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Article

Fekete-Szegő and Zalcman Functional Estimates for Subclasses of Alpha-Convex Functions Related to Trigonometric Functions ‡

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‡ Dedicated to the memory of Professor Miroslav Pavlović (1552–2021)

Abstract: In this study we introduce the new classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$ of α-convex functions associated with sine and cosine functions. Also, we obtain the initial coefficient bounds for the first five coefficients of the functions that belong to these classes. Further, we determine the upper bound of Zalcman functional for the class $\mathcal{M}_\alpha(\cos)$ for the case $n = 3$, showing that the Zalcman conjecture holds for this value. Moreover, the problem of the Fekete-Szegő functional estimate for these classes is studied.

Keywords: Analytic functions; subordination; Carathéodory functions; sine function; cosine function; alpha-convex functions; starlike and convex functions; coefficient bounds; Fekete-Szegő functional; Zalcman functional

MSC: 30C45, 30C50, 30C55

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1. Introduction and Preliminaries

Let $\mathcal{T}$ be the class consists of all analytic and normalized functions f, where f has the form

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots, \; z \in \mathbb{D}, \tag{1}$$

and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc; also, the subclass of $\mathcal{T}$ consisting of univalent functions is denoted by $\mathcal{S}$.

Let us consider two analytic functions P and Q in $\mathbb{D}$. The function P is said to be *subordinated* to Q, written symbolically as $\mathrm{P}(z) \prec \mathrm{Q}(z)$, if there exists an analytic function η in $\mathbb{D}$, with $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$, such that $\mathrm{P} = \mathrm{Q} \circ \eta$. Further, if Q is an univalent function in $\mathbb{D}$, then the following equivalence holds (see [1]):

$$\mathrm{P}(z) \prec \mathrm{Q}(z) \Leftrightarrow \mathrm{P}(0) = \mathrm{Q}(0) \text{ and } \mathrm{P}(\mathbb{D}) \subset \mathrm{Q}(\mathbb{D}).$$

The family of functions p analytic in $\mathbb{D}$ satisfying the condition $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$, and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} t_n z^n, \; z \in \mathbb{D}, \tag{2}$$

is denoted by $\mathcal{P}$, that represents the well-known *Carathéodory function class*.

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In [2] Mocanu introduced and studied the well-known class of *α-convex functions*, that is

$$\mathcal{M}_{\alpha} := \left\{ f \in \mathcal{T} : \operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{D} \right\},\ \alpha \geq 0,$$

and the properties of this class of functions was extensively studied during a long period by many researchers (see, for example [3–5]). In [6] it was proved that all α-convex functions are univalent and starlike, while the subclass $\mathcal{S}^* := \mathcal{M}_0$ is called the class of *starlike (normalized) functions* in $\mathbb{D}$ and $\mathcal{S}^c := \mathcal{M}_1$ represents the class of *convex (normalized) functions* in $\mathbb{D}$.

Definition 1. *Let us now define the new classes $\mathcal{M}_{\alpha}(\sin)$ and $\mathcal{M}_{\alpha}(\cos)$, with $\alpha \geq 0$, connected with the sine and cosine functions, respectively, as follows:*

$$\mathcal{M}_{\alpha}(\sin) := \left\{ f \in \mathcal{T} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + \sin z =: \Phi(z) \right\}, \tag{3}$$

$$\mathcal{M}_{\alpha}(\cos) := \left\{ f \in \mathcal{T} : (1-\alpha)\frac{zf'z}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \cos z =: \Psi(z) \right\}. \tag{4}$$

Remark 1. *(i) Substituting the value of $\alpha = 0$ and $\alpha = 1$ in (3) we obtain the following subclasses which were studied in [7–9], respectively, that are*

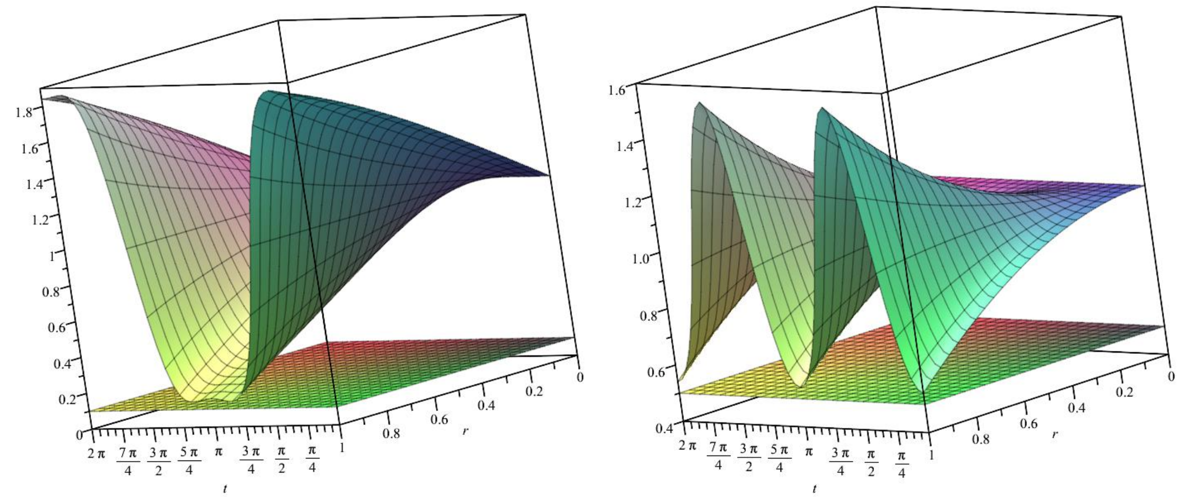
$$\mathcal{S}^*_{\sin} := \mathcal{M}_0(\sin), \quad \mathcal{S}^c_{\sin} := \mathcal{M}_1(\sin).$$

