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*Article*

# Global Non-Existence of a Coupled Parabolic-Hyperbolic System of Thermoelastic Type with History

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**Abstract:** We consider two abstract systems of parabolic-hyperbolic type that model thermoelastic problems. We study the influence of the physical constants and the initial data on the nonexistence of global solutions that in our framework are produced by the blow-up in finite time of the norm of the solution in the phase space. We employ a differential inequality to find sufficient conditions that produce the blow-up. To that end, we construct a set that is positive invariant for any positive value of the initial energy. As a result we found that the coupling with the parabolic equation stabilizes the system, as well as the damping term in the hyperbolic equation. Moreover, for any pair of positive values $(\xi, \epsilon)$, there exist initial data such that the corresponding solution with initial energy $\xi$ blows-up at a finite time less than $\epsilon$. Our purpose is to improve results previously published in the literature.

**Keywords:** blow up; evolution equations; parabolic-hyperbolic system; thermoelasticity

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## 1. Introduction

Let us first introduce two parabolic-hyperbolic systems related with two particular cases of the abstract framework and are the motivation of our analysis. We begin with a Cauchy problem of a one-dimensional thermoelastic model obeying the Fourier's law of heat flux and the theory due to Gurtin-Pipkin with short memory.

$$
(\mathbf{ThE})_1 \begin{cases} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ \text{find } u(x,t)\in\mathbb{R},\ \theta(x,t)\in\mathbb{R}, \text{ such that} \\ \\ u_{tt} - a u_{xx} - b u_x + \eta\theta_x + \delta u_t = f(t,u), & x\in(0,L),\ \ t>0, \\ c\theta_t - \kappa\theta_{xx} + \eta u_{xt} - h*\theta_{xx} - p u_x - q\theta_x = g(\theta), & x\in(0,L),\ \ t>0, \\ u = u_0,\ u_t = u_1,\ \theta = \theta_0, & x\in(0,L),\ \ t=0, \\ u = \partial_\nu u = 0,\ \ \theta = 0, & x = 0, L,\ \ t>0, \end{cases}
$$

where $u(x,t)$ is the vertical displacement of a rod of length $L$, $\theta(x,t)$ is the difference temperature, $t \geq 0$ is the temporal variable, $a, c, \eta, \kappa, \delta$ are positive numbers, $b, p, q$ are nonnegative constants, $(h*l)(t) = \int_0^t h(t-s)l(s)\,ds$, is the convolution and $h$ is a relaxation function. This problem has been studied in [1–5] without source term in the parabolic equation $g(\theta) = 0$. Here, we shall study this problem with $q = p = b = 0, c = 1$ and an autonomous source term $f$.

The second problem models the dynamics of a extensible plate equation with long thermal memory modeled by a heat flux theory due to Coleman-Gurtin with parameter $\omega \in (0,1)$. This theory has the limit cases: the Fourier's law when $\omega = 1$ and the theory due to Gurtin-Pipkin if $\omega = 0$

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$$
(\mathbf{ThE})_2 \begin{cases} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R},\ s \le 0, \\ \text{find } u(x,t) \in \mathbb{R}^n,\ \theta(x,t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} + \Delta^2 u - \phi(\|\nabla u\|_2^2)\Delta u(t) + \eta\Delta\theta + \delta u_t = f(u), & x \in \Omega,\ t > 0, \\ \theta_t - \omega\Delta\theta - (1-\omega)\int_0^\infty k(t-s)\Delta\theta(s)\, ds - \eta\Delta u_t = g(\theta), & x \in \Omega,\ t > 0, \\ u = u_0,\ u_t = u_1, & x \in \Omega,\ t = 0, \\ \theta = \theta_0, & x \in \Omega,\ t \le 0, \\ u = \partial_\nu u = 0,\ \ \theta = 0, & x \in \partial\Omega,\ t > 0, \end{cases}
$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, with smooth boundary $\partial\Omega$ and corresponding normal vector $\nu$, $k$ is the long memory relaxation kernel, $\eta, \delta$ are like before and

$$
\phi(\|\nabla u\|_2^2) = 1 + \|\nabla u\|_2^{2p}, \qquad p \ge 1.
$$

Similar problems have been studied in [6–8], and in [9,10] where a rotational term is considered.

In the systems introduced above we have two types of memory terms, a short one and a long or infinite memory. We also observe that the coupling terms in these systems, those with the coefficient $\eta$, are of two kinds. In $(\mathbf{ThE})_2$, the corresponding operator has opposite signs in the equations of the system, is symmetric and positive. Meanwhile that in $(\mathbf{ThE})_1$, the coupling operator is antisymmetric and the sign in both equations is the same. In order to handle these examples and some more, we shall work with two abstract systems. The first one is a system with short memory and an antisymmetric coupling operator. The second one is a system with long memory and a coupling positive and symmetric operator. Although more combinations of memory and coupling terms can be studied, their analysis can be performed in a similar way.

Let us consider the following semilinear problems associated with abstract parabolic-hyperbolic systems with short and long memory terms, respectively. The first one models problems with short memory. For every initial data $u_0, u_1, v_0$, find functions $t \mapsto (u(t), v(t)),\ t \ge 0$ , such that the following system holds, for every $t > 0$,

$$
(\mathbf{P})_1 \begin{cases} Pu_{tt}(t) + A_1 u(t) - \eta B v(t) + \delta P u_t(t) = f(u(t)), \\ \\ Pv_t(t) + A_2 v(t) + \int_0^t h(t-\tau) A_2 v(\tau)\, d\tau - \eta B u_t(t) = g(v(t)), \\ \\ u(0) = u_0,\ u_t(0) = u_1,\ v(0) = v_0. \end{cases}
$$

In the second problem, we consider infinite memory as follows. For every initial data $u_0, u_1, v_0(\cdot)$, find functions $t \mapsto (u(t), v(t)),\ t \ge 0$ , such that the following system holds, for every $t > 0$,

$$
(\mathbf{P})_2 \begin{cases} Pu_{tt}(t) + A_1 u(t) - \eta B v(t) + \delta P u_t(t) = f(u(t)), \\ \\ Pv_t(t) + \omega A_2 v(t) + (1-\omega)\int_0^\infty k(s) A_2 v(t-s)\, ds + \eta B u_t(t) = g(v(t)), \\ \\ u(0) = u_0,\ u_t(0) = u_1,\ v(-t) = v_0(-t),\ t \ge 0. \end{cases}
$$

Here, $\eta > 0$, $\delta > 0$ and $0 < \omega < 1$, are constants. The functions $h(t),\ t \ge 0$ and $k(t),\ t \in \mathbb{R}$ are short and long memory relaxation kernels, respectively. The functions $f(u)$ and $g(v)$ are nonlinear source terms. The following operators, defined on Banach spaces, are linear and continuous

$$
P : V_P \to V_P',\ A_j : V_{A_j} \to V_{A_j}', j = 1, 2,\ B : V_B \to V_B'.
$$

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We assume that

$$V_{A_1} \subset V_{A_2} \subset V_B \subset V_P \subset H,$$

are linear subspaces of a Hilbert space $H$ with inner product $(\cdot,\cdot)$, norm $\|\cdot\|$, and $H', V_P', V_{A_j}', j = 1,2, V_B'$, are the corresponding dual spaces. We identify $H = H'$, then

$$H \subset V_P' \subset V_B' \subset V_{A_2}' \subset V_{A_1}'.$$

In terms of the corresponding duality pairs, we have the following bilinear forms

$$\mathcal{P}(u,w) \equiv (Pu,w)_{V_P' \times V_P}, \ u,w \in V_P, \ \ \mathcal{B}(u,w) \equiv (Bu,w)_{V_B' \times V_B}, \ u,w \in V_B,$$
$$\mathcal{A}_j(u,w) \equiv (A_j u,w)_{V_{A_j}' \times V_{A_j}}, \ u,w \in V_{A_j}, j = 1,2.$$

We assume that $P$ and $A_j$, j = 1,2, are positive and symmetric, then we have the corresponding norms for $V_P, V_{A_j}, j = 1,2,$

$$\|u\|_{V_P}^2 \equiv \mathcal{P}(u,u), \ u \in V_P, \quad \|u\|_{V_{A_j}}^2 \equiv \mathcal{A}_j(u,u), \ u \in V_{A_j}, j = 1,2,$$

The following hypotheses are assumed to hold along the paper.

