzy soft topological space (U, τ_E) [16], the interior of f_A and the closure of f_A will be denoted by $I_\tau(e, f_A, r, r)$ and $C_\tau(e, f_A, r, r)$, respectively. The basic definitions and results which we need next sections are found in [15,16].

2. On r -Fuzzy Soft δ -Open Sets

In this section, we are going to give the concepts of r -fuzzy soft δ -open (resp. semi-open and α -open) sets on fuzzy soft topological space (U, τ_E) . Some properties of these sets along with their mutual relationships are investigated with the help of some examples.

Definition 2.1. Let (U, τ_E) be a FSTS. A fuzzy soft set $f_A \in \widetilde{(U, E)}$ is said to be r -fuzzy soft δ -open (resp. semi-open and α -open) if $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ (resp. $f_A \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ and $f_A \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r)$) for every $e \in E$ and $r \in I_0$.

Remark 2.1. The concept of r -fuzzy soft δ -open set and r -fuzzy soft β -open set [18] are independent concepts, as shown by Example 2.1 and 2.2.

Example 2.1. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$ and define $h_E, g_E, f_E \in \widetilde{(U, E)}$ as follows: $h_E = \{(e, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $g_E = \{(e, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\}), (k, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\}), (k, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\})\}$. Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ \frac{2}{3}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, h_E is $\frac{1}{3}$ -fuzzy soft β -open set, but it is neither $\frac{1}{3}$ -fuzzy soft δ -open nor $\frac{1}{3}$ -fuzzy soft semi-open.

Example 2.2. Let $U = \{u_1, u_2, u_3\}$, $E = \{e, k\}$ and define $h_E, g_E, f_E \in \widetilde{(U, E)}$ as follows: $h_E = \{(e, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\}), (k, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\})\}$, $g_E = \{(e, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\}), (k, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\})\}$, $f_E = \{(e, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\}), (k, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\})\}$. Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ \frac{1}{3}, & \text{if } m_E = h_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ \frac{1}{3}, & \text{if } m_E = h_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, h_E^c is $\frac{1}{4}$ -fuzzy soft δ -open set, but it is neither $\frac{1}{4}$ -fuzzy soft β -open nor $\frac{1}{4}$ -fuzzy soft semi-open.

Remark 2.2. The complement of r -fuzzy soft δ -open (resp. semi-open, α -open and β -open) set is said to be r -fuzzy soft δ -closed (resp. semi-closed, α -closed and β -closed).

Proposition 2.1. Let (U, τ_E) be a FSTS, $f_A \in \widetilde{(U, E)}$, $e \in E$ and $r \in I_0$. The following statements are equivalent,

- (i) f_A is r -fuzzy soft semi-open.
- (ii) f_A is r -fuzzy soft δ -open and r -fuzzy soft β -open.

Proof. (i) \Rightarrow (ii) Let f_A be r -fuzzy soft semi-open, then $f_A \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(C_\tau(e, f_A, r), r), r)$. This shows that f_A is r -fuzzy soft β -open. Moreover, $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, f_A, r) \sqsubseteq C_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r) = C_\tau(e, I_\tau(e, f_A, r), r)$. Therefore, f_A is r -fuzzy soft δ -open.

(ii) \Rightarrow (i) Let f_A be r -fuzzy soft δ -open and r -fuzzy soft β -open, $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ and $f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r)$. Thus, $f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r) \sqsubseteq C_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r) = C_\tau(e, I_\tau(e, f_A, r), r)$. This shows that f_A is r -fuzzy soft semi-open.

□

Proposition 2.2. Let (U, τ_E) be a FSTS, $f_A \in \widetilde{(U, E)}$, $e \in E$ and $r \in I_0$. The following statements are equivalent,

- (i) f_A is r -fuzzy soft α -open.
- (ii) f_A is r -fuzzy soft δ -open and r -fuzzy soft pre-open.

Proof. (i) \Rightarrow (ii) From Proposition 2.1 the proof is straightforward.

