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A Convergence Criterion for a Class of Stationary Inclusions in Hilbert Spaces

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Abstract: We consider a stationary inclusion in a real Hilbert space X, governed by a set of constraints K, a nonlinear operator A and an element $f \in X$. Under appropriate assumptions on the data the inclusion has a unique solution, denoted by u. We state and prove a covergence criterion, i.e., we provide necessary and sufficient conditions on a sequence $\{u_n\} \subset X$ which guarantee its convergence to the solution u. We then present several applications which provide the continuous dependence of the solution with respect to the data K, A and f, on one hand, and the convergence of an associate penalty problem, on the other hand. We use these abstract results in the study of a frictional contact problem with elastic materials which, in a weak formulation, leads to a stationary inclusion for the deformation field. Finally, we apply the abstract penalty method in the anlysis of two nonlinear elastic constitutive laws.

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MSC: 47J22; 47J20; 47J30; 74M10; 74C05

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1. Introduction

Besides existence and uniqueness results, convergence results represent an important topic in Functional Analysis, Numerical Analysis, Differential and Partial Differential Equations Theory. Some elementary examples are the following: the continuous dependence of the solution with respect to the data, the convergence of the solution of a penalty problem to the solution of the original problem as the penalty parameter converges, the convergence of the discrete solution to the solution of the continuous problem as the time step or the discretization parameter converges to zero. Convergence results are important in the study of mathematical models which arise in Mechanics and Engineering Sciences, as well. Thus, the convergence of the solution of a contact problem with a deformable foundation to the solution of a contact problem with a rigid foundation as the stiffness coeficient of the foundation goes to infinity, the convergence of the solution of a viscoelastic problem to the solution of an elastic problem as the viscosity vanishes, the convergence of the solution of a frictional problem to the solution of a frictionless problem as the coefficient of friction converges to zero are simple examples, among others.

For all these reasons, a considerable effort was done to obtain convergence results in the study of various mathematical problems including nonlinear equations, inequality problems, fixed point problems and optimization problems, for instance. Most of the convergence results obtained in the literature provide sufficient conditions which guarantee the convergence of a given sequence $\{u_n\}$ to the solution of the corresponding problem, denoted in what follows by $\mathcal{P}$. In other words, these results do not describe all the sequences which have this property. Therefore, we naturally arrive to consider the following problem.

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Problem $\mathcal{Q}_\mathcal{P}$. *Given a metric space (X, d), a Problem $\mathcal{P}$ which has a unique solution $u \in X$, provide necessary and sufficient conditions which guarantee the convergence of an arbitrary sequence $\{u_n\} \subset X$ to the solution u.*

In other words, Problem $\mathcal{Q}_\mathcal{P}$ consists to provide a convergence criterion to the solution of Problem $\mathcal{P}$.

Note that the solution of Problem $\mathcal{Q}_\mathcal{P}$ depends on the structure of the original problem $\mathcal{P}$, cannot be provided in this abstract framework, and requires additional assumptions. Results in solving Problem $\mathcal{Q}_\mathcal{P}$ have been obtained in [7] in the particular case when $\mathcal{P}$ is a variational inequality, in [16] when $\mathcal{P}$ is a minimization problem as well as in [17], in the case when $\mathcal{P}$ is a fixed point problem and an ordinary differential equation in a Banach or Hilbert space.

In this current paper we continue our research in [7,16,17] with the case when $\mathcal{P}$ is an inclusion problem of the form

$$-u \in N_K(Au + f). \tag{1}$$

Here and below in this paper K is a nonempty subset of a real Hilbert space X, N_K represents the outward normal cone of K, $A : X \to X$ is a nonlinear operator and $f \in X$. Our study is motivated by possible applications in Solid and Contact Mechanics, among others. Indeed, a large number of constitutive laws in nonlinear elasticity and plasticity can be cast in the form (1) and so do a number of mathematical models which describes the contact of a deformable body with a foundation. We shall provide such examples in the last two sections of the current paper. Moreover, we refer the reader to [13] as well as to the recent book [14] where inclusions of the form (1) have been considered, together with various applications in Contact Mechanics.

The rest of the manuscrit is structured as follows. In Section 2 we introduce some preliminary material. Then, in Section 3 we state and prove our main result, Theorem 2. It provides necesssary and sufficient conditions which guarantee the convergence of a sequence $\{u_n\}$ to the solution u of the inclusion (1). Next, in Section 4 we apply Theorem 2 in order to deduce the continuous dependence of the solution with respect to the data K, A and f, and to obtain a convergence result for a penalty problem, as well. In Section 5 we use these convergence results in the study of a specific inclusion problem which describes the frictional contact of an elastic body with a foundation. Finally, in Section 6 we provide an application of the abstract results in Section 4 in the study of two elastic constitutive laws. We complete the results in Section 5 and 6 with various mechanical interpretations.

2. Preliminaries

Most of the preliminary results we present here can be found in many books or surveys. For the convenience of the reader we mention here the books [1,3,9,19,20], for instance. There, details on the framework and notation we use as well as additional results in the field can be found.

Everywhere in this paper, unless it is specified otherwise, we use the functional framework described in Introduction. Therefore, X represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and its associated norm $\|\cdot\|_X := \sqrt{(\cdot,\cdot)_X}$. The set of parts of X will be denoted by 2^X and notation 0_X and I_X will represent the zero element and the identity operator of X, respectively. All the limits below are considered as $n \to \infty$, even if we do not mention it explicitely. The symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in various spaces which will be specified, except in the case when these convergence take place in $\mathbb{R}$. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \le \varepsilon_n \to 0$. Finally, we denote by $d(u, K)$ the distance between the element $u \in X$ and the set K, that is

$$d(u, K) = \inf_{v \in K} \|u - v\|_X. \tag{2}$$

For the convenience of the reader we also recall the following definition.