(ii) Taking $\alpha = 0$ in Equation (4) we obtain the subclass $\mathcal{S}^*_{\cos} := \mathcal{M}_0(\cos)$ defined in [10], and by taking $\alpha = 1$ in Equation (4) we obtain the subclass $\mathcal{S}^c_{\cos} := \mathcal{M}_1(\cos)$.

(iii) Since the functions Φ and Ψ defined above have real positive parts in $\mathbb{D}$, and moreover (see the Figure 1A,B made with MAPLE™ computer software)

$$\operatorname{Re}\Phi(z) > \frac{1}{10}, \quad \operatorname{Re}\Psi(z) > \frac{1}{2}, \ z \in \mathbb{D},$$

it follows that the classes $\mathcal{M}_{\alpha}(\sin)$ and $\mathcal{M}_{\alpha}(\cos)$ are subsets of the class $\mathcal{M}_{\alpha}$, that is $\mathcal{M}_{\alpha}(\sin), \mathcal{M}_{\alpha}(\cos) \subset \mathcal{M}_{\alpha} \subset \mathcal{S}^* \subset \mathcal{S}$, $\alpha \geq 0$.



(a) The image of $\operatorname{Re}\Phi\left(re^{it}\right)$, $r \in [0,1]$, $t \in [0, 2\pi]$ **(b)** The image of $\operatorname{Re}\Psi\left(re^{it}\right)$, $r \in [0,1]$, $t \in [0, 2\pi]$

Figure 1. Figures for the Remark 1 (iii).

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The following lemmas are necessary to understand the proofs of our main results.

Lemma 1. *If $p \in \mathcal{P}$ has the form (2), then*

$$|t_n| \leq 2, \text{ for } n \geq 1, \tag{5}$$
$$|t_{i+j} - \mu t_i t_j| \leq 2\max\{1; |1-2\mu|\}, \text{ for } \mu \in \mathbb{C},, \tag{6}$$

and for any complex number ζ we have

$$|t_2 - \zeta t_1^2| \leq 2\max\{1; |2\zeta - 1|\}. \tag{7}$$

The inequality (5) is the well-known Carathéodory's result (see [11,12]), while (6) may be found in [1], and the inequality (7) is from [13] (see also [14], Lemma 2).

Lemma 2 ([7](Lemma 2.2))**.** *If $p \in \mathcal{P}$ has the form (2), then*

$$|\alpha t_1^3 - \beta t_1 t_2 + \gamma t_3| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|. \tag{8}$$

2. Initial Coefficients Estimates for the Classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$

In this section the coefficients of the functions of the classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$ are analysed, and the upper bounds for the first five coefficients is obtained.

Theorem 1. *If $f \in \mathcal{M}_\alpha(\sin)$ has the form (1), then*

$$\begin{aligned}
|a_2| \leq& \frac{1}{1+\alpha}, \\
|a_3| \leq& \frac{1}{2(1+2\alpha)} \max\left\{1; \frac{3\alpha+1}{(1+\alpha)^2}\right\}, \\
|a_4| \leq& \frac{1}{6(1+3\alpha)} \left(\max\left\{1; \frac{6\alpha(\alpha-1)}{(1+2\alpha)(1+\alpha)}\right\} + \frac{-4\alpha^4+31\alpha^3+30\alpha^2+35\alpha+4}{3(1+2\alpha)(1+\alpha)^3}\right), \\
|a_5| \leq& \frac{1}{4(1+4\alpha)}\left[\frac{1}{2}\left(\max\left\{1; \frac{|27\alpha^2-20\alpha+1|}{9\alpha^2+12\alpha+3}\right\}+1\right)\right. \\
&\left. + \frac{|\Phi_1(\alpha)|\,|\Phi_2(\alpha)|}{18(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2} + \mathrm{M}(\alpha)\right].
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1(\alpha) &:= -108\alpha^7 + 200\alpha^6 + 556\alpha^5 + 1042\alpha^4 + 631\alpha^3 + 409\alpha^2 + 145\alpha + 5, \\
\Phi_2(\alpha) &:= -180\alpha^5 - 8\alpha^4 + 69\alpha^3 + 187\alpha^2 + 75\alpha + 1,
\end{aligned}$$

and

$$\mathrm{M}(\alpha) := \frac{8\alpha^2+1}{9(1+2\alpha)^2} \max\left\{1; \left|1 - \frac{\Phi_2(\alpha)}{(1+3\alpha)(1+\alpha)^2(8\alpha^2+1)}\right|\right\}.$$