(ii) \Rightarrow (i) Let f_A be r -fuzzy soft pre-open and r -fuzzy soft δ -open. Then, $f_A \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r)$. This shows that f_A is r -fuzzy soft α -open.

□

Then, we have the following implications:

$$\begin{array}{ccc} & \alpha - \text{open} & \\ & \downarrow \quad \downarrow & \\ \text{pre} - \text{open} & \text{semi} - \text{open} & \longrightarrow \delta - \text{open} \\ & \downarrow \quad \downarrow & \\ & \beta - \text{open} & \end{array}$$

Moreover, the converses of the above relationships are not true, in general, as shown by Example 2.1, 2.2, 2.3, 2.4 and 2.5.

Example 2.3. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$ and define $g_E, f_E, h_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (k, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$, $h_E = \{(e, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\})\}$.

Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{(U, E)}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ \frac{2}{3}, & \text{if } m_E = f_E, \\ \frac{2}{3}, & \text{if } m_E = g_E \sqcap f_E, \\ \frac{1}{2}, & \text{if } m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ \frac{1}{2}, & \text{if } m_E = g_E \sqcap f_E, \\ \frac{1}{3}, & \text{if } m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, h_E is $\frac{1}{3}$ -fuzzy soft semi-open set, but it is neither $\frac{1}{3}$ -fuzzy soft α -open nor $\frac{1}{3}$ -fuzzy soft pre-open.

Example 2.4. Let $U = \{u_1, u_2, u_3\}$, $E = \{e, k\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\}), (k, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\})\}$. Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{(U, E)}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, f_E is $\frac{1}{3}$ -fuzzy soft β -open set, but it is not $\frac{1}{3}$ -fuzzy soft pre-open.

Example 2.5. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topology $\tau_E : E \rightarrow [0, 1]^{(U, E)}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, f_E is $\frac{1}{4}$ -fuzzy soft pre-open set, but it is neither $\frac{1}{4}$ -fuzzy soft α -open nor $\frac{1}{4}$ -fuzzy soft semi-open.

Theorem 2.1. Let (U, τ_E) be a FSTS, $f_A, g_B \in \widetilde{(U, E)}$, $e \in E$ and $r \in I_0$. If f_A is r -fuzzy soft δ -open set such that $f_A \sqsubseteq g_B \sqsubseteq C_\tau(e, f_A, r)$, then g_B is also r -fuzzy soft δ -open.

Proof. Suppose that f_A is r -fuzzy soft δ -open and $f_A \sqsubseteq g_B \sqsubseteq C_\tau(e, f_A, r)$. Then, $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, g_B, r), r)$. Since $g_B \sqsubseteq C_\tau(e, f_A, r)$, $I_\tau(e, C_\tau(e, g_B, r), r) \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, g_B, r), r)$. This shows that g_B is r -fuzzy soft δ -open. \square

3. Continuity via r -Fuzzy Soft δ -Open Sets

In this section, we introduce the concepts of fuzzy soft δ -continuous (resp. β -continuous, semi-continuous, pre-continuous and α -continuous) functions on fuzzy soft topological space in Šostaks sense. Also, we study several relationships related to fuzzy soft δ -continuity with the help of some illustrative problems. In addition, a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α -continuity are obtained.

Definition 3.1. Let (U, τ_E) and (V, τ_F^*) be a FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U, E)} \rightarrow \widetilde{(V, F)}$ is said to be fuzzy soft δ -continuous (resp. β -continuous, semi-continuous, pre-continuous and α -continuous) if $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft δ -open (resp. β -open, semi-open, pre-open and α -open) set for every $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r, e \in E, (k = \psi(e)) \in F$ and $r \in I_0$.

Remark 3.1. Fuzzy soft δ -continuity and fuzzy soft β -continuity are independent concepts, as shown by Example 3.1 and 3.2.