(i) There are constants $c > 0, \ \tilde{c} > 0$, such that

$$(H0) \qquad \|u\|_{V_{A_1}}^2 \geq c\|u\|_{V_P}^2, \ u \in V_{A_1}, \ \ |\mathcal{B}(u,v)| \leq \tilde{c}\|u\|_{V_{A_1}}\|v\|_{V_P}, \ u \in V_{A_1}, \ v \in V_B.$$

For the problem $(\mathbf{P})_\mathbf{1}$, we assume that the operator $B$ is antisymmetric

$$(H0)_1 \qquad \mathcal{B}(u,v) = -\mathcal{B}(v,u), \ u,v \in V_B,$$

in particular

$$\mathcal{B}(v,v) = 0, \ v \in V_B.$$

For the problem $(\mathbf{P})_\mathbf{2}$, we assume that the operator $B$ is symmetric and positive, then we define a norm for $B$

$$(H0)_2 \qquad \|u\|_{V_B}^2 \equiv \mathcal{B}(v,v), \ v \in V_B.$$

(ii) The nonlinear source term $f : V_{A_1} \to H$ is a potential operator with potential $F : V_{A_1} \to \mathbb{R}$, that is, $f(u) = D_u F(u)$. We assume that $f(0) = 0 = g(0)$, and there exists a constant $r > 2$, such that

$$(H1)_1 \qquad (f(u),u) - rF(u) \geq 0, \ u \in V_{A_1},$$

and

$$(H1)_2 \qquad \|u\|_{V_{A_2}}^2 - (g(u),u) \geq 0, \ u \in V_{A_2}.$$

(iii) The relaxation kernel $h \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, satisfies the following conditions

$$(H2)_1 \qquad h(0) > 0, \ l \equiv 1 - \int_0^\infty h(t)\,dt > 0, \ \dot{h}(t) \equiv \frac{d}{dt}h(t) \leq 0, \ t \geq 0,$$

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and $h$ is a positive definitive kernel, that is

$$(H2)_2 \qquad \int_0^t (h*w)(s)w(s)\,ds \geq c_0 \int_0^t |(h*w)(s)|^2\,ds,$$

for every $w \in L^1_{loc}(\mathbb{R}^+, V_{A_2})$ and some constant $c_0 > 0$, where

$$(h*w)(s) \equiv \int_0^s h(s-\tau)w(\tau)\,d\tau \geq 0,$$

is the convolution of $h$ and $w$.

(iv) The long memory relaxation kernel $k \in C^1(\mathbb{R}, \mathbb{R}^+)$, satisfies the following hypotheses

$$(H2)_3 \qquad \xi(t) \equiv -(1-\omega)\dot{k}(t) \geq 0, \quad \dot{\xi}(t) \leq 0,$$

$$(H2)_4 \qquad \xi(t) \to 0, \;\; k(t) \to 0 \quad \text{as} \quad t \to \infty.$$

The phase space, where we study the dynamics of $(\mathbf{P})_1$, is

$$\mathcal{H}_1 \equiv V_{A_1} \times V_P \times V_P,$$

with corresponding square norm

$$\|(u,w,v)\|^2_{\mathcal{H}_1} \equiv \|u\|^2_{V_{A_1}} + \|w\|^2_{V_P} + \eta\|v\|^2_{V_P}.$$

For the problem $(\mathbf{P})_2$, we introduce the following new memory function

$$\rho(t,s) \equiv \int_0^s v(t-\tau)\,d\tau = \int_{t-s}^t v(y)\,dy,$$

for every $t \geq 0,\; s \geq 0$, and hence

$$\rho_t(t,s) + \rho_s(t,s) = v(t),$$

with

$$\rho(t,0) = 0, \;\; \rho(0,s) = \rho_0(s) \equiv \int_{-s}^0 v_0(y)\,dy.$$

With respect to this new function, we define the following space

$$\mathcal{M}_{V_{A_2}} \equiv L^2_{\xi}(\mathbb{R}^+, V_{A_2}) \equiv \left\{ w : \mathbb{R}^+ \to V_{A_2}, \; \left| \int_0^\infty \xi(s)\|w(s)\|^2_{V_{A_2}}\,ds \right| < \infty \right\},$$

with norm

$$\|w\|^2_{\mathcal{M}_{V_{A_2}}} \equiv \int_0^\infty \xi(s)\|w(s)\|^2_{V_{A_2}}\,ds.$$

We notice that, by an integration by parts

$$(1-\omega)\int_0^\infty k(s)A_2 v(t-s)\,ds = -(1-\omega)\int_0^\infty \dot{k}(s)A_2\rho(t,s)\,ds.$$

Then, the problem $(\mathbf{P})_2$ is equivalent to the following.

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For every initial data $u_0, u_1, v_0, \rho_0$, find functions $t \mapsto (u(t), v(t), \rho(t),\ t \geq 0$ , such that the following system holds, for every $t > 0$,

$$
(\mathbf{P})_{\mathbf{2}}^{*}
\begin{cases}
Pu_{tt}(t) + A_1 u(t) - \eta B v(t) + \delta P u_t(t) = f(u(t)), \\
Pv_t + \omega A_2 v(t) + \int_0^{\infty} \xi(s) A_2 \rho(t,s)\, ds + \eta B u_t(t) = g(v(t)), \\
\rho_t(t,s) + \rho_s(t,s) = v(t), \\
u(0) = u_0,\ u_t(0) = u_1,\ v(0) = v_0,\ \rho(0,\cdot) = \rho_0(\cdot)\ t \geq 0, \\
\text{such that} \quad \dot{\rho}_0(0) = v_0.
\end{cases}
$$

The phase space, where we study the dynamics of $(\mathbf{P})_{\mathbf{2}}^{*}$, is

$$\mathcal{H}_2 \equiv V_{A_1} \times V_P \times V_P \times \mathcal{M}_{V_{A_2}},$$

with corresponding square norm

$$\|(u, w, v, \rho)\|^2_{\mathcal{H}_2} \equiv \|u\|^2_{V_{A_1}} + \|w\|^2_{V_P} + \eta\|v\|^2_{V_P} + \eta\|\rho\|^2_{\mathcal{M}_{V_{A_2}}}.$$

The concavity argument, introduced by Professor Howard Levine [11,12] is one of the methods to study nonexistence of global solutions of evolution equations due to blow-up and has been generalized by means of the analysis of differential inequalities. See the book [13] and references therein, for an account of several methods to study blow-up in equations of mathematical physics. Here, we shall apply the differential inequality recently studied in [14].