Proof. If $f \in \mathcal{M}_\alpha(\sin)$, then there exists a function η that is analytic in $\mathbb{D}$ and satisfy the conditions $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$, such that

$$(1-\alpha)\frac{zf'z}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Phi(\eta(z)) = 1 + \sin\eta(z),\ z \in \mathbb{D}.$$

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Since f is of the form (1), it follows that

$$\begin{aligned}(1-\alpha)\frac{zf'z}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=&1+(1+\alpha)a_2z+\left(-3\alpha a_2^2+4\alpha a_3-a_2^2+2a_3\right)z^2\\&+\left(7\alpha a_2^3-15\alpha a_2a_3+a_2^3+9\alpha a_4-3a_2a_3+3a_4\right)z^3\\&+\left(-15\alpha a_2^4+44\alpha a_2^2a_3-a_2^4-28\alpha a_2a_4-16\alpha a_3^2\right.\\&\left.+4a_2^2a_3+16\alpha a_5-4a_2a_4-2a_3^2+4a_5\right)z^4+\ldots,\ z\in\mathbb{D}.\end{aligned} \tag{9}$$

From the fact that $\eta(0)=0$ and $|\eta(z)|<1$ for all $z\in\mathbb{D}$, if we define the function p by

$$p(z):=\frac{1+\eta(z)}{1-\eta(z)}=1+t_1z+t_2z^2+\ldots,\ z\in\mathbb{D},$$

we obtain that $p\in\mathcal{P}$ and

$$\eta(z)=\frac{p(z)-1}{p(z)+1}=\frac{t_1z+t_2z^2+\ldots}{2+t_1z+t_2z^2+\ldots},\ z\in\mathbb{D}.$$

According to the above relation we get

$$\begin{aligned}1+\sin\eta(z)=&1+\frac{1}{2}t_1z+\left(\frac{t_2}{2}-\frac{t_1^2}{4}\right)z^2+\left(\frac{1}{2}t_3-\frac{1}{2}t_1t_2+\frac{5}{48}t_1^3\right)z^3\\&+\left(\frac{5}{16}t_2t_1^2-\frac{1}{32}t_1^4+\frac{1}{2}t_4-\frac{1}{2}t_1t_3-\frac{1}{4}t_2^2\right)z^4+\ldots,\ z\in\mathbb{D},\end{aligned} \tag{10}$$

$$\begin{aligned}\cos\eta(z)=&1-\frac{t_1^2}{8}z^2+\left(-\frac{1}{4}t_1t_2+\frac{1}{8}t_1^3\right)z^3\\&+\left(-\frac{35}{384}t_1^4-\frac{1}{8}t_2^2+\frac{3}{8}t_2t_1^2-\frac{1}{4}t_1t_3\right)z^4+\ldots,\ z\in\mathbb{D},\end{aligned} \tag{11}$$

and equating the corresponding coefficients of (9) and (10) we obtain

$$a_2=\frac{t_1}{2(1+\alpha)}, \tag{12}$$

$$a_3=\frac{2(1+\alpha)^2t_2+\alpha(1-\alpha)t_1^2}{8(1+2\alpha)(1+\alpha)^2}, \tag{13}$$

$$\begin{aligned}a_4=&\frac{5(1+2\alpha)(1+\alpha)^3-6(1+7\alpha)(1+2\alpha)+9(1+5\alpha)(1-\alpha)\alpha}{144(1+3\alpha)(1+2\alpha)(1+\alpha)^3}t_1^3\\&+\frac{6(1+5\alpha)-8(1+2\alpha)(1+\alpha)}{48(1+3\alpha)(1+2\alpha)(1+\alpha)}t_1t_2+\frac{1}{6(1+3\alpha)}t_3,\end{aligned} \tag{14}$$

$$\begin{aligned}a_5=&\frac{1}{4(1+4\alpha)}\left[\frac{\Phi_1(\alpha)}{288(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4}t_1^4-\frac{\Phi_2(\alpha)}{48(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}t_1^2t_2\right.\\&\left.+\frac{2(1+7\alpha)-3(1+3\alpha)(1+\alpha)}{6(1+3\alpha)(1+\alpha)}t_1t_3+\frac{(1+8\alpha)-2(1+2\alpha)^2}{8(1+2\alpha)^2}t_2^2+\frac{1}{2}t_4\right].\end{aligned} \tag{15}$$

Using (12) we get

$$|a_2|=\frac{1}{2(1+\alpha)}|t_1|,$$

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and from (5) we have $|t_1| \leq 2$, hence
$$|a_2| \leq \frac{1}{1+\alpha}.$$