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Definition 1. *Let $\{K_n\}$ be a sequence of nonempty subsets of X and let K be a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K in the sense of Mosco ([11]) and we write $K_n \xrightarrow{M} K$, if the following conditions hold:*

(a) *for each $u \in K$, there exists a sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \mathbb{R}ightarrow u$ in X;*

(b) *for each sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.*

In the study of Problem (1) we consider the following assumptions on the data.

$$K \text{ is a nonemply closed convex subset of } X. \tag{3}$$

$$\begin{cases} A : X \to X \text{ is a strongly monotone and Lipschitz continuous operator,} \\ \text{i.e., there exist } m_A > 0 \text{ and } L_A > 0 \text{ such that} \\ \text{(a)} \qquad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X. \\ \text{(b)} \qquad \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall\, u, v \in X \end{cases} \tag{4}$$

$$f \in X. \tag{5}$$

Recall that in (1) and below $N_K : X \to 2^X$ is the outward normal cone of K in the sense of convex analysis and $P_K : X \to K$ represents the projection operator on K. Then, the following equivalences hold, for all $u, \xi \in X$:

$$\xi \in N_K(u) \iff u \in K, \quad (\xi, v - u)_X \leq 0 \qquad \forall\, v \in K. \tag{6}$$

$$u = P_K\xi \iff u \in K, \quad (\xi - u, v - u)_X \leq 0 \quad \forall\, v \in K. \tag{7}$$

Therefore, using (6) it follows that

$$-u \in N_K(Au + f) \iff Au + f \in K, \quad (Au + f - v, u)_X \leq 0 \quad \forall\, v \in K. \tag{8}$$

This equivalence will be repeatedly used in the rest of the manuscript. Moreover, recall that the projection operator is monotone and nonexpansive, i.e.,

$$(P_K\xi_1 - P_K\xi_2, \xi_1 - \xi_2)_X \geq 0 \qquad \forall\, \xi_1,\, \xi_2 \in X, \tag{9}$$

$$\|P_K\xi_1 - P_K\xi_2\|_X \leq \|\xi_1 - \xi_2\|_X \qquad \forall\, \xi_1,\, \xi_2 \in X. \tag{10}$$

In addition, using assumption (3) we deduce that for each $u \in X$ the following equality holds:

$$d(u, K) = \|u - P_K u\|_X. \tag{11}$$

On the other hand, it is well known that conditions (4) implies that the operator is invertible and, moreover, its inverse $A^{-1}: X \to X$ is a strongly monotone Lipschitz continuous operator with constants $\frac{m_A}{L_A^2}$ and $\frac{1}{m_A}$, respectively. A proof of this result can be found in [15, p. 23], for instance. Therefore, under assumption (4) the following inequalities hold:

$$(A^{-1}u - A^{-1}v, u - v)_X \geq \frac{m_A}{L_A^2} \|u - v\|_X^2 \qquad \forall\, u,\, v \in X, \tag{12}$$

$$\|A^{-1}u - A^{-1}v\|_X \leq \frac{1}{m_A} \|u - v\|_X \qquad \forall\, u,\, v \in X. \tag{13}$$

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The unique solvability of the inclusion (1) is provided by the following existence and uniqueness result.

Theorem 1. *Assume (3)–(5). Then there exists a unique element $u \in X$ such that (1) holds.*

Theorem 1 was proved in [13] by using a fixed point argument. There, various convergence results to the solution of thus inequality have been proved and an example arising in Contact Mechanics has been presended.

We now proceed with the following elementary result which will be used in the next section.

Proposition 1. *Let K be a closed convex nonempty subset of X and let $A = I_X$. Then, for each $f \in X$ the solution of the inclusion (1) is given by*

$$u = P_K f - f. \tag{14}$$

In addition, if K is the ball of radius 1 centered at 0_X, then

$$u = \begin{cases} \left(\frac{1}{\|f\|_X} - 1\right) f & \text{if } \|f\|_X > 1, \\ 0 & \text{if } \|f\|_X \leq 1. \end{cases} \tag{15}$$

Proof. We use the equivalence (6) to see that, in the particular case when $A = I_X$, u is a solution to (1) if and only if

$$u + f \in K, \quad (u + f - v, u)_X \leq 0 \qquad \forall\, v \in K$$

or, equivalently,

$$u + f \in K, \quad ((u + f) - v, (u + f) - f)_X \leq 0 \qquad \forall\, v \in K. \tag{16}$$

We now combine (16) and (7) to see that $u + f = P_K f$ which proves (14).

Assume now that K is the closed ball of radius 1 centered at 0_X, i.e.,

$$K = \Big\{ v \in X \,:\, \|v\|_X \leq 1 \Big\}.$$

Then, using (7) it is easy to see that

$$P_K f = \begin{cases} \frac{f}{\|f\|_X} & \text{if } \|f\|_X > 1, \\ f & \text{if } \|f\|_X \leq 1 \end{cases}$$

and, using (14), we deduce (15). □

Note that the solution of the inclusion (1) depends on the data A, K and f. For this reasons, sometimes below we shall use the notation $u(f)$ or $u(K)$, for instance. This dependence was studied in [14] where the following results have been proved.

Proposition 2. *Assume (3)–(5). Then the solution $u = u(f)$ of inequality (1) depends continuosly on f, i.e., if $u_n = u(f_n)$ denotes the solution of (1) with $f = f_n \in X$, for each $n \in \mathbb{N}$, then*

$$f_n \to f \quad \text{in } X \quad \Longrightarrow \quad u_n \to u \quad \text{in } X. \tag{17}$$

Proposition 3. *Assume (3)–(5). Then the solution $u = u(K)$ of inequality (1) depends continuosly on K, i.e., if for each $n \in \mathbb{N}$, K_n is a non empty closed convex subset of X and $u_n = u(K_n)$ denotes the solution of (1) with $K = K_n$, then*

$$K_n \xrightarrow{M} K \quad \text{in } X \quad \Longrightarrow \quad u_n \to u \quad \text{in } X. \tag{18}$$

Note that Propositions 2 and 3 provide sequences $\{u_n\} \subset X$ which converge to the solution of the inclusion (1). Nevertheless, these proposition do not describe all the sequences which have this property, as it results from the two elementary examples below.

Example 1. *Consider the inclusion* (1) *in the particular case* $X = \mathbb{R}$, $K = [-1, 1]$, $A = I_X$, *and* $f = 0$. *Then, using* (14) *we deduce that the solution of inclusion* (1) *is* $u = P_K f - f = 0$. *Let* $\{u_n\} \subset \mathbb{R}$ *be the sequence given by* $u_n = -\frac{1}{n}$ *for all* $n \in \mathbb{N}$. *Then* $u_n \to u$ *but we cannot find a sequence* $\{f_n\} \subset \mathbb{R}$ *such that* $f_n \to 0$ *and* $u_n = u(f_n)$. *Indeed, assume that* $u_n = u(f_n)$ *and* $f_n \to 0$. *Then* $u_n = P_K f_n - f_n = -\frac{1}{n}$ *and, using the analytic expression of the function* $x \mapsto P_K x - x$, *we deduce that either* $f_n = \frac{1}{n} + 1$ *or* $f_n = \frac{1}{n} - 1$ *which contradicts the assumption* $f_n \to 0$. *It follows from here that the convergence* $u_n \to u$ *above cannot be deduced as a consequence of Proposition* 2.

