

Review

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Review

An Overview of Kerr Blackhole

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Abstract: In this article, we review about the nature of Kerr Blackhole, Kerr metric and express kerr metric in Boyer-Lindquist co-ordinates.

Keywords: Kerr metric; Boyer-Lindquist co-ordinate; event horizon

1. Kerr Blackhole

The Kerr metric, also known as Kerr geometry, characterizes the space-time surrounding an axially symmetric, rotating, and uncharged ellipsoidal body. It's important to note that any spherically symmetric body, when set into rotation, assumes an ellipsoidal shape, typically becoming oblate at the equator. Consequently, the Kerr metric serves as a comprehensive description of the spacetime geometry surrounding a rotating, uncharged black hole [6].

Despite the differences between the Kerr metric, which describes the geometry around a rotating massive object, and the Schwarzschild metric, which pertains to non-rotating massive objects, there are noteworthy similarities between them. Both metrics can be applied to describe objects known as black holes, characterized by true curvature singularities [4]. Additionally, both metrics exhibit coordinate singularities at their event horizons and involve phenomena such as time dilation and curvature near the event horizon. However, these coordinate singularities can be mitigated by transitioning to appropriate coordinate systems [2].

In 1963, Kerr introduced the solution to Einstein's field equations for a rotating massive object without charge, now referred to as the Kerr metric. This solution accounts for the rotation of a massive, spherically symmetric object with a non-zero angular momentum L along an axis passing through its center [3]. We will adopt the Boyer-Lindquist coordinates, which are expressed in spherical coordinates, effectively capturing the characteristics of a rotating black hole [1]. However, it is important to address the coordinate singularity introduced by Boyer-Lindquist coordinates, which may necessitate the introduction of alternative coordinate systems to resolve such issues [3,5].

$$ds^2 = - \left(1 - \frac{2GMr}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{2aGMr \sin^2 \theta}{\rho} dt d\theta + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) d\phi^2 \quad (1)$$

Where:

$$\Delta(r) = r^2 + a^2 - 2GMr \quad (2)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad (3)$$

M = Mass of the rotating black hole

G = Gravitational constant

$$a = \frac{J}{M}$$

J = Angular momentum

If $J = 0, a \rightarrow 0$ the Kerr metric reduces to Schwarzschild metric.

1.1. Kerr Metric in Boyer-Lindquist co-ordinates

By employing oblate spheroidal or ellipsoidal coordinates as a foundation, we can derive the Kerr metric in Boyer-Lindquist coordinates, taking advantage of the equator's bulging in the equatorial plane of a slowly rotating massive body with mass M .

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \phi \quad (4)$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \phi \quad (5)$$

$$z = r \cos \theta \quad (6)$$

Expressing Minwowski metric in oblate spheroidal co-ordinate

$$x \rightarrow x(r, \theta, \phi), y \rightarrow y(r, \theta, \phi), z \rightarrow z(r, \theta, \phi) \quad (7)$$

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi \quad (8)$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi \quad (9)$$

$$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \phi} d\phi \quad (10)$$

here,

$$\begin{aligned} dx &= \frac{r}{\sqrt{r^2 + a^2}} \sin \theta \cos \phi dr + \sqrt{r^2 + a^2} \cos \theta \cos \phi d\theta - \sqrt{r^2 + a^2} \sin \theta \sin \phi d\phi \\ &= \frac{r}{\sqrt{r^2 + a^2}} \sin \theta \cos \phi dr + \sqrt{r^2 + a^2} (\cos \phi \cos \theta d\theta - \sin \theta \sin \phi d\phi) \end{aligned} \quad (11)$$

Then,

$$\begin{aligned} dx^2 &= \frac{r^2}{\sqrt{r^2 + a^2}} \sin^2 \theta \cos^2 \phi dr^2 + 2r \sin \theta \cos \theta \cos^2 \phi dr d\theta \\ &\quad - 2r \sin^2 \theta \sin \phi \cos \phi dr d\phi + (r^2 + a^2) \cos^2 \theta \cos^2 \phi d\theta^2 - \\ &\quad 2(r^2 + a^2) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi + (r^2 + a^2) \sin^2 \theta \sin^2 \phi d\phi^2 \end{aligned} \quad (12)$$

Again,

$$\begin{aligned} dy &= \frac{r}{\sqrt{r^2 + a^2}} \sin \theta \sin \phi dr + \sqrt{r^2 + a^2} \cos \theta \sin \phi d\theta + \sqrt{r^2 + a^2} \sin \theta \cos \phi d\phi \\ &= \frac{r}{\sqrt{r^2 + a^2}} \sin \theta \sin \phi dr + \sqrt{r^2 + a^2} (\cos \phi \sin \theta d\theta + \sin \theta \cos \phi d\phi) \end{aligned} \quad (13)$$

Then,

$$\begin{aligned} dy^2 &= \frac{r^2}{\sqrt{r^2 + a^2}} \sin^2 \theta \sin^2 \phi dr^2 + 2r \sin \theta \sin \theta \sin^2 \phi dr d\theta \\ &\quad + 2r \sin^2 \theta \sin \phi \cos \phi dr d\phi + (r^2 + a^2) \cos^2 \theta \sin^2 \phi d\theta^2 \\ &\quad + 2(r^2 + a^2) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi + (r^2 + a^2) \sin^2 \theta \cos^2 \phi d\phi^2 \end{aligned} \quad (14)$$