The relation (13) leads to
$$|a_3| = \left| \frac{t_2}{4(1+2\alpha)} - \frac{\alpha(\alpha-1)\,t_1^2}{8(1+2\alpha)(1+\alpha)^2} \right|,$$
using triangle inequality we get
$$|a_3| \leq \frac{1}{4(1+2\alpha)} \left| t_2 - \frac{\alpha(\alpha-1)\,t_1^2}{2(1+\alpha)^2} \right|,$$
and according to (7), since $\alpha \geq 0$, we obtain
$$\begin{aligned} |a_3| = \frac{1}{4(1+2\alpha)} \left| t_2 - \frac{\alpha(\alpha-1)\,t_1^2}{2(1+\alpha)^2} \right| &\leq \frac{1}{4(1+2\alpha)} 2\max\left\{1; \left| 2\cdot\frac{\alpha(\alpha-1)}{2(1+\alpha)^2} - 1 \right| \right\} \\ &= \frac{1}{2(1+2\alpha)} \max\left\{1; \frac{3\alpha+1}{(1+\alpha)^2}\right\}. \end{aligned}$$

The equality (14) leads to
$$\begin{aligned} |a_4| = &\frac{1}{3(1+3\alpha)} \left| \frac{1}{4}\left(t_3 - \frac{4(1+2\alpha)(1+\alpha) - 3(1+5\alpha)}{2(1+2\alpha)(1+\alpha)} t_1 t_2 \right) \right. \\ &\left. + \frac{t_3}{4} - \frac{6(1+7\alpha)(1+2\alpha) - 5(1+2\alpha)(1+\alpha)^3 - 9(1+5\alpha)(1-\alpha)\alpha}{12(1+2\alpha)(1+\alpha)^3} t_1^3 \right|, \end{aligned}$$
and using the triangle inequality we get
$$\begin{aligned} |a_4| \leq &\frac{1}{3(1+3\alpha)} \left[\frac{1}{4} \left| t_3 - \frac{4(1+2\alpha)(1+\alpha) - 3(1+5\alpha)}{2(1+2\alpha)(1+\alpha)} t_1 t_2 \right| \right. \\ &\left. + \frac{|t_3|}{4} + \frac{6(1+7\alpha)(1+2\alpha) - 5(1+2\alpha)(1+\alpha)^3 - 9(1+5\alpha)(1-\alpha)\alpha}{12(1+2\alpha)(1+\alpha)^3} |t_1|^3 \right]. \end{aligned}$$
From (5), (6) and of Lemma 1, the above inequality implies that
$$|a_4| \leq \frac{1}{6(1+3\alpha)} \left(\max\left\{1; \frac{6\alpha(\alpha-1)}{(1+2\alpha)(1+\alpha)}\right\} + \frac{-4\alpha^4 + 31\alpha^3 + 30\alpha^2 + 35\alpha + 4}{3(1+2\alpha)(1+\alpha)^3} \right).$$

By rearranging (15) we get

$$\begin{aligned}|a_5| =& \frac{1}{4(1+4\alpha)}\left|\frac{1}{4}\left(t_4 - \frac{6(1+3\alpha)(1+\alpha)-4(1+7\alpha)}{3(1+3\alpha)(1+\alpha)}t_1t_3\right)\right.\\ &+\frac{1}{4}\left(t_4 - \frac{-7\left((1+8\alpha)-2(1+2\alpha)^2\right)}{18(1+2\alpha)^2}t_2^2\right)\\ &-\frac{5\Phi_2(\alpha)}{144(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}t_1^2t_2+\frac{\Phi_1(\alpha)}{288(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4}t_1^4\\ &\left.-\frac{2(1+2\alpha)^2-(1+8\alpha)}{36(1+2\alpha)^2}t_2\left(t_2-\frac{\Phi_2(\alpha)}{2(1+3\alpha)(1+\alpha)^2\left(2(1+2\alpha)^2-(1+8\alpha)\right)}t_1^2\right)\right|,\end{aligned}$$

and using triangle inequality we get

$$\begin{aligned}|a_5| \le& \frac{1}{4(1+4\alpha)}\left[\frac{1}{4}\left|t_4 - \frac{6(1+3\alpha)(1+\alpha)-4(1+7\alpha)}{3(1+3\alpha)(1+\alpha)}t_1t_3\right|\right.\\ &+\frac{1}{4}\left|t_4 - \frac{7\left(2(1+2\alpha)^2-(1+8\alpha)\right)}{18(1+2\alpha)^2}t_2^2\right|\\ &+\frac{5|\Phi_2(\alpha)|}{144(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}|t_1|^2|t_2|+\frac{|\Phi_1(\alpha)|}{288(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4}|t_1|^4\\ &\left.+\frac{8\alpha^2+1}{36(1+2\alpha)^2}|t_2|\left|t_2-\frac{\Phi_2(\alpha)}{2(1+3\alpha)(1+\alpha)^2(8\alpha^2+1))}t_1^2\right|\right]. \end{aligned} \tag{16}$$

Now we will find an upper bound for the each term of the right hand side of the above inequality, as follows.