Example 2. *Keep the same notation as those in Example* 1. *We claim that we cannot find a sequence* $\{K_n\} \subset \mathbb{R}$ *such that* $K_n \xrightarrow{M} K$ *and* u_n *is the solution of the inclusion* (1) *with* K_n *instead of* K. *Indeed, arguing by contradiction, assume that there exists* K_n *such that* $u_n = u(K_n)$ *and* $K_n \xrightarrow{M} K$. *Then* $u_n = P_{K_n} f - f = P_{K_n} 0 = -\frac{1}{n}$. *Therefore,* K_n *is an interval of the form* $(-\infty, -\frac{1}{n}]$ *or* $[a, -\frac{1}{n}]$ *with* $a \in \mathbb{R}$, $a \le -\frac{1}{n}$. *In both cases we arrive to a contradiction, since the Mosco convergence* $K_n \xrightarrow{M} K$ *does not hold. We conclude that the convergence* $u_n \to u$ *above cannot be deduced as a consequence of Proposition* 3.

3. A convergence criterion

In this section we state and prove a convergence criterion for the solution of the inclusion (1). To this end, under the assumption of Theorem 1, we denote by u the solution of the inclusion (1). Moreover, given an arbitrary sequence $\{u_n\} \subset X$, we consider the following statements:

$$u_n \to u \quad \text{in } X. \tag{19}$$

$$\begin{cases} \text{there exists } 0 \le \varepsilon_n \to 0 \text{ such that } d(Au_n + f, K) \le \varepsilon_n \text{ and} \\ (Au_n + f - v, u_n)_X \le \varepsilon_n (\|v\|_X + 1) \quad \forall\, v \in K, \ \forall\, n \in \mathbb{N}. \end{cases} \tag{20}$$

Our main result in this section is the following.

Theorem 2. *Assume* (3)–(5). *Then the statements* (19) *and* (20) *are equivalent.*

Proof. Assume that (19) holds and let $n \in \mathbb{N}$. Then, the regularity $Au + f \in K$ implies that

$$d(Au_n + f, K) \le \|(Au_n + f) - (Au + f)\|_X = \|Au_n - Au\|_X$$

and, using assumption (4)(b), we find that

$$d(Au_n + f, K) \le L_A \|u_n - u\|_X. \tag{21}$$

Consider now an arbitrary element $v \in K$. Then, using the identity

$$\begin{aligned} &(Au_n + f - v, u_n)_X = \\ &\quad = (Au_n - Au, u_n) + (Au + f - v, u_n - u)_X + (Au + f - v, u)_X \end{aligned}$$

as well as inequality

$$(Au + f - v, u)_X \le 0$$

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in (8) we find that

$$(Au_n + f - v, u_n)_X \le (Au_n - Au, u_n)_X + (Au + f - v, u_n - u)_X \tag{22}$$
$$\le \|Au_n - Au\|_X \|u_n\|_X + \|Au + f\|_X \|u_n - u\|_X + \|v\|_X \|u_n - u\|_X.$$

Note also that assumption (19) implies that the sequence $\{u_n\}$ is bounded in X. Therefore, using assumption (4)(b) we deduce that there exists a constant C which does not depeend on n such that

$$\|Au_n - Au\|_X \|u_n\|_X \le C\|u_n - u\|_X. \tag{23}$$

We now combine inequalities (22) and (23) to see that

$$(Au_n + f - v, u_n)_X \le \big(C + \|Au + f\|_X\big)\|u_n - u\|_X + \|v\|_X \|u_n - u\|_X. \tag{24}$$

Denote

$$\varepsilon_n = \max\Big\{(C + \|Au + f\|_X)\|u_n - u\|_X, \|u_n - u\|_X, L_A\|u_n - u\|_X)\Big\} \tag{25}$$

and note that, using assumption (19) it follows that

$$0 \le \varepsilon_n \to 0. \tag{26}$$

Finally, we use (26), (25), (21) and (24) to see that (20) holds.

Conversely, assume that (20) holds. Let $n \in \mathbb{N}$ and denote

$$v_n = P_K(Au_n + f), \qquad w_n = Au_n + f - v_n. \tag{27}$$

Then, we have

$$v_n = Au_n + f - w_n \tag{28}$$

and, using (11), (20), we deduce that

$$w_n \to 0_X, \tag{29}$$
$$(Au_n + f - v, u_n) \le \varepsilon_n(1 + \|v\|_X) \qquad \forall\, v \in K. \tag{30}$$

We now take $v = Au + f$ in (30) and $v = v_n = Au_n + f - w_n$ in (8) to deduce that

$$(Au_n - Au, u_n) \le \varepsilon_n(1 + \|Au + f\|_X), \quad (Au_n - Au, u)_X + (w_n, u)_X \le 0$$

and, adding these inequalities, we find that

$$(Au_n - Au, u_n - u) \le \varepsilon_n(1 + \|Au + f\|_X) - (w_n, u)_X.$$

Next, we use assumption (4)(a) on the operator A to find that

$$m_A\|u_n - u\|_X^2 \le \varepsilon_n(1 + \|Au + f\|_X) + \|w_n\|_X\|u\|_X. \tag{31}$$

Finally, we combine inequality (31) with the convergences $\varepsilon_n \to 0$ and $w_n \to 0_X$, guaranted by (20) and (29). As a result we deduce that $u_n \to u$ in X, which concludes the proof. $\square$

We remark that Theorem 2 provides an answer to Problem $\mathcal{Q}_{\mathcal{P}}$ in the particular case when Problem $\mathcal{P}$ is the inclusion problem (1). Indeed, it provides a convergence criterion to the solution of this problem.

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4. Some applications

Theorem 2 is useful to obtain various convergence results in the study of the inclusion (1). In this section we present two types of such results : results concerning the continuous dependence of the solution with respect to the data and a result concerning the convergence of the solution of a penalty problem.

a) *We start with a continuous dependence result of the solution with respect to the data A and f.* To this end we consider two sequences $\{A_n\}$ and $\{f_n\}$ such that

$$\begin{cases} A_n\colon X \to X \text{ satisfies condition (4) with } m_n > 0 \text{ and } L_n > 0 \\ \text{and, moreover, there exists } a_n \geq 0,\, m_0 > 0 \text{ such that:} \\ \text{(a)}\ \ \|A_n v - Av\|_X \leq a_n(\|v\|_X + 1) \quad \forall\, v \in X,\ n \in \mathbb{N}. \\ \text{(b)}\ \ a_n \to 0 \ \text{ as } \ n \to \infty. \\ \text{(c)}\ \ m_n \geq m_0 \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{32}$$

$$f_n \in X, \qquad f_n \to f \quad \text{in } \ X. \tag{33}$$

Then, using Theorem 1 it follows that for each $n \in \mathbb{N}$ the exists a unique solution to the inclusion problem.

$$-\,u_n \in N_K(A_n u_n + f_n). \tag{34}$$

Moreover, the solution satisfies

$$A_n u_n + f_n \in K, \quad (A_n u_n + f_n - v, u_n)_X \leq 0 \quad \forall\, v \in K. \tag{35}$$

Our first result in this section is the following.