Some authors have considered other mathematical models of thermoelasticity. For instance, in [15,16] chemical potentials are included. Time fractional parabolic-hyperbolic and time fractional hyperbolic thermo-elasticity equations are studied in [17]. Other nonlinearities like p-Laplacian and fractional powers of operators are worked in [18–22]. Equations with delay terms are studied in [23,24]. Similar parabolic-hyperbolic systems to the ones presented in the introduction have been analyzed in [25,26]. Thermoelasticity system in n dimensions with short memory are studied in [27,28] and are commented in last section.

The analysis of problem $(\mathbf{P})_{\mathbf{1}}$ will be done for weak solutions in the following sense,

**Definition 1.1.** For every initial data

$$(u_0, u_1, v_0) \in \mathcal{H}_1,$$

the map, for $t > 0$,

$$(u_0, u_1, v_0) \mapsto (u(t), \dot{u}(t), v(t)) \in \mathcal{D}_1 \equiv V_{A_1} \times V_B \times V_{A_2} \subset \mathcal{H}_1,$$

is a weak local solution of problem $(\mathbf{P})_{\mathbf{1}}$, if there exists some $T > 0$, such that

$$(u, \dot{u}, v) \in C([0, T]; \mathcal{H}_1)$$

with

$$\dot{u} \in L^2([0, T]; V_P),\ v \in L^2([0, T]; V_{A_2}),$$
$$u(0) = u_0,\ \dot{u}(0) = u_1,\ v(0) = v_0,$$

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and

$$\frac{d}{dt}\mathcal{P}(\dot{u}(t),w)+\mathcal{A}_1(u(t),w)-\eta\mathcal{B}(v(t),w)+\delta\mathcal{P}(\dot{u}(t),w)=(f(u(t)),w),$$

$$\frac{d}{dt}\mathcal{P}(v(t),\tilde{w})+\mathcal{A}_2(v(t),\tilde{w})+\int_0^t h(t-\tau)\mathcal{A}_2(v(\tau),\tilde{w})\,d\tau-\eta\mathcal{B}(\dot{u}(t),\tilde{w})$$
$$=(g(v(t)),\tilde{w}),$$

a. e. $t\in(0,T)$, for every $w\in V_{A_1}$, $\tilde{w}\in V_{A_2}$.

We shall consider that the solution in this sense is unique and satisfies the following energy equation for $T>t\geq t_0\geq 0$,

$$E(u(t),\dot{u}(t),v(t))-E(u(t_0),\dot{u}(t_0),v(t_0))=-\delta\int_{t_0}^t\|\dot{u}(s)\|^2_{V_P}\,ds$$
$$-\eta\int_{t_0}^t\left(\|v(s)\|^2_{V_{A_2}}-(g(v(s)),v(s))\right)\,ds-\eta\int_{t_0}^t\mathcal{A}_2((h*v)(s),v(s))\,ds,$$

where

$$E(t)\equiv E(u(t),\dot{u}(t),v(t))\equiv\frac{1}{2}\|\dot{u}(t)\|^2_{V_P}+J(u(t),v(t)),$$
$$J(u(t),v(t))\equiv\frac{1}{2}\left(\|u(t)\|^2_{V_{A_1}}+\eta\|v(t)\|^2_{V_P}\right)-F(u(t)).$$

Due to $(H1)_2$ and $(H2)_2$,

$$E(u(t),\dot{u}(t),v(t))-E(u(t_0),\dot{u}(t_0),v(t_0))\leq-\delta\int_{t_0}^t\|\dot{u}(s)\|^2_{V_P}\,ds$$
$$-\eta\int_{t_0}^t\left(\|v(s)\|^2_{V_{A_2}}-(g(v(s)),v(s))+c_0\|(h*v)(s)\|^2_{A_2}\right)\,ds\leq 0.$$

Then,

$$E(t)=\frac{1}{2}\|(u(t),\dot{u}(t),v(t))\|^2_{\mathcal{H}_1}-F(u(t))\leq E_0$$

where

$$E_0\equiv E(0)=\frac{1}{2}\|(u_0,u_1,v_0)\|^2_{\mathcal{H}_1}-F(u_0).$$

Furthermore, if the maximal time of existence $T_{MAX}<\infty$, then

$$\lim_{t\to T_{MAX}}\|(u(t),\dot{u}(t),v(t))\|_{\mathcal{H}_1}=\infty,$$

consequently,

$$\lim_{t\to T_{MAX}}F(u(t))=\infty.$$

The analysis of problem $(\mathbf{P})_2^*$ will be done for weak solutions in the following sense,

**Definition 1.2.** For every initial data

$$(u_0,u_1,v_0,\rho_0)\in\mathcal{H}_2,$$

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the map, for $t > 0$,

$$(u_0, u_1, v_0, \rho) \mapsto (u(t), \dot{u}(t), v(t), \rho(t) \in \mathcal{D}_2 \equiv V_{A_1} \times V_B \times V_{A_2} \times \mathcal{M}_{V_{A_2}} \subset \mathcal{H}_2,$$

where $\rho(t) \equiv \rho(t, \cdot)$, is a weak local solution of problem $(\mathbf{P})_2^*$, if there exists some $T > 0$, such that

$$(u, \dot{u}, v, \rho) \in C([0, T]; \mathcal{H}_2)$$

with

$$\dot{u} \in L^2([0, T]; V_P), \ v \in L^2([0, T]; V_{A_2}),$$
$$u(0) = u_0, \ \dot{u}(0) = u_1, \ v(0) = v_0, \ \dot{\rho}_0(0) = v_0,$$

and

$$\begin{aligned}
&\frac{d}{dt}\mathcal{P}(\dot{u}(t), w) + \mathcal{A}_1(u(t), w) - \eta\mathcal{B}(v(t), w) + \delta\mathcal{P}(\dot{u}(t), w) = (f(u(t)), w), \\
&\frac{d}{dt}\mathcal{P}(v(t), \tilde{w}) + \omega\mathcal{A}_2(v(t), \tilde{w}) + \int_0^\infty \xi(s)\mathcal{A}_2(\rho(t,s), \tilde{w}) \, ds + \eta\mathcal{B}(\dot{u}(t), \tilde{w}) \\
&\qquad\qquad = (g(v(t)), \tilde{w}), \\
&\mathcal{A}_2(\rho_t(t,s), \tilde{w}) + \mathcal{A}_2(\rho_s(t,s), \tilde{w}) = \mathcal{A}_2(v(t), \tilde{w}),
\end{aligned}$$

a. e. $t \in (0, T)$, $s \geq 0$, for every $w \in V_{A_1}$, $\tilde{w} \in V_{A_2}$.