(i) According to (6) we have

$$\begin{aligned}\left|t_4 - \frac{6(1+3\alpha)(1+\alpha)-4(1+7\alpha)}{3(1+3\alpha)(1+\alpha)}t_1t_3\right| &\le 2\max\left\{1;\left|1-\frac{12(1+3\alpha)(1+\alpha)-8(1+7\alpha)}{3(1+3\alpha)(1+\alpha)}\right|\right\}\\ &= 2\max\left\{1;\left|\frac{27\alpha^2-20\alpha+1}{9\alpha^2+12\alpha+3}\right|\right\},\end{aligned}$$

whenever $\alpha \ge 0$.

(ii) Using again the inequality (6) a simple computation shows that

$$\begin{aligned}\left|t_4 - \frac{7\left(2(1+2\alpha)^2-(1+8\alpha)\right)}{18(1+2\alpha)^2}t_2^2\right| &\le 2\max\left\{1;\left|1-\frac{7\left(2(1+2\alpha)^2-(1+8\alpha)\right)}{9(1+2\alpha)^2}\right|\right\}\\ &= 2\max\left\{1;\left|\frac{2(10\alpha^2-18\alpha-1)}{9(1+2\alpha)^2}\right|\right\} = 2,\end{aligned}$$

whenever $\alpha \ge 0$.

(iii) For the sum of the third with the fourth term, using the inequality (5) we obtain

$$\begin{aligned}&\frac{5|\Phi_2(\alpha)|}{144(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}|t_1|^2|t_2|+\frac{|\Phi_1(\alpha)|}{288(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4}|t_1|^4\\ &\qquad\le \frac{|\Phi_1(\alpha)|\,|\Phi_2(\alpha)|}{18(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}.\end{aligned}$$

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(v) To get a majorant for the last term of the sum, according to (5) and (7) we have

$$\begin{aligned}
&\frac{8\alpha^2+1}{36\,(1+2\alpha)^2}|t_2|\left|t_2-\frac{\Phi_2(\alpha)}{2\,(1+3\alpha)\,(1+\alpha)^2\left(2\,(1+2\alpha)^2-(1+8\alpha)\right)}t_1^2\right|\Bigg\}\\
&\leq\frac{8\alpha^2+1}{9\,(1+2\alpha)^2}\max\left\{1;\left|1-\frac{\Phi_2(\alpha)}{(1+3\alpha)\,(1+\alpha)^2\,(8\alpha^2+1)}\right|\right\}:=M(\alpha).
\end{aligned}$$

Finally, using the upper bounds found to the items (i)–(v), from the inequality (16) we conclude that

$$\begin{aligned}
|a_5|\leq&\frac{1}{4\,(1+4\alpha)}\left\{\frac{1}{2}\left(\max\left\{1;\frac{|27\alpha^2-20\alpha+1|}{9\alpha^2+12\alpha+3}\right\}+1\right)\right.\\
&\left.+\frac{|\Phi_1(\alpha)|\,|\Phi_2(\alpha)|}{18\,(1+3\alpha)\,(1+2\alpha)^2\,(1+\alpha)^2}+\mathrm{M}(\alpha)\right\}.
\end{aligned}$$

□

Remark 2. *A simple computation shows that the upper bounds obtained in the Theorem 1 could be written in the following forms:*

$$|a_3|\leq\begin{cases}\dfrac{3\alpha+1}{2(1+2\alpha)(1+\alpha)^2}, & \text{if}\quad 0\leq\alpha\leq 1,\\[2mm] \dfrac{1}{2(1+2\alpha)}, & \text{if}\quad \alpha\geq 1,\end{cases}$$

and

$$|a_4|\leq\begin{cases}\dfrac{2\alpha^4+52\alpha^3+57\alpha^2+50\alpha+7}{18\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)^3}, & \text{if}\quad 0\leq\alpha\leq\dfrac{9+\sqrt{97}}{8},\\[2mm] \dfrac{14\alpha^4+49\alpha^3+12\alpha^2+17\alpha+4}{18\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)^3}, & \text{if}\quad \alpha\geq\dfrac{9+\sqrt{97}}{8}.\end{cases}$$

For $\alpha=0$ and $\alpha=1$, Theorem 1 reduces to the following corollary:

Corollary 1. *(i) If $f\in\mathcal{S}^*_{\sin}$ has the form (1), then*

$$|a_2|\leq 1,\quad |a_3|\leq\frac{1}{2},\quad |a_4|\leq\frac{7}{18},\quad |a_5|\leq\frac{25}{72}.$$

(ii) If $f\in\mathcal{S}^c_{\sin}$ has the form (1), then

$$|a_2|\leq\frac{1}{2},\quad |a_3|\leq\frac{1}{6},\quad |a_4|\leq\frac{7}{72},\quad |a_5|\leq\frac{145}{18}.$$

Remark 3. *The upper bounds given by Theorem 1 are not the best possible, excepting those for the first two coefficients.*