Theorem 3. *Assume (3)–(5) and (32), (33). Then $u_n \to u$ in X.*

Proof. Let $n \in \mathbb{N}$ and $v_0 \in K$ be fixed. We use inequality (35) to write

$$(A_n u_n - A_n 0_X, u_n)_X \leq (v_0 - f_n, u_n)_X - (A_n 0_X, u_n)_X$$

and, using assumption (32)(a),(c) we deduce that

$$\begin{aligned} m_0\|u_n\|_X^2 \leq m_n\|u_n\|_X^2 &\leq \|v_0 - f_n\|_X\|u_n\|_X + \|A_n 0_X\|_X\|u_n\|_X \\ &\leq \|v_0 - f_n\|_X\|u_n\|_X + \|A_n 0_X - A0_X\|_X\|u_n\|_X + \|A0_X\|_X\|u_n\|_X \\ &\leq \|v_0 - f_n\|_X\|u_n\|_X + a_n\|u_n\|_X + \|A0_X\|_X\|u_n\|_X. \end{aligned}$$

It follows from here that

$$\|u_n\|_X \leq \frac{1}{m_0}\Big(\|v_0 - f_n\|_X + a_n + \|A0_X\|_X\Big)$$

and, using assumptions (32)(b), (33) we deduce that there exists $M > 0$ which does not depend on n such that

$$\|u_n\|_X \leq M \qquad \forall\, n \in \mathbb{N}. \tag{36}$$

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Next, we use the regularity $A_n u_n + f_n \in K$ in (35), definition (2) and assumption (32)(a) to see that

$$d(Au_n + f, K) \le \|Au_n + f - A_n u_n - f_n\|_X$$
$$\le \|Au_n - A_n u_n\|_X + \|f - f_n\|_X \le a_n(\|u_n\|_X + 1) + \|f - f_n\|_X$$

and, using the bound (36) we deduce that

$$d(Au_n + f, K) \le a_n(M+1) + \|f - f_n\|_X \qquad \forall\, n \in \mathbb{N}. \tag{37}$$

Consider now an arbitrary element $v \in K$ and let $n \in \mathbb{N}$. Then, using the identity

$$(Au_n + f - v, u_n)_X =$$
$$= (Au_n - A_n u_n + f - f_n, u_n)_X + (A_n u_n + f_n - v, u_n)_X$$

as well as inequality in (35) we find that

$$(Au_n + f - v, u_n)_X \le (Au_n - A_n u_n + f - f_n, u_n)_X$$

and, therefore,

$$(Au_n + f - v, u_n)_X \le \|Au_n - A_n u_n\|_X \|u_n\|_X + \|f_n - f\|_X \|u_n\|_X.$$

We now use assumption (32)(a) and the bound (36) to deduce that

$$(Au_n + f - v, u_n)_X \le a_n(M+1)M + M\|f_n - f\|_X. \tag{38}$$

Denote

$$\varepsilon_n = \max\Big\{a_n(M+1) + \|f_n - f\|_X,\ a_n M(M+1) + M\|f_n - f\|_X\Big\}. \tag{39}$$

and note that, using assumptions (32)(b), (33) it follows that

$$0 \le \varepsilon_n \to 0. \tag{40}$$

Finally, we use (37)–(40) to see that condition (20) is satisfied. We are now in a position to use Theorem 2 to deduce the convergence $u_n \to u$ in X, which concludes the proof. □

b) *We proceed with a result which shows the dependence of the solution with respect to the set of constraints.* To this end we consider two sequences of $\{a_n\} \subset \mathbb{R}$ and $\{b_n\} \subset X$ such that

$$\begin{cases} \text{(a)} \quad a_n \ne 0 \quad \forall\, n \in \mathbb{N}, \quad a_n \to 1 \ \text{ as } \ n \to \infty. \\ \text{(b)} \quad b_n \to 0_X \ \text{ as } \ n \to \infty. \end{cases} \tag{41}$$

Then, we define the set K_n by equality

$$K_n = a_n K + b_n \tag{42}$$

and, using Theorem 1 it follows that for each $n \in \mathbb{N}$ the exists a unique solution u_n to the inclusion problem

$$-u_n \in N_{K_n}(Au_n + f). \tag{43}$$

Moreover, the solution satisfies

$$Au_n + f \in K_n, \quad (Au_n + f - v, u_n)_X \le 0 \quad \forall\, v \in K_n. \tag{44}$$

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Our second result in this section is the following.

Theorem 4. *Assume (3)–(5) and (41), (42). Then $u_n \to u$ in X.*

Proof. We use Theorem 2 and, to this end we check in what follows that condition (20) is satisfied. Let $n \in \mathbb{N}$. Since $Au_n + f \in K_n$ it follows from (42) that there exists $v_n \in K$ such that $Au_n + f = a_n v_n + b_n$ which implies that

$$v_n = \frac{1}{a_n}\,(Au_n + f - b_n). \tag{45}$$

Therefore,

$$d(Au_n + f, K) \le \|Au_n + f - v_n\|_X = \Big\|Au_n + f - \frac{1}{a_n}\,(Au_n + f - b_n)\Big\|_X$$

$$= \Big\|(1 - \frac{1}{a_n})(Au_n + f) + \frac{1}{a_n} b_n\Big\|_X$$

which implies that

$$d(Au_n + f, K) \le \Big|1 - \frac{1}{a_n}\Big|\,\|Au_n + f\|_X + \frac{1}{|a_n|}\,\|b_n\|_X. \tag{46}$$

Now, using (42) and arguments similar to those used in the proof of inequality (36) we find that the sequence $\{u_n\}$ is bounded in X and, therefore, there exists $N > 0$ which does not depend on n such that

$$\|u_n\|_X \le N, \qquad \|Au_n + f\|_X \le N. \tag{47}$$

Thus, it follows from (46) that

$$d(Au_n + f, K) \le N\Big|1 - \frac{1}{a_n}\Big| + \frac{1}{|a_n|}\|b_n\|_X. \tag{48}$$