Example 3.1. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and define $h_E, g_E, f_E \in \widetilde{(U, E)}$ as follows: $h_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $g_E = \{(e_1, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\}), (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\}), (e_2, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = g_E, \\ \frac{2}{3}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = h_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \rightarrow (U, \tau_E^*)$ is fuzzy soft β -continuous, but it is neither fuzzy soft δ -continuous nor fuzzy soft semi-continuous.

Example 3.2. Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$ and define $h_E, g_E, f_E \in \widetilde{(U, E)}$ as follows: $h_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\})\}$, $g_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = g_E, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ \frac{1}{3}, & \text{if } m_E = h_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = h_E^c, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau_E^*)$ is fuzzy soft δ -continuous, but it is neither fuzzy soft β -continuous nor fuzzy soft semi-continuous.

Now, we have the following decomposition of fuzzy soft semi-continuity and decomposition of fuzzy soft α -continuity, according to Propositions 2.1 and 2.2.

Proposition 3.1. Let (U, τ_E) and (V, τ_F^*) be a FSTSs. $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is fuzzy soft semi-continuous function iff it is both fuzzy soft δ -continuous and fuzzy soft β -continuous.

Proof. The proof is obvious by Proposition 2.1. \square

Proposition 3.2. Let (U, τ_E) and (V, τ_F^*) be a FSTSs. $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is fuzzy soft α -continuous function iff it is both fuzzy soft δ -continuous and fuzzy soft pre-continuous.

Proof. The proof is obvious by Proposition 2.2. \square

Theorem 3.1. Let (U, τ_E) and (V, τ_F^*) be a FSTSs, and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. The following statements are equivalent for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$ and $r \in I_0$:

- (i) φ_ψ is fuzzy soft β -continuous.
- (ii) $I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_k^*(g_B^c) \geq r$.
- (iii) $I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r))$.
- (iv) $\varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$.

Proof. (i) \Rightarrow (ii) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B^c) \geq r$. Then by Definition 3.1,

$$(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B^c), r), r), r) = (I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r))^c. \text{ Thus, } I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(g_B).$$

(ii) \Rightarrow (iii) Obvious.

(iii) \Rightarrow (iv) Since $(I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r))^c = C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B^c), r), r), r)$ and $(\varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)))^c = \varphi_\psi^{-1}(I_{\tau^*}(k, g_B^c, r))$. Then, $\varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$, for each $g_B \in \widetilde{(V, F)}$.

(iv) \Rightarrow (i) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Then by (iv) and $g_B = I_{\tau^*}(k, g_B, r)$, $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$. Thus, φ_ψ is fuzzy soft β -continuous.

The following theorem is similarly proved as in Theorem 3.1. \square

Theorem 3.2. Let (U, τ_E) and (V, τ_F^*) be a FSTSs, and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. The following statements are equivalent for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$ and $r \in I_0$:

- (i) φ_ψ is fuzzy soft δ -continuous.
- (ii) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$, if $\tau_k^*(g_B^c) \geq r$.
- (iii) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)), r), r)$
- (iv) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$.

Then, we have the following implications:

$$\begin{array}{ccc}
 \alpha - \text{continuity} & & \\
 \downarrow & \downarrow & \\
 \text{pre} - \text{continuity} & \text{semi} - \text{continuity} \longrightarrow \delta - \text{continuity} & \\
 \downarrow & \downarrow & \\
 \beta - \text{continuity} & &
 \end{array}$$

Moreover, the converses of the above relationships are not true, in general, as shown by Example 3.1, 3.2, 3.3, 3.4 and 3.5.

Example 3.3. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and define $g_E, f_E, h_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (e_2, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$, $h_E = \{(e_1, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{(U, E)}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ \frac{2}{3}, & \text{if } m_E = f_E, \\ \frac{2}{3}, & \text{if } m_E = g_E \sqcap f_E, \\ \frac{1}{2}, & \text{if } m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = h_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \rightarrow (U, \tau_E^*)$ is fuzzy soft semi-continuous, but it is neither fuzzy soft α -continuous nor fuzzy soft pre-continuous.