(i) Thus, for the case $\alpha=0$, the function

$$\widehat{f}(z):=z\exp\left(\int_0^z\frac{\sin(c\,t)}{t}\,\mathrm{d}t\right)=z+cz^2+\frac{c^2}{2}z^3+\frac{c^3}{9}z^4-\frac{c^4}{72}z^5-\ldots,\ z\in\mathbb{D},\ \textit{with}\ |c|=1,$$

is the solution $\widehat{f} \in \mathcal{T}$ *of the differential equation* $\dfrac{z\widehat{f}'(z)}{\widehat{f}(z)} = 1 + \sin(c\,z)$, $|c| = 1$, *therefore* $\widehat{f} \in \mathcal{S}^*_{\sin} := \mathcal{M}_0(\sin)$. *For* $\widehat{f}$ *we have*

$$|a_2| = 1, \quad |a_3| = \frac{1}{2}, \quad |a_4| = \frac{1}{9} < \frac{7}{18}, \quad |a_5| = \frac{1}{72} < \frac{25}{72},$$

hence the estimations given by Theorem 1 are not sharp for $|a_4|$ *and* $|a_5|$.

(ii) Similarly, for $\alpha = 1$, *the function*

$$\begin{aligned}\widetilde{f}(z) &:= \int_0^z \left[\exp\left(\int_0^x \frac{\sin(c\,t)}{t}\,\mathrm{d}t\right)\right]\mathrm{d}x\\ &= z + \frac{c}{2}z^2 + \frac{c^2}{6}z^3 + \frac{c^3}{36}z^4 - \frac{c^4}{360}z^5 - - \ldots,\ z \in \mathbb{D},\ \textit{with}\ |c| = 1,\end{aligned}$$

is the solution $\widetilde{f} \in \mathcal{T}$ *of the differential equation* $1 + \dfrac{z\widehat{f}''(z)}{\widehat{f}'(z)} = 1 + \sin(c\,z)$, $|c| = 1$, *hence* $\widetilde{f} \in \mathcal{S}^c_{\sin} := \mathcal{M}_1(\sin)$. *For this function*

$$|a_2| = \frac{1}{2}, \quad |a_3| = \frac{1}{6}, \quad |a_4| = \frac{1}{36} < \frac{7}{72}, \quad |a_5| = \frac{1}{360} < \frac{145}{18},$$

thus the estimations of Theorem 1 are not sharp for $|a_4|$ *and* $|a_5|$.

Theorem 2. *If* $f \in \mathcal{M}_\alpha(\cos)$ *has the form* (1), *then*

$$\begin{aligned}&|a_2| = 0, \quad |a_3| \le \frac{1}{4(1+2\alpha)},\\ &|a_4| \le \frac{1}{3\,(1+3\alpha)}, \quad |a_5| \le \frac{1}{8(1+4\alpha)}\left(\frac{3\alpha|1-\alpha|}{(1+2\alpha)^2} + \frac{11}{3}\right).\end{aligned}$$

Proof. If $f \in \mathcal{M}_\alpha(\cos)$, then equating the corresponding coefficients of (9) and (11) we obtain

$$a_2 = 0, \tag{17}$$

$$a_3 = -\frac{t_1^2}{16(1+2\alpha)}, \tag{18}$$

$$a_4 = -\frac{t_1}{12\,(1+3\alpha)}\left(t_2 - \frac{t_1^2}{2}\right), \tag{19}$$

$$a_5 = \frac{4\alpha(1-\alpha)}{512\,(1+4\alpha)\,(1+2\alpha)^2}t_1^4 - \frac{t_1}{8(1+4\alpha)}\left(\frac{1}{2}t_3 - \frac{3}{4}t_1t_2 + \frac{1}{6}t_1^3\right) - \frac{t_2^2}{32\,(1+4\alpha)}. \tag{20}$$

Using (18) we get

$$|a_3| = \frac{1}{16\,(1+2\alpha)}|t_1|^2,$$

and from (5) we have $|t_1| \le 2$, hence

$$|a_3| \le \frac{1}{4\,(1+2\alpha)}.$$

The relation (19) leads to

$$|a_4| = \left|\frac{t_1}{12\,(1+3\alpha)}\left(t_2 - \frac{t_1^2}{2}\right)\right|,$$

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and according to (5) and (7), we obtain

$$|a_4| \leq \frac{1}{3\,(1+3\alpha)}.$$

From the equality (20) we have

$$|a_5| = \left| \frac{4\alpha(1-\alpha)}{512\,(1+4\alpha)\,(1+2\alpha)^2} t_1^4 - \frac{t_1}{8(1+4\alpha)} \left(\frac{1}{2}t_3 - \frac{3}{4}t_1 t_2 + \frac{1}{6}t_1^3 \right) - \frac{t_2^2}{32\,(1+4\alpha)} \right|,$$

and using the triangle inequality we get

$$|a_5| \leq \frac{4\alpha\,|1-\alpha|}{512\,(1+4\alpha)\,(1+2\alpha)^2} |t_1|^4 + \frac{|t_1|}{8(1+4\alpha)} \left| \frac{1}{2}t_3 - \frac{3}{4}t_1 t_2 + \frac{1}{6}t_1^3 \right| + \frac{|t_2|^2}{32\,(1+4\alpha)}.$$

From (5) and Lemma 2 for the appropriate values $\alpha = \frac{1}{6}$, $\beta = \frac{3}{4}$, and $\gamma = \frac{1}{2}$, the above inequality implies that

$$|a_5| \leq \frac{1}{8(1+4\alpha)} \left(\frac{3\alpha|1-\alpha|}{(1+2\alpha)^2} + \frac{11}{3} \right),$$

and all the estimations are proved. □

For $\alpha = 0$ and $\alpha = 1$ the Theorem 2 leads us to the following corollary.