Assume now that $v \in K$. We write

$$(Au_n + f - v, u_n)_X = (Au_n + f - a_n v - b_n, u_n)_X + ((a_n - 1)v + b_n, u_n)_X \tag{49}$$

and, since $a_n v + b_n \in K_n$, using (44) we deduce that

$$(Au_n + f - a_n v - b_n, u_n)_X \le 0. \tag{50}$$

We now combine (49) and (50) to see that

$$(Au_n + f - v, u_n)_X \le (|a_n - 1|\|v\|_X + \|b_n\|_X)\|u_n\|_X$$

and, using (47) we find that

$$(Au_n + f - v, u_n)_X \le N(|a_n - 1|\|v\|_X + \|b_n\|_X). \tag{51}$$

Denote

$$\varepsilon_n = \max\Big\{N\Big|1 - \frac{1}{a_n}\Big| + \frac{1}{|a_n|}\|b_n\|_X, N|a_n - 1|, N\|b_n\|_X\Big\}. \tag{52}$$

and note that, using assumptions (41), it follows that

$$0 \le \varepsilon_n \to 0. \tag{53}$$

Finally, we use (53), (52), (48) and (51) to see that condition (20) is satisfied. We are now in a position to use Theorem 2 to deduce the convergence $u_n \to u$ in X which concludes the proof. □

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c) *We now present a convergence result concening a penalty method.* To this end we consider a numerical sequence $\{\lambda_n\}$ such that

$$\lambda_n > 0 \quad \forall\, n \in \mathbb{N}, \quad \lambda_n \to 0 \ \text{ as } \ n \to \infty, \tag{54}$$

together with the problem of finding u_n such that

$$u_n \in X, \quad u_n + \frac{1}{\lambda_n}\big(Au_n + f - P_K(Au_n + f)\big) = 0_X. \tag{55}$$

Our third result in this section is the following.

Theorem 5. *Assume (3)–(5) and (54). Then, for each $n \in \mathbb{N}$ equation (55) has a unique solution. Moreover, $u_n \to u$ in X.*

Proof. The proof is structured in several steps, as follows.

Step i) We prove the unique solvability of equation (55). Let $n \in \mathbb{N}$, $u_n \in X$ and denote

$$\sigma_n = Au_n + f. \tag{56}$$

Then, since $A : X \to X$ is invertible, we have

$$u_n = A^{-1}(\sigma_n - f). \tag{57}$$

Using these equalities it is easy to see that u_n is a solution of equation (55) if and only if σ_n is a solution of the equation

$$A^{-1}(\sigma_n - f) + \frac{1}{\lambda_n}(\sigma_n - P_K\sigma_n) = 0_X. \tag{58}$$

Consider now the operator $B_n : X \to X$ defined by

$$B_n\sigma = A^{-1}(\sigma - f) + \frac{1}{\lambda_n}(\sigma - P_K\sigma) \qquad \forall\, \boldsymbol{\sigma} \in X. \tag{59}$$

Then, using the properties (9), (10) and (12), (13) of the operators P_K and A, it is easy to see that the operator B_n is strongly monotone and Lipschitz continuous with constants $\frac{m_A}{L_A^2}$ and $\frac{1}{m_A} + \frac{2}{\lambda_n}$, that is

$$(B_n\sigma_1 - B_n\sigma_2, \sigma_1 - \sigma_2)_X \geq \frac{m_A}{L_A^2}\,\|\sigma_1 - \sigma_2\|_X^2 \qquad \forall\, \sigma_1, \sigma_2 \in X, \tag{60}$$

$$\|B_n\sigma_1 - B_n\sigma_2\|_X \leq \Big(\frac{1}{m_A} + \frac{2}{\lambda_n}\Big)\|\sigma_1 - \sigma_2\|_X \qquad \forall\, \sigma_1, \sigma_2 \in X. \tag{61}$$

Therefore, it is invertible and its inverse, denoted by B_n^{-1}, is defined on X with values in X. We conclude from here that there exists a unique element σ_n such that $B_n\sigma_n = 0_X$. Using now the definition (59) we obtain the unique solvability of the nonlinear equation (58) and, equivalently, the unique solvability of the nonlinear equation (55).

Step ii) We prove the boundedness of the sequences $\{\sigma_n\}$ and $\{u_n\}$. Let $n \in \mathbb{N}$ and let v_0 be a fixed element in K. We use (60) to deduce that

$$\frac{m_A}{L_A^2}\,\|\sigma_n - v_0\|_X^2 \leq (B_n\sigma_n - B_nv_0, \sigma_n - v_0)_X$$

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and, since $B_n\sigma_n = 0_X$, $B_n v_0 = A^{-1}(v_0 - f)$, we find that

$$\frac{m_A}{L_A^2}\,\|\sigma_n - v_0\|_X^2 \le (A^{-1}(v_0 - f), v_0 - \sigma_n)_X \le \|A^{-1}(v_0 - f)\|_X \|\sigma_n - v_0\|_X$$

which proves that the sequence $\{\sigma_n - v_0\}$ is bounded in X. This implies that the sequence $\{\sigma_n\}$ is bounded in X and, using (57) we deduce that $\{u_n\}$ is a bounded sequence in X which concludes the proof of this step.

Step iii) We prove the inequality

$$(Au_n + f - v, u_n)_X \le 0 \qquad \forall\, v \in K,\ n \in \mathbb{N}. \tag{62}$$

Let $n \in \mathbb{N}$ and $v \in K$. We use (56)–(58) and equality $v = P_K v$ to see that

$$\begin{aligned}(Au_n + f - v, u_n)_X &= (\sigma_n - v, A^{-1}(\sigma_n - f))_X \\ &= -\frac{1}{\lambda_n}(\sigma_n - v, \sigma_n - P_K\sigma_n)_X = -\frac{1}{\lambda_n}(\sigma_n - v, (\sigma_n - P_K\sigma_n) - (v - P_K v))_X\end{aligned}$$

which shows that

$$(Au_n + f - v, u_n)_X = -\frac{1}{\lambda_n}\Big[\|\sigma_n - v\|_X^2 - (P_K\sigma_n - P_K v, \sigma_n - v)_X\Big]. \tag{63}$$

Recall that $\lambda_n > 0$ and, moreover, (10) implies that

$$(P_K\sigma_n - P_K v, \sigma_n - v)_X \le \|P_K\sigma_n - P_K v\|_X \|\sigma_n - v\|_X \le \|\sigma_n - v\|_X^2.$$

Therefore, using (63) we deduce that (62) holds.