Finally,

$$dz = \cos \theta dr - r \sin \theta d\theta \quad (15)$$

$$dz^2 = \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2 \quad (16)$$

Adding,

$$dx^2 + dy^2 + dz^2 = \frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \quad (17)$$

where,

$$\rho^2 = r^2 + a^2 \cos^2 \theta d\phi^2$$

Adding t-direction in metric.

we get,

$$ds^2 = -c^2 dt^2 \frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \quad (18)$$

$$\begin{aligned} &\approx -A(c^2 dt^2 - 2a \sin^2 \theta c dt d\phi + a^2 \sin^4 \theta d\phi^2) \\ &\quad + \frac{\rho}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 \\ &\quad B(r^2 + a^2)^2 d\phi - 2ac dt(r^2 + a^2) d\phi + a^2 c^2 dt^2 \\ &\approx \frac{-(r^2 + a^2)}{\rho^2} (c dt - a \sin^2 \theta d\phi)^2 + \frac{\rho}{r^2 + a^2} dr \\ &\quad + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2) d\phi - ac dt)^2 \end{aligned} \quad (19)$$

Hence,

$$\begin{aligned} ds^2 &= \frac{-(r^2 + a^2)}{\rho^2} (c dt - a \sin^2 \theta d\phi)^2 + \frac{\rho}{r^2 + a^2} dr \\ &\quad + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2) d\phi - ac dt)^2 \end{aligned} \quad (20)$$

When $a \rightarrow 0$ the co-ordinate system changes to Schwarzschild coordinate.

$$ds_{shz}^2 = - \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (21)$$

Here $c^2 dt^2$ and dr^2 contains M source. So assume

$$-\frac{r^2 + a^2}{\rho^2} \rightarrow \frac{-(r^2 + a^2) + k}{\rho^2}$$

and

$$\frac{\rho^2}{r^2 + a^2} \rightarrow \frac{\rho^2}{r^2 + a^2 + h}$$

So

$$\begin{aligned} ds^2 &= \frac{-(r^2 + a^2) + k}{\rho^2} (c dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{r^2 + a^2 + h} dr + \rho^2 d\theta^2 \\ &\quad + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2) d\phi - ac dt)^2 \end{aligned} \quad (22)$$

$$\begin{aligned} ds^2 &= \frac{-(r^2) + k}{\rho^2} (c dt)^2 + \frac{\rho^2}{r^2 + h} dr + \rho^2 d\theta^2 \\ &\quad + \frac{\sin^2 \theta}{\rho^2} ((r^2) d\phi - ac dt)^2 \end{aligned} \quad (23)$$

applying $\lim_{a \rightarrow 0}$

comparing with ds_{shz}^2

$$k = \frac{2GMr}{c^2} \quad (24)$$

$$h = \frac{-2GMr}{c^2} \quad (25)$$

$$\therefore k = -h \quad (26)$$

$$ds^2 = \frac{-(r^2 + a^2) + \frac{2GMr}{c^2}}{\rho^2} (cdt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{r^2 + a^2 - \frac{2GMr}{c^2}} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2) d\phi - a cdt)^2 \quad (27)$$

Redefining

$$\Delta = r^2 + a^2 - \frac{2GMr}{c^2} \quad (28)$$

Also solving,

$$\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} = 1 - \frac{2GMr}{c^2 \rho^2} \quad (29)$$

Then the metric becomes

$$ds^2 = - \left(1 - \frac{2GMr}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{2aGMr \sin^2 \theta}{\rho} dt d\theta + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) d\phi^2 \quad (30)$$

Rewriting in matrix form

$$g_{\mu\nu} = \begin{bmatrix} - \left(1 - \frac{2GMr}{\rho^2} \right) & 0 & 0 & \frac{2aGMr \sin^2 \theta}{\rho} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{2aGMr \sin^2 \theta}{\rho} & 0 & 0 & \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) \end{bmatrix} \quad (31)$$

1.2. Kerr Coordinates

Boyer-Lindquist coordinates represent the spherical coordinate system associated with a rotating black hole, while Kerr coordinates are the coordinates that trace the trajectory of a 'radial' infalling photon.

$$d\tilde{V} = dt + \frac{r^2 + a^2}{\Delta} dr \quad (32)$$

$$d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr \quad (33)$$

Then

$$ds^2 = \left(1 - \frac{2mr}{\rho^2} \right) d\tilde{V}^2 - 2dr d\tilde{V} - \rho^2 d\theta^2 \quad (34)$$

$$- \rho^{-2} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \sin^2 \theta d\tilde{\phi} dr - \frac{4amr \sin^2 \theta}{\rho^2} d\tilde{\phi} d\tilde{V}$$

with

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad (35)$$

$$\Delta = r^2 + a^2 - 2mr \quad (36)$$

Kerr coordinates, unlike Boyer-Lindquist coordinates, avoid a coordinate singularity at the event horizon, making them suitable for describing the behavior of ingoing freely falling photons in the Kerr metric.

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