We shall consider that the solution in this sense is unique and satisfies the following energy equation for $T > t \geq t_0 \geq 0$,

$$\begin{aligned}
&E(u(t), \dot{u}(t), v(t), \rho(t)) - E(u(t_0), \dot{u}(t_0), v(t_0), \rho(t_0)) = -\delta \int_{t_0}^t \|\dot{u}(\tau)\|_{V_P}^2 \, d\tau \\
&- \eta \int_{t_0}^t \left(\omega\|v(\tau)\|_{V_{A_2}}^2 - (g(v(\tau)), v(\tau))\right) \, d\tau + \frac{\eta}{2} \int_{t_0}^t \int_0^\infty \dot{\xi}(s)\|\rho(\tau, s)\|_{\mathcal{A}_2}^2 \, ds \, d\tau,
\end{aligned}$$

where

$$E(t) \equiv E(u(t), \dot{u}(t), v(t)\rho(t)) \equiv \frac{1}{2}\|\dot{u}(t)\|_{V_P}^2 + J(u(t), v(t), \rho(t)),$$
$$J(u(t), v(t), \rho(t)) \equiv \frac{1}{2}\left(\|u(t)\|_{V_{A_1}}^2 + \eta\|v(t)\|_{V_P}^2 + \eta\|\rho(t)\|_{\mathcal{M}_{V_{A_2}}}^2\right) - F(u(t)).$$

Indeed, from last definition

$$\begin{aligned}
\frac{d}{dt}E(t) = &-\delta \int_{t_0}^t \|\dot{u}(s)\|_{V_P}^2 \, ds - \eta \int_{t_0}^t \left(\omega\|v(s)\|_{V_{A_2}}^2 - (g(v(s)), v(s))\right) \, ds \\
&-\eta \int_0^\infty \xi(s)\mathcal{A}_2(\rho(t,s), v(t)) \, ds + \frac{\eta}{2}\frac{d}{dt} \int_0^\infty \xi(s)\|\rho(t,s)\|_{V_{A_2}}^2 \, ds.
\end{aligned}$$

But,

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$$
\begin{aligned}
&-\int_0^\infty \xi(s)\mathcal{A}_2(\rho(t,s),v(t))\,ds + \frac{1}{2}\frac{d}{dt}\int_0^\infty \xi(s)\|\rho(t,s)\|^2_{V_{A_2}}\,ds\\
&=\int_0^\infty \xi(s)\mathcal{A}_2(\rho(t,s),\rho_t(t,s)-v(t))\,ds = -\int_0^\infty \xi(s)\mathcal{A}_2(\rho(t,s),\rho_s(t,s))\,ds\\
&= -\frac{1}{2}\int_0^\infty \xi(s)\frac{\partial}{\partial s}\|\rho(t,s)\|^2_{V_{A_2}}\,ds = \frac{1}{2}\int_0^\infty \dot{\xi}(s)\|\rho(t,s)\|^2_{V_{A_2}}\,ds
\end{aligned}
$$

Due to $(H1)_2$ and $(H2)_3$,

$$E(t) - E(t_0) \le 0.$$

Then,

$$E(t) = \frac{1}{2}\|(u(t),\dot{u}(t),v(t),\rho(t))\|^2_{\mathcal{H}_2} - F(u(t)) \le E_0$$

where

$$E_0 \equiv E(0) = \frac{1}{2}\|(u_0,u_1,v_0,\rho_0)\|^2_{\mathcal{H}_2} - F(u_0).$$

Furthermore, if the maximal time of existence $T_{MAX} < \infty$, then

$$\lim_{t\to T_{MAX}} \|(u(t),\dot{u}(t),v(t),\rho(t)\|_{\mathcal{H}_2} = \infty,$$

consequently,

$$\lim_{t\to T_{MAX}} F(u(t)) = \infty.$$

## 2. Main result

In this section we shall analyze the nonexistence of global solutions for both problems introduced in the introduction and any positive value of the initial energy. To this end, we define the following constants

$$
\left.
\begin{aligned}
&\alpha \equiv \tfrac{1}{4}(r-2), \qquad \gamma \equiv 2rE_0,\\
&\beta \equiv c\left(r-2-\tfrac{1}{2}\tilde{c}\eta\right),
\end{aligned}
\right\} \tag{2.1}
$$

We assume that

$$(H3) \qquad r \ge \frac{1}{2}\tilde{c}, \quad r > 2 + \frac{1}{2}\tilde{c}\eta, \quad \text{then} \quad \alpha > 0, \quad \beta > 0.$$

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Also, we define the functions

$$\left.\begin{aligned}
&\psi(t) \equiv \|u(t)\|^2_{V_P},\\
&\phi(t) \equiv \left( \frac{\frac{d}{dt}\psi(t)}{\psi^{\frac{1}{2}}(t)} - \frac{\delta}{\alpha}\psi^{\frac{1}{2}}(t) \right)^2 + \frac{\beta}{\alpha}\psi(t),\\
&\sigma_\nu(t) \equiv \frac{1+2\alpha}{2}\left(\phi(t) - \frac{\beta\nu}{\alpha}\psi(t)\right),\\
&\mu_\lambda(t) \equiv \frac{1+2\alpha}{2}\left(\phi(t) - \frac{\beta}{\alpha(1+2\alpha)}\psi(t)\left(\frac{\lambda\beta\psi(t)}{\alpha\phi(t)}\right)^{2\alpha}\right),
\end{aligned}\right\} \tag{2.2}$$

for $t \geq 0,\ \nu > 0, \lambda \in (0,1)$, and

$$\psi_0 \equiv \psi(0),\ \ \phi_0 \equiv \phi(0) = \left( \frac{\dot{\psi}_0}{\psi_0^{\frac{1}{2}}} - \frac{\delta}{\alpha}\psi_0^{\frac{1}{2}} \right)^2 + \frac{\beta}{\alpha}\psi_0,\ \ \dot{\psi}_0 \equiv \frac{d}{dt}\psi(0). \tag{2.3}$$

**Theorem 2.1.** *Consider any solution either from problem $(\mathbf{P})_\mathbf{1}$ or problem $(\mathbf{P})^*_\mathbf{2}$, in the sense of Definitions 1.1, 1.2, respectively. Assume that hypotheses $(H0)-(H3)$ hold. If*

$$\dot{\psi}_0 > \frac{\delta}{\alpha}\psi_0 > 0, \tag{2.4}$$

*is satisfied, then there exists a nonempty interval*

$$\mathcal{I} \equiv (\mathbf{a}, \mathbf{b}) \subset \left(0, \frac{1+2\alpha}{2}\phi_0\right),$$

*with the following consequences:*

*(i)* *If $\gamma = 2rE_0 \in \mathcal{I}$, then $\psi(t)$ blows-up at a finite time $t^* > 0$, that is*

$$\lim_{t \to t^*} \psi(t) = \infty.$$

*Hence, the corresponding solution of anyone of the two problems is not global.*

*(ii)* $\mathbf{a} = \sigma_\nu(0)$ *and* $\mathbf{b} = \mu_\lambda(0)$, *moreover*

$$\mathbf{a} = \frac{\beta\psi_0}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}} < \frac{\beta\psi_0}{(1+2\alpha)^{\frac{1}{\alpha}}},$$

$$\mathbf{b} = \frac{\alpha\phi_0}{\lambda^*} > \frac{1+2\alpha}{2}\phi_0 - \left(\frac{1+2\alpha}{2\alpha} - \chi(\lambda^*)\right)\beta\psi_0 > \frac{1+2\alpha}{2}\phi_0 - \frac{\beta}{2\alpha}\psi_0,$$

*for some $\frac{2\alpha}{1+2\alpha} < \lambda^* < 1$ and $\nu^* > 1+2\alpha$, where $1 < \chi(\lambda^*) < \frac{1+2\alpha}{2\alpha}$ is a function of $\lambda^*$.*

*(iii)* *For fixed $\psi_0, \dot{\psi}_0$,*

$$\delta \mapsto t^*,\ \textit{is strictly increasing, and}$$
$$\delta \mapsto |\mathcal{I}| = \mathbf{b} - \mathbf{a},\ \textit{is strictly decreasing.}$$