Step iv) We prove that there exists $M > 0$ such that

$$d(Au_n + f, K)_X \le M\lambda_n \qquad \forall\, n \in \mathbb{N}. \tag{64}$$

Let $n \in \mathbb{N}$. We use (56) and (58) to see that

$$d(Au_n + f, K) = d(\sigma_n, K) = \|\sigma_n - P_K\sigma_n\|_X = \lambda_n \|A^{-1}(\sigma_n - f)\|_X. \tag{65}$$

On the other hand, it follows from the proof of Step ii) that the sequence $\{\sigma_n\}$ is bounded in X. Therefore, using the properties of the operator A^{-1} we deduce that there exists $M > 0$ which does not depend on n such that

$$\|A^{-1}(\sigma_n - f)\|_X \le M \qquad \forall\, n \in \mathbb{N}. \tag{66}$$

Inequality (64) is now a consequence of relations (65) and (66).

Step v) End of proof. We now combine inequalities (62) and (64) with assumptions (54) to see that condition (20) is satisfied with $\varepsilon_n = M\lambda_n$. Finally, we use Theorem 2 to conclude that the convergence $u_n \to u$ in X holds. $\square$

5. An example in Contact Mechanics

In this section we apply the abstract results in Sections 3 and 4 in the variational analysis of a mathematical model which describes the bilateral contact between an elastic body and a foundation. The classical formulation of the problem is the following.

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Problem $\mathcal{M}$. *Find a displacement field $\boldsymbol{u}:\Omega\to\mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}:\Omega\to\mathbb{S}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{67}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{68}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{69}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{70}$$

$$u_\nu = 0 \quad \text{on} \quad \Gamma_3, \tag{71}$$

$$\|\boldsymbol{\sigma}_\tau\| \le g, \quad \boldsymbol{\sigma}_\tau = -g\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{ if } \ \boldsymbol{u}_\tau \neq \boldsymbol{0} \quad \text{on} \quad \Gamma_3. \tag{72}$$

Here $\Omega\subset\mathbb{R}^d$ ($d\in\{2,3\}$) is a domain with smooth boundary Γ divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1)>0$. It represents the reference configuration of the elastic body. Moreover, $\boldsymbol{\nu}$ is the unit outward normal to Γ, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ and, below, we use the notation "$\cdot$", $\|\cdot\|$, $\boldsymbol{0}$ for the inner product, the norm and the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. A generic point in $\Omega\cup\Gamma$ will be denoted by $\boldsymbol{x}=(x_i)$.

We now provide a short description of the equations and boundary conditions in Problem $\mathcal{M}$ and send the reader to [2,5,6,8,10,12] for more details and comments. First, equation (67) represents the constitutive law of the material in which $\mathcal{A}$ is the elasticity operator and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor. Equation (68) is the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces acting on the body. The boundary condition (69) is the displacement condition which models the fact that the body is held fixed on the part Γ_1 on its boundary. Condition (70) is the traction boundary condition. It models the fact that a traction of density $\boldsymbol{f}_2$ is acting on the part Γ_2 of the surface of the body. The boundary conditions (71) and (72) are the interface laws on Γ_3 where the body is assumed to be in contact with an obstacle, the so-called foundation. Here u_ν and $\boldsymbol{u}_\tau$ denote the normal and the tangential displacement, respectively, and $\boldsymbol{\sigma}_\tau$ is the tangential part of the stress vector $\boldsymbol{\sigma\nu}$. Condition (71) shows that the contact is bilateral, i.e., there is no separation between the body and the foundation. Finally, condition (72) represents the Tresca friction law, in which g denotes the friction bound.

In the analysis of Problem $\mathcal{M}$ we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, for an element $\boldsymbol{v}\in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. In addition, recall that $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma\nu}-\sigma_\nu\boldsymbol{\nu}$ with $\sigma_\nu=\boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$.

Next, for the displacement field we need the space V and for the stress and strain fields we need the space Q, defined as follows:

$$V=\{\,\boldsymbol{v}\in H^1(\Omega)^d:\ \boldsymbol{v}=\boldsymbol{0}\ \text{ on }\ \Gamma_1,\ v_\nu=0\ \text{ on }\ \Gamma_3\,\},$$
$$Q=\{\,\boldsymbol{\sigma}=(\sigma_{ij}):\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\quad \forall\, i,j=1,\ldots,d\,\}.$$

The spaces V and Q are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,\qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx. \tag{73}$$

Here and below $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\,(u_{i,j}+u_{j,i}),$$

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where an index that follows a comma denotes the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. The associated norms on these spaces will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption *meas* $(\Gamma_1) > 0$, which allows the use of Korn's inequality. Note also that, by the definition of the inner product in the spaces V and Q, we have

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall\, \boldsymbol{v} \in V. \tag{74}$$

In the study of Problem $\mathcal{M}$ we assume that the operator $\mathcal{A}$ satisfies the following condition.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}) \text{ belongs to } Q.
\end{array}\right. \tag{75}$$

Moreover, the density of body forces and the friction bound are such that

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{76}$$

$$g > 0. \tag{77}$$

Assume now that $(\boldsymbol{u}, \boldsymbol{\sigma})$ represents a couple of regular functions which satisfy (67)–(72). Then, using standard arguments it follows that

$$\begin{aligned}
\boldsymbol{u} \in V, \quad & \int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx + g \int_{\Gamma_3} \|\boldsymbol{v}_\tau\|\, da - g \int_{\Gamma_3} \|\boldsymbol{u}_\tau\|\, da \\
& \ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, da \quad \forall\, \boldsymbol{v} \in V.
\end{aligned} \tag{78}$$

We now introduce the operator $A : Q \to Q$, the functional $j : V \to \mathbb{R}$ the element $\boldsymbol{f} \in V$ and the set K defined by

$$(A\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \mathcal{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx \qquad \forall\, \boldsymbol{\sigma},\ \boldsymbol{\tau} \in Q, \tag{79}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} \|\boldsymbol{v}_\tau\|\, da, \qquad \forall\, \boldsymbol{v} \in V, \tag{80}$$

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in V, \tag{81}$$

$$K = \Big\{ \boldsymbol{\tau} \in Q \,:\, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + gj(\boldsymbol{v}) \ge (\boldsymbol{f}, \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V \Big\}. \tag{82}$$

Then, using (78) and notation (80), (81) we obtain that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + gj(\boldsymbol{v}) - gj(\boldsymbol{u}) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V. \tag{83}$$

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We now test in (83) with $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \boldsymbol{0}_V$ to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + gj(\boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{84}$$

Therefore, using (83) and (84) we find that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + gj(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V.$$

This inequality combined with the definition (82) implies that

$$\boldsymbol{\sigma} \in K. \tag{85}$$

To proceed, we use (82) and (84) to see that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \geq 0 \qquad \forall\, \boldsymbol{\tau} \in K$$

and, using notation $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ for the strain field, we find that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\omega})_Q \geq 0 \qquad \forall\, \boldsymbol{\tau} \in K. \tag{86}$$

On the other hand, the constitutive law (67), definition (79) and equality $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ show that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = (A\boldsymbol{\omega}, \boldsymbol{\tau})_Q \qquad \forall\, \boldsymbol{\tau} \in Q$$

and, therefore,

$$\boldsymbol{\sigma} = A\boldsymbol{\omega}. \tag{87}$$

We now combine (85)–(87) to deduce that

$$A\boldsymbol{\omega} \in K, \qquad (A\boldsymbol{\omega} - \tau, \boldsymbol{\omega})_Q \leq 0 \qquad \forall\, \boldsymbol{\tau} \in K. \tag{88}$$

Finally, inequality (88) and (6) lead to the following variational formulation of Problem $\mathcal{M}$, in terms of the strain field.