Example 3.4. Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\}), (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{(U, E)}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{3}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \rightarrow (U, \tau_E^*)$ is fuzzy soft β -continuous, but it is not fuzzy soft pre-continuous.

Example 3.5. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{(U, E)}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \tilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau_E^*)$ is fuzzy soft pre-continuous, but it is neither fuzzy soft α -continuous nor fuzzy soft semi-continuous.

Proposition 3.3. Let (U, τ_E) , (V, τ_F^*) and (W, γ_H) be a FSTSs, and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$, $\varphi_{\psi^*} : \widetilde{(V, F)} \longrightarrow \widetilde{(W, H)}$ be two fuzzy soft functions. Then $\varphi_{\psi^*} \circ \varphi_\psi$ is fuzzy soft δ -continuous (resp. β -continuous) if φ_ψ is fuzzy soft δ -continuous (resp. β -continuous) and φ_{ψ^*} is fuzzy soft continuous.

Proof. Obvious. \square

4. New Types of Fuzzy Soft Functions

In this section, we introduce a weaker form of fuzzy soft continuous function [15] called fuzzy soft almost (resp. weakly) continuous function and study some properties of these functions. Also, we show that fuzzy soft continuity \Rightarrow fuzzy soft almost continuity \Rightarrow fuzzy soft weakly continuity, but the converse need not be true. Finally, we introduce a concept of continuity in a very general setting called fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuous functions.

Definition 4.1. Let (U, τ_E) and (V, τ_F^*) be a FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is said to be fuzzy soft almost (resp. weakly) continuous if, for each $e_{u_i} \in P_t(U)$ and each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$ containing $\varphi_\psi(e_{u_i})$, there is $f_A \in \widetilde{(U, E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_i} such that $\varphi_\psi(f_A) \sqsubseteq I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)$ (resp. $\varphi_\psi(f_A) \sqsubseteq C_{\tau^*}(k, g_B, r)$).

Theorem 4.1. Let (U, τ_E) and (V, τ_F^*) be a FSTSs, and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. Suppose that one of the following holds for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$ and $r \in I_0$:

- (i) If $\tau_k^*(g_B) \geq r$, $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$.
- (ii) $C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_k^*(g_B) \geq r$.

Then, φ_ψ is fuzzy soft almost continuous.

Proof. (i) \Rightarrow (ii) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. From (i), it follows

$$\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B^c, r), r)), r) = I_\tau(e, \varphi_\psi^{-1}((C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r))^c), r) = I_\tau(e, (\varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)))^c, r) = (C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r))^c.$$

Hence, $C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$. Similarly, we get (ii) \Rightarrow (i).

Suppose that (i) holds. Let $e_{u_i} \in P_t(U)$ and $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$ containing $\varphi_\psi(e_{u_i})$. Then, by (i), $e_{u_i} \in I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$, and so there is $f_A \in \widetilde{(U, E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_i} such that $f_A \sqsubseteq \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r))$. Hence, $\varphi_\psi(f_A) \sqsubseteq I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)$. Then, φ_ψ is fuzzy soft almost continuous.

\square

Lemma 4.1. Every fuzzy soft continuous function [15] is fuzzy soft almost continuous.

Proof. It follows from Theorem 4.1. \square

The converse of the above Lemma is not true, in general, as shown by Example 4.1.

Example 4.1. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if } m_E \in \{f_E, g_E\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \rightarrow (U, \tau_E^*)$ is fuzzy soft almost continuous, but it is not fuzzy soft continuous.

Theorem 4.2. Let (U, τ_E) and (V, τ_F^*) be a FSTSs, and $\varphi_\psi : \widetilde{(U, E)} \rightarrow \widetilde{(V, F)}$ be a fuzzy soft function.

Suppose that one of the following holds for every $g_B \in (V, F)$, $e \in E$, $(k = \psi(e)) \in F$ and $r \in I_0$:

(i) $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)), r)$, if $\tau_k^*(g_B) \geq r$.