Corollary 2. *(i) If $f \in \mathcal{S}^*_{\cos}$ has the form (1), then*

$$|a_2| = 0, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{3}, \quad |a_5| \leq \frac{11}{24}.$$

(ii) If $f \in \mathcal{S}^c_{\cos}$ has the form (1), then

$$|a_2| = 0, \quad |a_3| \leq \frac{1}{12}, \quad |a_4| \leq \frac{1}{12}, \quad |a_5| \leq \frac{11}{120}.$$

Remark 4. *The estimations given by Theorem 2 are not the best possible, excepting those for the first two coefficients.*

(i) Thus, for $\alpha = 0$ and if $f \in \mathcal{S}^*_{\cos}$, then the inequality $|a_3| \leq \frac{1}{4}$ is sharp and it is attained for the function $f_* \in \mathcal{S}^*_{\cos}$ that satisfies the differential equation $\frac{z f_*'(z)}{f_*(z)} = \cos(c\,z)$, $|c| = 1$, that is

$$f_*(z) := z \exp\left(\int_0^z \frac{\cos(c\,t) - 1}{t}\, \mathrm{d}t \right) = z - \frac{c^2}{4}z^3 + \frac{c^4}{24}z^5 - \frac{47c^6}{8640}z^7 + \ldots,\ z \in \mathbb{D},\ \textit{with } |c| = 1.$$

(ii) Also, for $\alpha = 1$ and if $f \in \mathcal{S}^c_{\cos}$, then the inequality $|a_3| \leq \frac{1}{12}$ is sharp being attained for the function $f_\circ \in \mathcal{S}^*_{\cos}$ that it is the solution of the differential equation $1 + \frac{z f_\circ''(z)}{f_\circ'(z)} = \cos(c\,z)$, $|c| = 1$, and

$$\begin{aligned} f_\circ(z) &:= \int_0^z \left[\exp\left(\int_0^x \frac{\cos t - 1}{t}\, \mathrm{d}t \right) \right] \mathrm{d}x \\ &= z - \frac{c^2}{12}z^3 + \frac{c^4}{120}z^5 - \frac{47c^6}{60480}z^7 + \ldots,\ z \in \mathbb{D},\ \textit{with } |c| = 1. \end{aligned}$$

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3. The Fekete-Szegő Inequality for the Classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$

In this section we determine upper bounds for the Fekete-Szegő functional for the new defined classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$.

Theorem 3. *If $f \in \mathcal{M}_\alpha(\sin)$ has the form (1), then*

$$|a_3 - \rho a_2^2| \leq \frac{1}{2(1+2\alpha)} \max\left\{1; \frac{|2\rho(1+2\alpha) - (1+3\alpha)|}{(1+\alpha)^2}\right\}, \ \rho \in \mathbb{C}.$$

Proof. If $f \in \mathcal{M}_\alpha(\sin)$, then from (12) and (13) we get

$$\begin{aligned} a_3 - \rho a_2^2 =& \frac{1+3\alpha}{2(1+2\alpha)} \frac{t_1^2}{4(1+\alpha)^2} + \frac{1}{4(1+2\alpha)}\left(t_2 - \frac{t_1^2}{2}\right) - \rho \frac{t_1^2}{4(1+\alpha)^2}, \\ =& \frac{1}{4(1+2\alpha)}\left(t_2 - \frac{2\rho(1+2\alpha) - \alpha(1-\alpha)}{2(1+\alpha)^2} t_1^2\right), \end{aligned}$$

and using (7) it follows that

$$\begin{aligned} |a_3 - \rho a_2^2| \leq & \frac{1}{4(1+2\alpha)} 2 \max\left\{1; \left|\frac{2\rho(1+2\alpha) - \alpha(1-\alpha)}{(1+\alpha)^2} - 1\right|\right\}, \\ =& \frac{1}{2(1+2\alpha)} \max\left\{1; \frac{|2\rho(1+2\alpha) - (1+3\alpha)|}{(1+\alpha)^2}\right\}. \end{aligned}$$

□

For $\alpha = 0$ and $\alpha = 1$, the following special are obtained.