Problem $\mathcal{M}^V$. *Find a strain field* $\boldsymbol{\omega} \in Q$ *such that*

$$-\,\boldsymbol{\omega} \in N_K(A\boldsymbol{\omega}). \tag{89}$$

We now consider the sequences $\{\boldsymbol{f}_{0n}\}$, $\{\boldsymbol{f}_{2n}\}$, $\{g_n\}$ such that, for each $n \in \mathbb{N}$, the following hold.

$$\boldsymbol{f}_{0n} \in L^2(\Omega)^d, \qquad \boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d. \tag{90}$$

$$g_n \geq g. \tag{91}$$

$$\boldsymbol{f}_{0n} \to \boldsymbol{f} \quad \text{in } L^2(\Omega)^d, \qquad \boldsymbol{f}_{2n} \to \boldsymbol{f}_2 \quad \text{in } L^2(\Gamma_2)^d. \tag{92}$$

$$g_n \to g. \tag{93}$$

Then, for each $n \in \mathbb{N}$ we consider the element $\boldsymbol{f}_n \in V$ and the set K_n given by

$$(\boldsymbol{f}_n, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0n} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot \boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in V, \tag{94}$$

$$K_n = \Big\{ \boldsymbol{\tau} \in Q \,:\, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + g_n j(\boldsymbol{v}) \geq (\boldsymbol{f}_n, \boldsymbol{v})_V \;\; \forall\, \boldsymbol{v} \in V \Big\}, \tag{95}$$

together with the following problem.

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Problem $\mathcal{M}_n^V$. *Find a strain field* $\boldsymbol{\omega}_n \in Q$ *such that*

$$-\boldsymbol{\omega}_n \in N_{K_n}(\mathcal{A}\boldsymbol{\omega}_n). \tag{96}$$

Our main result in this section is the following.

Theorem 6. *Assume* (75)–(77), (90) *and* (91). *Then, Problem* $\mathcal{M}^V$ *has a unique solution* $\boldsymbol{\omega}$ *and, for each* $n \in \mathbb{N}$, *Problem* $\mathcal{M}_n^V$ *has a unique solution* $\boldsymbol{\omega}_n$. *Moreover, if* (92) *and* (93) *hold, then* $\boldsymbol{\omega}_n \to \boldsymbol{\omega}$ *in* Q.

Proof. For the existence part we use Theorem 1 on the space $X = Q$. First, we note that

$$(\boldsymbol{w}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \forall\, \boldsymbol{w},\, \boldsymbol{v} \in V \tag{97}$$

and, since $gj(\boldsymbol{v}) \geq 0$ for each $\boldsymbol{v} \in V$, using definition (82) we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{f}) \in K$ and, therefore, K is nonempty. On the other hand, it is easy to see that K is a convex subset of Q. We conclude from here that condition (3) is satisfied. In addition, using assumption (75) we see that

$$(\mathcal{A}\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\tau}, \boldsymbol{\sigma} - \boldsymbol{\tau})_Q \geq m_{\mathcal{A}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_Q^2, \qquad \|\mathcal{A}\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\tau}\|_Q \leq L_{\mathcal{A}}\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_Q$$

for all $\boldsymbol{\sigma},\, \boldsymbol{\tau} \in Q$. Therefore, condition (4) holds with $m_A = m_{\mathcal{A}}$ and $L_A = L_{\mathcal{A}}$. We are now in a position to use Theorem 1 with $f = \mathbf{0}_Q$ to deduce the unique solvability of the inclusion (89). The unique solvability of the inclusion and (96) follows from the same argument.

Assume now that the convergences (92) and (93) hold. Then, using the definitions (94) and (81) it is easy to see that $\boldsymbol{f}_n \to \boldsymbol{f}$ in V and, therefore, (74) implies that

$$\boldsymbol{\varepsilon}(\boldsymbol{f}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{f}) \quad \text{in } Q. \tag{98}$$

On the other hand using the definitions (82) and (95) of the sets K and K_n together with equality (97), it is easy to set that the following equivalence holds, for each $n \in \mathbb{N}$:

$$\boldsymbol{\sigma} \in K \quad \Longleftrightarrow \quad \frac{g_n}{g}\,\boldsymbol{\sigma} - \frac{g_n}{g}\,\boldsymbol{\varepsilon}(\boldsymbol{f}) + \boldsymbol{\varepsilon}(\boldsymbol{f}_n) \in K_n.$$

We deduce from here that

$$K_n = a_n K + b_n \quad \text{with} \quad a_n = \frac{g_n}{g} \quad \text{and} \quad b_n = \boldsymbol{\varepsilon}(\boldsymbol{f}_n) - \frac{g_n}{g}\,\boldsymbol{\varepsilon}(\boldsymbol{f}).$$

It follows from here that condition (42) is satisfied. Moreover, the convergences (93) and (98) guarantee that the sequences $\{a_n\}$ and $\{b_n\}$ defined above satisfy conditions (41). The convergence result in Theorem 6 is now a direct consequence of Theorem 4. □

In addition to the mathematical interest in this theorem, it is important from mechanical point of view since it shows that the weak solution of the contact problems $\mathcal{M}$ depends continuously on the density of body forces, the density of traction forces and the friction bound, as well.

6. An application in Solid Mechanics

In this section we provide an example of inclusion in Solid Mechanics for which the results in Theorem 5 work. More precisely, we introduce and analize two nonlinear constitutive laws for elastic materials. To this end, again, we use notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ with $d \in \{1, 2, 3\}$ and recall that the indices i, j, k, l run between 1 and d. Our construction below is based on rheological arguments which can be found in [4], for instance.