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*For fixed $\psi_0, \delta$,*

$$\dot{\psi}_0 \mapsto t^*, \text{ is strictly decreasing, and}$$

*and*

$$\dot{\psi}_0 \mapsto |\mathcal{I}|, \text{ is strictly increasing.}$$

*We have the bounds*

$$0 < \frac{1+2\alpha}{2}\phi_0 - |\mathcal{I}| < \left(\frac{1+2\alpha}{2\alpha} - \chi(\lambda^*) + \frac{1}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}}\right)\beta\psi_0,$$

$$t^* \geq \left(\alpha\frac{\dot{\psi}_0}{\psi_0} - \delta\right)^{-1}.$$

*(iv) Furthermore, for $\psi_0$ fixed, we have the limit values as $\dot{\psi}_0 \to \infty$,*

$$\mathbf{a} \to 0, \qquad \left|\mathbf{b} - \frac{1+2\alpha}{2}\phi_0\right| \to 0, \qquad t^* \to 0,$$

$$\nu^* \to \infty, \qquad \lambda^* \to \frac{2\alpha}{1+2\alpha}, \qquad \chi(\lambda^*) \to \frac{1+2\alpha}{2\alpha}.$$

**Corollary 2.2.** *Consider any solution either from problem $(\mathbf{P})_1$ or problem $(\mathbf{P})_2^*$, in the sense of Definitions 1.1, 1.2, respectively. Assume that hypotheses of Theorem 2.1 are met. Given any numbers $\xi > 0, \epsilon > 0$, we can choose initial data with $\mathcal{P}(u_0, u_1)$ large enough, so that the conclusions of Theorem 2.1 are satisfied for initial energy with $E_0 = \xi$ at a blow-up time $t^* < \epsilon$.*

For the proof we will employ the following definitions and remarks.
First, we define the following orthogonal decomposition of the velocity

$$\dot{u} = \frac{\mathcal{P}(\dot{u}, u)}{\|u\|_{V_P}^2}u + h, \quad \mathcal{P}(u, h) = 0,$$
$$\|\dot{u}\|_{V_P}^2 = \|h\|_{V_P}^2 + \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{V_P}^2} \geq \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{V_P}^2} \equiv \mathcal{Q}(\dot{u}, u). \tag{2.5}$$

Second, since the conditions on the initial data that produce the nonexistence of global solution in both problems are only on $u_0, u_1$, we define the auxiliary space

$$\tilde{\mathcal{H}} \equiv V_{A_1} \times V_P,$$

then the phase spaces for the problems, in the sense of Definitions 1.1, 1.2 become, respectively

$$\mathcal{H}_1 = \tilde{\mathcal{H}} \times V_P, \quad \mathcal{H}_2 = \tilde{\mathcal{H}} \times V_P \times \mathcal{M}_{V_{A_2}}.$$

Third, we define the concept of a positive invariant set with respect to any solution either from problem $(\mathbf{P})_1$ or problem $(\mathbf{P})_2^*$, in the sense of Definitions 1.1, 1.2, respectively. Indeed, $\mathcal{W}_1 \in \mathcal{H}_1$ is a

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positive invariant set, along $(u, \dot{u}, v)$, respectively $\mathcal{W}_2 \in \mathcal{H}_2$ is a positive invariant set, along $(u, \dot{u}, v, \rho)$, if

$$(u_0, u_1, v_0) \equiv (u(0), \dot{u}(0), v(0)) \in \mathcal{W}_1$$
$$\Rightarrow (u(t), \dot{u}(t), v(t)) \in \mathcal{W}_1, \text{ for any } t > 0,$$

respectively

$$(u_0, u_1, v_0, \rho_0) \equiv (u(0), \dot{u}(0), v(0), \rho(0)) \in \mathcal{W}_2$$
$$\Rightarrow (u(t), \dot{u}(t), v(t), \rho(t)) \in \mathcal{W}_2, \text{ for any } t > 0.$$

Fourth, from (2.2)-(2.3) and by introducing the function,

$$\mathcal{G}(t) \equiv \psi^{-\alpha}(t)e^{\delta t},$$

the inequality in (2.4) has the equivalent forms

$$2\mathcal{P}(u, \dot{u}) > \frac{\delta}{\alpha}\|u\|^2_{V_P} \Leftrightarrow \dot{\psi}_0 > \frac{\delta}{\alpha}\psi_0 \Leftrightarrow \dot{\mathcal{G}}_0 < 0.$$

Finally, we define the sets

$$\mathcal{V} \equiv \left\{(u, \dot{u}) \in \tilde{\mathcal{H}} : 2\mathcal{P}(u, \dot{u}) > \frac{\delta}{\alpha}\|u\|^2_{V_P}\right\}$$
$$= \left\{(u, \dot{u}) \in \tilde{\mathcal{H}} : \dot{\psi} > \frac{\delta}{\alpha}\psi\right\} = \{(u, \dot{u}) \in \tilde{\mathcal{H}} : \dot{\mathcal{G}} < 0\}.$$

$$\mathcal{V}_1 \equiv \{(u, \dot{u}, v) \in \mathcal{H}_1 : (u, \dot{u}) \in \mathcal{V}\} \quad \mathcal{V}_2 \equiv \{(u, \dot{u}, v, \rho) \in \mathcal{H}_2 : (u, \dot{u}) \in \mathcal{V}\}$$

**Lemma 2.3.** *Consider any solution either from problem $(\mathbf{P})_1$ or problem $(\mathbf{P})^*_2$, in the sense of Definitions 1.1, 1.2, respectively. Assume that hypotheses $(H0) - (H3)$ hold and (2.4) is satisfied. If there exists a constant $\kappa_0^2 > 0$ such that*

$$\mathcal{J}(s) \geq \kappa_0^2 > 0, \ \ s \geq 0, \tag{2.6}$$

*where the function $\mathcal{J}(s)$ defined by*

$$\mathcal{J}(s) \equiv \frac{2\alpha^2}{1+2\alpha}\gamma s^{\frac{1+2\alpha}{\alpha}} - \alpha\beta s^2 + \alpha^2\psi_0^{-(1+2\alpha)}\left(\phi_0 - \frac{2\gamma}{1+2\alpha}\right), \ \ s \geq 0.$$

*then, along the solution, the corresponding set $\mathcal{V}_j, j = 1, 2$ is positive invariant. Furthermore,*

$$\dot{\mathcal{G}}(t) \leq -\kappa_0 < 0, \ \ \text{for any } \ t \geq 0.$$

*Proof.* (of Lemma 2.3.) Consider a solution either from problem $(\mathbf{P})_1$ or problem $(\mathbf{P})^*_2$, such that the initial data are in the corresponding $\mathcal{V}_j, j = 1, 2$. Then, in any case $(u_0, u_1) \equiv (u(0), \dot{u}(0)) \in \mathcal{V}$. To show the invariance property, we proceed by contradiction. Assume that there exists some $\hat{t} > 0$ such that

$$(u(t), \dot{u}(t)) \in \mathcal{V}, \text{ for } t \in [0, \hat{t}), \quad \text{and} \quad (u(\hat{t}), \dot{u}(\hat{t})) \notin \mathcal{V},$$

that is

$$\dot{\mathcal{G}}(t) < 0, \ \ t \in [0, \hat{t}), \ \ \dot{\mathcal{G}}(\hat{t}) = 0.$$