Corollary 3. *(i) If $f \in \mathcal{S}^*_{\sin}$, then*

$$|a_3 - \rho a_2^2| \leq \frac{1}{2} \max\{1; |2\rho - 1|\}, \ \rho \in \mathbb{C}.$$

(ii) If $f \in \mathcal{S}^c_{\sin}$, then

$$|a_3 - \rho a_2^2| \leq \frac{1}{6} \max\left\{1; \frac{|3\rho - 2|}{2}\right\}, \ \rho \in \mathbb{C}.$$

Remark 5. *1. According to the Remark 3, the upper bounds given by Theorem 3 are the best possible for $\alpha = 0$ and $\alpha = 1$.*

2. If $f \in \mathcal{M}_\alpha(\cos)$ has the form (1), from (17), (18) and (5) we get

$$\left|a_3 - \rho a_2^2\right| = |a_3| = \frac{|t_1|^2}{16(1+2\alpha)} \leq \frac{1}{4(1+2\alpha)}, \ \rho \in \mathbb{C},$$

hence to find the upper bound of the Fekete-Szegő functional is obvious.

4. The Zalcman Functional Estimate for the Class $\mathcal{M}_\alpha(\cos)$

Zalcman conjectured in 1960 that the coefficients of the functions $f \in \mathcal{S}$ having the form (1) satisfies the inequality

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2, \ n \geq 2.$$

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Further, the equality is obtained only for the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ and its rotations. Like it was shown in [15,16] it implies the Bieberbach conjecture, that is $|a_n| \leq n$, $n \geq 2$. It is noteworthy that for $n = 2$ the above inequality is a well-known consequence of the *Area Theorem* and could be found in [1, Theorem 1.5]. In the recent years the Zalcman functional has been given a special interest by many researchers (see, for example, [17–19]).

In the next result, for $n = 3$ we find the Zalcman functional upper bound for the class $\mathcal{M}_\alpha(\cos)$ that allows us to prove that the Zalcman conjecture is holds in this case.

Theorem 4. *If $f \in \mathcal{M}_\alpha(\cos)$ has the form (1), then*

$$\left|a_3^2 - a_5\right| \leq \frac{170\alpha^2 + 170\alpha + 41}{96\,(1+4\alpha)\,(1+2\alpha)^2}. \tag{21}$$

Proof. For $f \in \mathcal{M}_\alpha(\cos)$, using the equalities (18) and (20) it follows that

$$\begin{aligned} a_3^2 - a_5 =& \frac{70\alpha^2 + 70\alpha + 19}{768\,(2\alpha+1)^2\,(4\alpha+1)} t_1^4 + \frac{1}{8\,(1+4\alpha)}\left(\frac{1}{2}t_1 t_3 + \frac{1}{4}t_2^2 - \frac{3}{4}t_1^2 t_2\right), \\ =& \frac{t_1}{8\,(1+4\alpha)}\left(\frac{70\alpha^2 + 70\alpha + 19}{96\,(2\alpha+1)^2} t_1^3 - \frac{3}{4}t_1 t_2 + \frac{1}{2}t_3\right) + \frac{1}{32\,(1+4\alpha)} t_2^2, \end{aligned}$$

and from the triangle inequality we get

$$|a_3^2 - a_5| \leq \frac{|t_1|}{8\,(1+4\alpha)}\left|\frac{70\alpha^2 + 70\alpha + 19}{96\,(2\alpha+1)^2} t_1^3 - \frac{3}{4}t_1 t_2 + \frac{1}{2}t_3\right| + \frac{1}{32\,(1+4\alpha)} |t_2|^2.$$

Using the inequalities (5) of Lemma 1 and (8) of Lemma 2, the above relation leads easily to (21). □

Since

$$4 - \frac{170\alpha^2 + 170\alpha + 41}{(96 + 384\alpha)\,(1+2\alpha)^2} = \frac{6144\alpha^3 + 7510\alpha^2 + 2902\alpha + 343}{(96 + 384\alpha)\,(1+2\alpha)^2} > 0, \text{ for all } \alpha \geq 0,$$

using the result of the Theorem 4 we deduce that:

Corollary 4. *If $f \in \mathcal{M}_\alpha(\cos)$ has the form (1), then*

$$\left|a_3^2 - a_5\right| \leq 4,$$

therefore the Zalcman conjecture hold for the class $\mathcal{M}_\alpha(\cos)$ if $n = 3$.

5. Conclusions

This paper mainly focuses on finding the upper bounds of the first five coefficients for the classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$ of α-convex functions connected with the sine and cosine function. Also, we obtained the estimate for Fekete-Szegő functional for these classes, we found the upper bound for Zalcman functional for these class $\mathcal{M}_\alpha(\cos)$ for the case $n = 3$, and this allows us to prove that the Zalcman inequality holds for this case.

Like we mentioned in the Remarks 3 and 4 the upper bounds we get for $|a_4|$ and $|a_5|$ for the functions that belong to the classes $\mathcal{M}_\alpha(\sin)$ and $\mathcal{M}_\alpha(\cos)$ are not the best possible, hence the estimation given in Theorem 4 is not sharp. The problem of finding the best bounds of the above mentioned coefficients and functionals for these classes remains an interesting open question.

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