(ii) $C_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_k^*(g_B^c) \geq r$.

Then, φ_ψ is fuzzy soft weakly continuous.

Proof. (i) \Rightarrow (ii) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B^c) \geq r$. From (i), it follows

$$\varphi_\psi^{-1}(g_B^c) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B^c, r)), r) = I_\tau(e, \varphi_\psi^{-1}((I_{\tau^*}(k, g_B, r))^c), r) = I_\tau(e, (\varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)))^c, r) = (C_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)), r))^c.$$

Hence, $C_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$. Similarly, we get (ii) \Rightarrow (i).

Suppose that (i) holds. Let $e_{u_t} \in \widetilde{P_t(U)}$ and $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$ containing $\varphi_\psi(e_{u_t})$. Then, by (i), $e_{u_t} \tilde{\in} I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)), r)$, and so there is $f_A \in \widetilde{(U, E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_t} such that $f_A \sqsubseteq \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r))$. Thus, $\varphi_\psi(f_A) \sqsubseteq C_{\tau^*}(k, g_B, r)$. Hence, φ_ψ is fuzzy soft weakly continuous. \square

Lemma 4.2. Every fuzzy soft almost continuous function is fuzzy soft weakly continuous.

Proof. It follows from Definition 4.1. \square

The converse of the above Lemma is not true, in general, as shown by Example 4.2.

Example 4.2. Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$ and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.6}, \frac{u_2}{0.6}, \frac{u_3}{0.5}\}), (e_2, \{\frac{u_1}{0.6}, \frac{u_2}{0.6}, \frac{u_3}{0.5}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0}, \frac{u_3}{0.5}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0}, \frac{u_3}{0.5}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \rightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = g_E, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_e^*(m_E) = \begin{cases} 1, & \text{if } m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if } m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau_E^*)$ is fuzzy soft weakly continuous, but it is not fuzzy soft almost continuous.

Then, we have the following implications:

$$\begin{array}{c} \text{fuzzy soft continuity} \\ \Downarrow \\ \text{fuzzy soft almost continuity} \\ \Downarrow \\ \text{fuzzy soft weakly continuity.} \end{array}$$

In [20], the difference between f_A and g_B is a fuzzy soft set defined as follows:

$$(f_A \bar{\cap} g_B)(e) = \begin{cases} 0, & \text{if } f_A(e) \leq g_B(e), \\ f_A(e) \wedge (g_B(e))^c, & \text{otherwise,} \end{cases} \quad \forall e \in E.$$

Let \mathcal{L} and $\mathcal{M} : E \times \widetilde{(U, E)} \times I_o \rightarrow I^U$ be operators on $\widetilde{(U, E)}$, and \mathcal{N} and $\mathcal{O} : F \times \widetilde{(V, F)} \times I_o \rightarrow I^V$ be operators on $\widetilde{(V, F)}$.

Definition 4.2. Let (U, τ_E) and (V, τ_F^*) be a FSTSs. $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is said to be fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuous function if $\mathcal{L}[e, \varphi_\psi^{-1}(\mathcal{O}(k, g_B, r)), r] \bar{\cap} \mathcal{M}[e, \varphi_\psi^{-1}(\mathcal{N}(k, g_B, r)), r] = \Phi$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$, $e \in E$ and $(k = \psi(e)) \in F$.

In (2014), Aygünoğlu et al. [15] defined the concept of fuzzy soft continuous functions: $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau_k^*(g_B)$ for every $g_B \in \widetilde{(V, F)}$, $e \in E$ and $(k = \psi(e)) \in F$. We can see that Definition 4.2 generalizes the concept of fuzzy soft continuous functions, when we choose \mathcal{L} = identity operator, \mathcal{M} = interior operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.