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We shall prove that the time $\hat{t}$ is never reached. To this end, we first construct a differential inequality for the function

$$\psi(t) \equiv \|u(t)\|^2_{V_P} \in \mathbb{R}^+, \quad t \geq 0.$$

We calculate the first and second derivatives of $\psi(t)$, and we use Definitions 1.1 and 1.2. First, we only use the hyperbolic equation, which is the same in both problems. Then, we conclude the following for $t \geq 0$,

$$\left.\begin{aligned} \frac{d}{dt}\psi(t) &= 2\mathcal{P}(u(t),\dot{u}(t)) \\ \frac{d^2}{dt^2}\psi(t) &= 2(\|\dot{u}(t)\|^2_{V_P} - \|u(t)\|^2_{V_{A_1}} + \eta\mathcal{B}(v(t),u(t)) + (f(u(t)),u(t)) \\ &\quad -2\delta\mathcal{P}(u(t),\dot{u}(t)). \end{aligned}\right\} \tag{2.7}$$

We shall estimate the terms of the right hand side of the second derivative of $\psi(t)$. First, we consider the problem $(\mathbf{P})_\mathbf{1}$. By the corresponding energy equation and hypothesis $(H1)$, we obtain the following

$$\begin{aligned} &2(\|\dot{u}(t)\|^2_{V_P} - \|u(t)\|^2_{V_{A_1}} + (f(u(t),u(t))) + 2r(E(t) - E(t)) \\ &\geq (r+2)\|\dot{u}(t)\|^2_{V_P} + (r-2)\|u(t)\|^2_{V_{A_1}} + r\eta\|v(t)\|^2_{V_P} - 2rE_0. \end{aligned}$$

For the problem $(\mathbf{P})^*_\mathbf{2}$, we estimate in a similar way the terms of the right hand side of the second derivative of $\psi(t)$

$$\begin{aligned} &2(\|\dot{u}(t)\|^2_{V_P} - \|u(t)\|^2_{V_{A_1}} + (f(u(t),u(t))) + 2r(E(t) - E(t)) \\ &\geq (r+2)\|\dot{u}(t)\|^2_{V_P} + (r-2)\|u(t)\|^2_{V_{A_1}} + r\eta\|v(t)\|^2_{V_P} + r\eta\|\rho(t)\|^2_{\mathcal{M}_{V_{A_2}}} - 2rE_0 \\ &\geq (r+2)\|\dot{u}(t)\|^2_{V_P} + (r-2)\|u(t)\|^2_{V_{A_1}} + r\eta\|v(t)\|^2_{V_P} - 2rE_0. \end{aligned}$$

Consequently, from (2.7), hypothesis $(H0)$, $(H0)_1$, $(H0)_2$, (2.5) and last inequalities, we get for both problems and for $t \geq 0$,

$$\begin{aligned} \frac{d^2}{dt^2}\psi(t) &\geq (r+2)\,\|\dot{u}(t)\|^2_{V_P} - 2\delta\mathcal{P}(u(t),\dot{u}(t)) \\ &\quad + (r-2)\|u(t)\|^2_{V_{A_1}} + r\eta\|v(t)\|^2_{V_P} \\ &\quad - \eta\tilde{c}\frac{1}{2}\left(\|u(t)\|^2_{V_{A_1}} + \|v(t)\|^2_{V_P}\right) - 2rE_0 \\ &\geq -\delta\frac{d}{dt}\psi(t) + (r+2)\,\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} \\ &\quad + \left(r - 2 - \frac{1}{2}\eta\tilde{c}\right)\|u(t)\|^2_{V_{A_1}} + \eta\left(r - \frac{1}{2}\tilde{c}\right)\|v(t)\|^2_{V_P} - 2rE_0 \\ &\geq -\delta\frac{d}{dt}\psi(t) + (r+2)\,\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} + c\left(r - 2 - \frac{1}{2}\eta\tilde{c}\right)\psi(t) - 2rE_0. \end{aligned}$$

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That is, for $t \geq 0$, the following inequality is satisfied

$$\frac{d^2}{dt^2}\psi(t) + \delta\frac{d}{dt}\psi(t) - (r+2)\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} - c\left(r - 2 - \frac{1}{2}\eta\tilde{c}\right)\psi(t) + 2rE_0 \geq 0. \tag{2.8}$$

From $(H3)$, we can simplify the notation by substituting the constants defined in (2.1). After multiplying the differential inequality (2.8) by $\psi(t)$, we obtain

$$\psi(t)\frac{d^2}{dt^2}\psi(t) + \delta\psi(t)\frac{d}{dt}\psi(t) - (1+\alpha)\left(\frac{d}{dt}\psi(t)\right)^2 - \beta\psi^2(t) + \gamma\psi(t) \geq 0, \quad t \geq 0. \tag{2.9}$$

If we now introduce $\mathcal{G}(t) \equiv \psi^{-\alpha}(t)e^{\delta t}$, this inequality becomes

$$\frac{d^2}{dt^2}\mathcal{G}(t) - \delta\frac{d}{dt}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) - \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq 0, \quad t \geq 0.$$

From the definition of $\hat{t} > 0$,

$$-\delta\frac{d}{dt}\mathcal{G}(t) > 0, \quad \text{for} \quad t \in [0, \hat{t}).$$

Hence, and from the differential inequality for $\mathcal{G}(t)$ we obtain

$$\frac{d^2}{dt^2}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) < \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t), \quad t \in [0, \hat{t}).$$

Consequently, we arrive to

$$\frac{d^2}{dt^2}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) - \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t) < 0, \qquad t \in [0, \hat{t}). \tag{2.10}$$

Multiplying (2.10) by $\frac{d}{dt}\mathcal{G}(t) < 0$, we conclude the following integral

$$\left(\frac{d}{dt}\mathcal{G}(t)\right)^2 < \mathcal{J}\left(\mathcal{G}(t)\right), \quad t \in [0, \hat{t}). \tag{2.11}$$

By hypotheses, there is a constant $\kappa_0^2 > 0$ such that

$$\mathcal{J}(s) \geq \kappa_0^2 > 0, \quad s \geq 0,$$

then, from (2.11)

$$\dot{\mathcal{G}}(t) = \frac{d}{dt}\mathcal{G}(t) < -\kappa_0 < 0, \quad t \in [0, \hat{t}).$$

By continuity, when $t \to \hat{t}$,

$$\dot{\mathcal{G}}(\hat{t}) \leq -\kappa_0 < 0,$$

which contradicts the definition of $\hat{t}$. Hence, as long as the solution exits,

$$\dot{\mathcal{G}}(t) \leq -\kappa_0 < 0,$$

and the corresponding $\mathcal{V}_j, j = 1, 2$, is positive invariant.
□

*Proof.* (of Theorem 2.1.) If the solution is global, then

$$t \to \psi(t) \equiv \|u(t)\|^2_{V_p} \in \mathbb{R}^+,$$

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that is, it is well defined for any $t \geq 0$. The conclusions of Theorem 2.1 are derived from the analysis of (2.9), as was made in [5]. However, for completeness we shall sketch the proof. First, from Lemma 2.3,

$$\frac{d}{dt}\mathcal{G}(t) \leq -\kappa_0 < 0.$$

And consequently,

$$0 \leq \psi^{-\alpha}(t)e^{\delta t} = \mathcal{G}(t) \leq \psi_0^{-\alpha} - t\kappa_0.$$

Then $t \to t^* \equiv (\kappa_0 \psi_0^{\alpha})^{-1}$ implies that $\psi(t) \to \infty$. That is, $\psi(t)$ blows-up at $t^*$.