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The first constitutive law is obtained by connecting in parallel an elastic rheological element with a rigid-elastic one with constraints. Therefore, we have an additive decomposition of the total stress $\boldsymbol{\sigma} \in \mathbb{S}^d$, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{RC}. \tag{99}$$

Here $\boldsymbol{\sigma}^E$ is the stress in the elastic element and $\boldsymbol{\sigma}^{RP}$ is the stress in the rigid-elastic element with constraints. We denote by $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ the strain tensor and we recall that, since the connexion is in parallel, this tensor is the same in the two rheological components we consider. We also assume that the constitutive law of the elastic element is given by

$$\boldsymbol{\sigma}^E = A\boldsymbol{\varepsilon} \tag{100}$$

in which $A = (A_{ijkl}) : \mathbb{S}^d \to \mathbb{S}^d$ is a fourth order tensor. Moreover we assume that, the constitutive law of the rigid-elastic element is given by

$$\boldsymbol{\varepsilon} \in N_K(\boldsymbol{\sigma}^{RC}) \tag{101}$$

where $K \subset \mathbb{S}^d$ represents the set of constraints and, as usual, N_K represent the outward normal cone to K. Denote by $int\,K$ the interior of K in the topology of $\mathbb{S}^d$. Then, for stress fields $\boldsymbol{\sigma}^{RC}$ such that $\boldsymbol{\sigma}^{RC} \in int\,K$ we have $N_K(\boldsymbol{\sigma}^{RC}) = \mathbf{0}$ and, therefore, equation (101) implies that $\boldsymbol{\varepsilon} = \mathbf{0}$. We conclude that this equation describes a rigid behavior. For stress fields $\boldsymbol{\sigma}^{RC}$ such that $\boldsymbol{\sigma}^{RC} \in K - int\,K$ we can have $\boldsymbol{\varepsilon} \neq \mathbf{0}$ and therefore, (101) describes a nonlinear elastic behaviour. An example of set of constraints is the von Mises convex used in [5,18], for instance. It is given by

$$K = \{\, \boldsymbol{\tau} \in \mathbb{S}^d \ : \|\boldsymbol{\tau}_D\| \leq k \,\} \tag{102}$$

where $\boldsymbol{\tau}_D$ represents the deviatoric part of the tensor $\boldsymbol{\tau}$ and k is a given yield limit.

We now use relations (99)–(101) to write

$$\boldsymbol{\varepsilon} \in N_K(\boldsymbol{\sigma}^{RC}) = N_K(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E) = N_K(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon})$$

and, using notation $\boldsymbol{\omega} = -\boldsymbol{\varepsilon}$ we obtain the following constitutive law:

$$-\,\boldsymbol{\omega} = N_K(A\boldsymbol{\omega} + \boldsymbol{\sigma}) \tag{103}$$

The second consitutive law is obtained by connecting in parallel a linearly elastic rheological element with a rigid-elastic rheological element without constraints. Therefore, we keep the notation $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^E$ and $\boldsymbol{\varepsilon}$ introduced above and we denote by $\boldsymbol{\sigma}^{RE}$ the stress in the rigid-elastic element. We have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{RE}. \tag{104}$$

and we assume now that the constitutive law of the rigid-elastic element is given by

$$\boldsymbol{\varepsilon} = \frac{1}{\lambda}(\boldsymbol{\sigma}^{RE} - P_K\boldsymbol{\sigma}^{RE}). \tag{105}$$

Here, again K represents the domain of rigidity of the material, assumed to be a nonempty closed convex subset of $\mathbb{S}^d$ and, in addition, P_K denotes the projection operator on K and $\lambda > 0$ is a given elastic coefficient. Note that for stress fields $\boldsymbol{\sigma}^{RE}$ such that $\boldsymbol{\sigma}^{RE} \in K$ we have $\boldsymbol{\sigma}^{RE} = P_K\boldsymbol{\sigma}^{RE}$ and, therefore, (105) implies that $\boldsymbol{\varepsilon} = \mathbf{0}$ which shows that this equation describes a rigid behavior. For stress fields $\boldsymbol{\sigma}^{RE}$ such that $\boldsymbol{\sigma}^{RE} \notin K$ we have $\boldsymbol{\varepsilon} \neq \mathbf{0}$.

We now use relations (105), (104) and (100) to write

$$\begin{aligned} \boldsymbol{\varepsilon} &= \frac{1}{\lambda}(\boldsymbol{\sigma}^{RE} - P_K\boldsymbol{\sigma}^{RE}) = \frac{1}{\lambda}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E - P_K(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E)\big) \\ &= \frac{1}{\lambda}\big(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon} - P_K(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon})\big) \end{aligned}$$

and, using notation $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\lambda = -\boldsymbol{\omega}_\lambda$ in order to underline the dependence of the strain field on the coeficient λ, we obtain the following constitutive law:

$$\boldsymbol{\omega}_\lambda + \frac{1}{\lambda}\big(A\boldsymbol{\omega}_\lambda + \boldsymbol{\sigma} - P_K(A\boldsymbol{\omega}_\lambda + \boldsymbol{\sigma})\big) = 0 \tag{106}$$

A brief comparation between the constitutive laws (106) and (103) reveals the fact that (103) is expressed in terms of inclusions and involves unilateral constraints. In contrast, the law (106) is in a form of an equality and does not involves unilateral constraints. For these reasons we say that the (106) is more regular that the constitutive law (103). Consider now the following assumptions.

$$A : \mathbb{S}^d \to \mathbb{S}^d \text{ is a positively symmetric fourth order tensor.} \tag{107}$$

$$K \subset \mathbb{S}^d \text{ is an nonempty closed subset.} \tag{108}$$

Our main result in this section is the following.

Theorem 7. *Assume* (107) *and* (108)*. Then, for every stress tensor* $\boldsymbol{\sigma} \in \mathbb{S}^d$ *there exists a unique solution* $\boldsymbol{\omega} \in \mathbb{S}^d$ *to inclusion* (103) *and, for every* $\boldsymbol{\sigma} \in \mathbb{S}^d$ *and* $\lambda > 0$*, there exists a unique solution* $\boldsymbol{\omega}_\lambda \in \mathbb{S}^d$ *to equation* (106)*. Moreover,* $\boldsymbol{\omega}_\lambda \to \boldsymbol{\omega}$ *in* $\mathbb{S}^d$ *as* $\lambda \to 0$*.*

Theorem 7 is a direct consequence of Theorem 5. In addition to the mathematical interest in this theorem, it is important from mechanical point of view since it shows that:

- a stress field $\boldsymbol{\sigma} \in \mathbb{S}^d$ gives rise to a unique strain field $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ associated with the constitutive law (103);
- a stress field $\boldsymbol{\sigma} \in \mathbb{S}^d$ gives rise to a unique strain field $\boldsymbol{\varepsilon}_\lambda \in \mathbb{S}^d$ associated with the constitutive law (106);
- the constitutive law (103) can be approached by the more regular constitutive law (106) for a small elasticity coefficient λ.

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