A historical justification of Definition 4.2:

- (1) In Section 3, we introduced the concept of fuzzy soft δ -continuous functions: $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = interior closure operator, \mathcal{M} = closure interior operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.
- (2) In Section 3, we introduced the concept of fuzzy soft β -continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = closure interior closure operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.
- (3) In Section 3, we introduced the concept of fuzzy soft semi-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = closure interior operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.
- (4) In Section 3, we introduced the concept of fuzzy soft pre-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = interior closure operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.
- (5) In Section 3, we introduced the concept of fuzzy soft α -continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = interior closure interior operator, \mathcal{N} = identity operator and \mathcal{O} = identity operator.
- (6) In Section 4, we introduced the concept of fuzzy soft almost continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = interior operator, \mathcal{N} = interior closure operator and \mathcal{O} = identity operator.

(7) In Section 4, we introduced the concept of fuzzy soft weakly continuous functions: $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_{\tau}(e, \varphi_{\psi}^{-1}(C_{\tau^*}(k, g_B, r)), r)$ for each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Here, \mathcal{L} = identity operator, \mathcal{M} = interior operator, \mathcal{N} = closure operator and \mathcal{O} = identity operator.

5. Conclusion and Future Work

This article is lay out as follows:

I. In Section 2, some new types of fuzzy soft sets namely r -fuzzy soft δ -open (resp. semi-open and α -open) sets are introduced on fuzzy soft topological space based on the paper Aygünoğlu et al. [15]. Also, we have the following relationships:

$$\begin{array}{ccc}
 \alpha - open & & \\
 \downarrow & \downarrow & \\
 pre - open & semi - open & \longrightarrow \delta - open \\
 \downarrow & \downarrow & \\
 \beta - open & &
 \end{array}$$

but the converse need not be true in general.

II. In Section 3, we introduce the concepts of fuzzy soft δ -continuous (resp. β -continuous, semi-continuous, pre-continuous and α -continuous) functions, and the relations of these functions with each other are investigated with the help of some illustrative examples. Moreover, a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α -continuity are given.

III. In Section 4, as a weaker form of fuzzy soft continuity [15], the concepts of fuzzy soft almost (resp. weakly) continuous functions are introduced, and some properties are obtained. Also, we show that fuzzy soft continuity \Rightarrow fuzzy soft almost continuity \Rightarrow fuzzy soft weakly continuity, but the converse need not be true in general. Finally, we explore the notion of continuity in a very general setting namely fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$ -continuous functions. Then, we have the following results:

- (1) Fuzzy soft $(id_U, I_{\tau}, id_V, id_V)$ -continuous function is a fuzzy soft continuous function [15].
- (2) Fuzzy soft $(I_{\tau}(C_{\tau}), C_{\tau}(I_{\tau}), id_V, id_V)$ -continuous function is a fuzzy soft δ -continuous function.
- (3) Fuzzy soft $(id_U, C_{\tau}(I_{\tau}(C_{\tau})), id_V, id_V)$ -continuous function is a fuzzy soft β -continuous function.
- (4) Fuzzy soft $(id_U, C_{\tau}(I_{\tau}), id_V, id_V)$ -continuous function is a fuzzy soft semi-continuous function.
- (5) Fuzzy soft $(id_U, I_{\tau}(C_{\tau}), id_V, id_V)$ -continuous function is a fuzzy soft pre-continuous function.
- (6) Fuzzy soft $(id_U, I_{\tau}(C_{\tau}(I_{\tau})), id_V, id_V)$ -continuous function is a fuzzy soft α -continuous function.
- (7) Fuzzy soft $(id_U, I_{\tau}, I_{\tau^*}(C_{\tau^*}), id_V)$ -continuous function is a fuzzy soft almost continuous function.
- (8) Fuzzy soft $(id_U, I_{\tau}, C_{\tau^*}, id_V)$ -continuous function is a fuzzy soft weakly continuous function.

In upcoming articles, we will use the r -fuzzy soft δ -open sets to introduce some new separation axioms and to define the concept of δ -compact spaces on fuzzy soft topological space based on the paper Aygünoğlu et al. [15].

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