The proof of (2.6) is as follows. First, we notice that $\mathcal{J}(s)$ attains an absolute minimum at $s_0 \equiv \left(\frac{\beta}{\gamma}\right)^{\alpha}$, that is

$$\mathcal{J}(s) \geq \mathcal{J}(s_0) = \alpha^2 \psi_0^{-(1+2\alpha)} \left(\phi_0 - \mathcal{K}(\gamma)\right),$$

where

$$\mathcal{K}(\gamma) \equiv \frac{2\gamma}{1+2\alpha} + \frac{\beta}{\alpha(1+2\alpha)} \left(\frac{\beta}{\gamma}\right)^{2\alpha} \psi_0^{1+2\alpha}.$$

We define $\kappa_0^2 \equiv \mathcal{J}(s_0)$. Then, (2.6) holds if and only if

$$\mathcal{K}(\gamma) < \phi_0. \tag{2.12}$$

We notice that

$$\mathcal{K}(\gamma) \to \infty \qquad \text{as either} \qquad \gamma \to 0 \;\text{ or }\; \gamma \to \infty.$$

Furthermore,

$$\mathcal{K}(\gamma) \geq \mathcal{K}(\gamma_0) = \frac{\beta}{\alpha}\psi_0, \qquad \gamma > 0,$$

where $\gamma_0 \equiv \beta\psi_0$. Hence, there exist two different roots, denoted by **a** and **b**, of

$$\mathcal{K}(\gamma) = \phi_0.$$

That is, there exists a nonempty interval $\mathcal{I} \equiv (\mathbf{a}, \mathbf{b})$, such that

$$0 < \mathbf{a} < \gamma_0 < \mathbf{b} < \frac{1+2\alpha}{2}\phi_0,$$

and

$$\frac{\beta}{\alpha}\psi_0 < \mathcal{K}(\gamma) < \phi_0 \Leftrightarrow \gamma \in \mathcal{I} \equiv (\mathbf{a}, \mathbf{b}), \gamma \neq \gamma_0.$$

Then, (2.6) holds if and only if $\gamma \in \mathcal{I}$. The strict monotonicity of $\mathcal{K}$ for $\gamma < \gamma_0$ and $\gamma > \gamma_0$, implies that, for fixed $\psi_0$, the interval $\mathcal{I}$ grows as $\dot{\psi}_0$ grows. That is,

$$\lim_{\dot{\psi}_0 \to \infty} \left| \frac{1+2\alpha}{2}\phi_0 - \mathbf{b} \right| = 0 = \lim_{\dot{\psi}_0 \to \infty} \mathbf{a}.$$

The rest of the conclusions follow as in [5].

□

*Proof.* (of Corollary 2.3.) Since $\dot{\psi}_0 \to \infty \Rightarrow \mathbf{a} \to 0$, $\mathbf{b} \to \infty$ and $t^* \to 0$, then, for every $\xi > 0$ there exists $\eta_1 > 0$, such that $\dot{\psi}_0 > \eta_1 \Rightarrow \gamma = 2r\xi \in \mathcal{I} = (\mathbf{a}, \mathbf{b})$. Also, for every $\epsilon > 0$ there exists $\eta_2 > 0$, such that $\dot{\psi}_0 > \eta_2 \Rightarrow t^* < \epsilon$. Hence, any solution with $\gamma/2r = E_0 = \xi$ blows-up at a finite time $t^* < \epsilon$ if $\dot{\psi}_0 > \eta \equiv \max\{\eta_1, \eta_2\}$.

□

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**Remark 2.1.** Notice that $\frac{\beta}{\alpha}\psi_0 < \phi_0$, the assumption (2.4) in Theorem 2.1, is the condition that allows the existence of $\mathcal{I}$, and $\gamma \in \mathcal{I}$ characterizes the condition that implies the positive invariance of $\mathcal{V}_j, j = 1, 2$ and hence the blow up of $\psi(t)$ in finite time and consequently the nonexistence of global solutions.

**Remark 2.2.** The blow-up of the solution it comes from two different sources of the system. (i) The physical properties of the model: $\delta, r, \eta$. (ii) The initial data: $\psi_0, \dot{\psi}_0$. The blow-up property is reached for a larger set of values of $r$ as long as $\tilde{c}\eta$ decreases. If we decouple the system, $\eta = 0$, then the blow-up is reached as if the parabolic equation did not exist. The coupling with the parabolic equation stabilizes the system, as does the damping term in the hyperbolic equation. Indeed, the numbers **a** and **b** are closer one each other as the damping coefficient $\delta$ or the coupling factor $\tilde{c}\eta$ grows. Hence, the length of the blow-up interval $\mathcal{I}$ decreases as $\delta$ or $\tilde{c}\eta$ increases. Therefore, as the damping coefficient or the coupling factor grows, then the set of initial energies where we can have global non existence becomes smaller. On the other hand, a notable property that should be highlighted is that the blow-up time approaches zero and the length of the blow-up interval $\mathcal{I}$ becomes infinity as $\dot{\psi}_0$ goes to infinity.

## 3. Applications and some extensions

First, we shall apply the result proved in last section to the following problems, related to the ones introduced at the beginning of this work. Then, we will present some problems where our analysis can be extended.

### *3.1. Cauchy problem of a one-dimensional thermoelastic model with a Gurtin-Pipkin short memory*

$$
(\mathbf{ThE})_1^* \begin{cases} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ \text{find } u(x,t) \in \mathbb{R},\ \theta(x,t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} - a u_{xx} + \eta\theta_x + \delta u_t = f(u), & x \in \Omega,\ \ t > 0, \\ \theta_t - \kappa\theta_{xx} + \eta u_{xt} - \int_0^t h(t-s)\theta_{xx}(s)\, ds = g(\theta), & x \in \Omega,\ \ t > 0, \\ u = u_0,\ u_t = u_1,\ \theta = \theta_0, & x \in \Omega,\ \ t = 0, \\ u = 0,\ \theta = 0, & x \in \partial\Omega\ \ t > 0. \end{cases}
$$

Here, $\Omega = (0, L)$, the short memory kernel $h$ satisfies the hypotheses $(H2)_1, (H2)_2$, and

$$
\begin{aligned}
&\mathcal{P}(u,w) = (u,w), \ \ u, w \in V_p = H = L_2(\Omega), \\
&\mathcal{A}_1(u,w) = a(u_x, w_x), \ \ u, w \in V_{A_1} = H_0^1(\Omega), \\
&\mathcal{A}_2(\theta,w) = \kappa(\theta_x, w_x), \ \ u, w \in V_{A_2} = H_0^1(\Omega), \\
&\mathcal{B}(u,\theta) = -\eta(u_x,\theta)_{V_B'\times V_B} = \eta(\theta_x, u)_{V_B'\times V_B} = -\mathcal{B}(\theta,u), \ \ u,\theta \in V_B = H^{1/2}(\Omega), \\
&|\mathcal{B}(u,\theta)| \le \tilde{c}\|u\|_{H_0^1(\Omega)}\|\theta\|_{L_2(\Omega)}, \ \ u \in V_{A_1},\ \theta \in V_B.
\end{aligned}
$$

Then, the hypotheses $(H0)$ and $(H0)_1$ are satisfied. Moreover, the nonlinearities satisfy the hypotheses $(H1)_1, (H1)_2$. We assume that $(H3)$ holds. Consider a solution in the sense of Definition 1.1 such that the initial data satisfy

$$
(u_0, u_1) > \frac{2\delta}{r-2}\|u_0\|^2 > 0,
$$

then there exists a nonempty blow-up interval $\mathcal{I}$, given by Theorem 2.1. If the initial energy is such that $2rE_0 \in \mathcal{I}$, then the corresponding solution is not global. In fact, blows-up up in finite time. Furthermore, for every positive value of the initial energy, there exists initial data such that the corresponding solution blows-up.

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*3.2. Plate equation with a Coleman-Gurtin long thermal memory*

$$
(\mathbf{ThE})_2^* \begin{cases} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ s \leq 0, \\ \text{find } u(x,t) \in \mathbb{R},\ \theta(x,t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} + \Delta^2 u - \Delta u(t) - \|\nabla u\|_2^{2p} \Delta u(t) + \eta \Delta\theta + \delta u_t = \hat{f}(u), & x \in \Omega,\ t > 0, \\ \theta_t - \omega\Delta\theta - (1-\omega) \int_0^\infty k(t-s)\Delta\theta(s)\, ds - \eta\Delta u_t = g(\theta), & x \in \Omega,\ t > 0, \\ u = u_0,\ u_t = u_1, & x \in \Omega,\ t = 0, \\ \theta = \theta_0, & x \in \Omega,\ t \leq 0, \\ u = \partial_\nu u = 0,\ \ \theta = 0, & x \in \partial\Omega,\ t > 0, \end{cases}
$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain, with smooth boundary $\partial\Omega$ and normal vector $\nu$, the long memory kernel $k$ satisfies the hypotheses $(H2)_3$, $(H2)_4$ with $\omega \in (0,1)$, and

$$
\begin{aligned}
&\mathcal{P}(u,w) = (u,w),\ \ u,w \in V_p = H = L_2(\Omega), \\
&\mathcal{A}_1(u,w) = (\Delta u, \Delta w) + (\nabla\theta, \nabla w),\ \ u,w \in V_{A_1} = H_0^2(\Omega), \\
&\mathcal{A}_2(\theta,w) = \mathcal{B}(\theta,w) = (\nabla\theta, \nabla w),\ \ u,w \in V_{A_2} = V_B = H_0^1(\Omega), \\
&|\mathcal{B}(u,\theta)| \leq \tilde{c}\|u\|_{H_0^2(\Omega)}\|\theta\|_{L_2(\Omega)},\ \ u \in V_{A_1},\ \theta \in V_B.
\end{aligned}
$$

Then, the hypotheses $(H0)$ and $(H0)_2$ are satisfied. The nonlinearity $g$ satisfies $(H1)_2$. The nonlinear term in the hyperbolic equation satisies $(H1)_1$. Indeed, the nonlinear source $\hat{f}$ is such that

$$
(\hat{f}(u), u) - \rho\hat{F}(u) \geq 0, \quad \rho > 2,
$$

where $\hat{F}$ is the potential of $\hat{f}$. Then,

$$
f(u) = \hat{f}(u) + \|\nabla u\|_{2p}^2 \Delta u,
$$

has the potential

$$
F(u) = \hat{F}(u) - \frac{1}{2(p+1)}\|\nabla u\|^{2(p+1)}.
$$

Hence, $(H1)_1$ is satisfied if

$$
\rho \geq r \geq 2(p+1).
$$

That is, the nonlinearity of the source term $\hat{f}$ is stronger than the one of $\phi$.

We assume that $(H3)$ holds. Consider a solution in the sense of Definition 1.2 such that the initial data satisfy

$$
(u_0, u_1) > \frac{2\delta}{r-2}\|u_0\|^2 > 0,
$$

then there exists a nonempty blow-up interval $\mathcal{I}$, given by Theorem 2.1. If the initial energy is such that $2rE_0 \in \mathcal{I}$, then the corresponding solution is not global. In fact, blows-up up in finite time. Furthermore, for every positive value of the initial energy, there exists initial data such that the corresponding solution blows-up.

In both problems above

$$
\mathcal{I} \subset \left(0, \left\{\frac{\sqrt{r}}{2\|u_0\|}\left[(u_0, u_1) - \frac{2\delta}{r-2}\|u_0\|^2\right]\right\}^2\right),
$$

and as $(u_0, u_1) \to \infty, \mathcal{I} \to (0, \infty)$.

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### 3.3. Thermoelasticity system in n dimensions with short memory

Consider the following n-dimensional system of thermoelasticity under the Fourier's law of heat flux.

$$
(\mathbf{ThE})_3 \begin{cases} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \\ \text{find } u(x,t) \in \mathbb{R}^n,\ \theta(x,t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\nabla\cdot u) + \eta\nabla\theta + \delta u_t = f(u), & x\in\Omega,\ t>0, \\ c\theta_t - \kappa\Delta\theta - \kappa\int_0^t \Delta h(t-\tau)\theta(\tau)\,d\tau + \eta\nabla\cdot u_t = g(\theta), & x\in\Omega,\ t>0, \\ u = u_0,\ u_t = u_1,\ \theta = \theta_0, & x\in\Omega,\ t=0, \\ u = 0,\ \ \theta = 0, & x\in\partial\Omega,\ t>0, \end{cases}
$$

where $u(x,t)$ is the displacement vector, $\theta(x,t)$ is the difference temperature, $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary $\partial\Omega$. Here, $\lambda, \mu$ are the Lame moduli, $\kappa$ is the Fourier heat conduction coefficient, $\delta > 0$ is the damping coefficient and $c, \eta$, are positive constants. Finally, $f(u), g(v)$ are nonlinear source terms. This problem has been studied in [27] with second sound and in [28–32] additionally with a viscoelastic dissipation acting on a part of the boundary, but without source term in the parabolic equation $g(\theta) = 0$, without damping $\delta = 0$ and without the viscoelastic term in the parabolic equation. An abstract formulation is studied in [33] with Cattaneo's law and inertial terms. Notice that the hyperbolic component is a system of equations in contrast with the parabolic one which is a single equation, then our formulation can not be applied directly and must be adapted. However, we can extend our results and get the same conclusions. That is, if the initial data satisfy

$$
(u_0, u_1) > \frac{2\delta}{r-2}\|u_0\|^2 > 0,
$$

then $\mathcal{I} \neq \emptyset$ and the solution blows-up if $2rE_0 \in \mathcal{I}$.

### 3.4. Kirchhoff equation with a Coleman-Gurtin long thermal memory

$$
(\mathbf{ThE})_4 \begin{cases} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ s \le 0, \\ \text{find } u(x,t) \in \mathbb{R},\ \theta(x,t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} - \Delta u(t) - \|\nabla u\|_2^{2p}\Delta u(t) + \eta\Delta\theta + \delta u_t = \hat{f}(u), & x\in\Omega,\ t>0, \\ \theta_t - \omega\Delta\theta - (1-\omega)\int_0^\infty k(t-s)\Delta\theta(s)\,ds - \eta\Delta u_t = g(\theta), & x\in\Omega,\ t>0, \\ u = u_0,\ u_t = u_1, & x\in\Omega,\ t=0, \\ \theta = \theta_0, & x\in\Omega,\ t\le 0, \\ u = 0,\ \ \theta = 0, & x\in\partial\Omega,\ t>0, \end{cases}
$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain, with smooth boundary $\partial\Omega$, the long memory kernel $k$ satisfies the hypotheses $(H2)_3, (H2)_4$ with $\omega \in (0,1)$. This is a quasilinear problem and our functional framework does not cover it. However, we can extend our analysis and conclude blow-up if the nonlinearities hold the same relations than in $(\mathbf{ThE})_2^*$. That is, the nonlinearity of the source term $\hat{f}$ is stronger than the one of $\phi$. Then, if the initial data satisfy

$$
(u_0, u_1) > \frac{2\delta}{r-2}\|u_0\|^2 > 0,
$$

then $\mathcal{I} \neq \emptyset$ and the solution blows-up if $2rE_0 \in \mathcal{I}$.

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