Article

Not peer-reviewed version

The Existence and Averaging Principle for Caputo Fractional Stochastic Delay Differential Systems with Poisson Jumps

Zhenyu Bai and Chuanzhi Bai *

Posted Date: 27 November 2023

doi: 10.20944/preprints202311.1668.v1

Keywords: Stochastic fractional delay differential systems; Delayed Mittag-Leffler type matrix function; Existence and uniqueness; averaging principle; Lp convergence

Article

The Existence and Averaging Principle for Caputo Fractional Stochastic Delay Differential Systems with Poisson Jumps

Zhenyu Bai [1] and Chuanzhi Bai [2,*]

[1] XJTLU Wisdom Lake Academy of Pharmacy, Xi'an Jiaotong-Liverpool University, Suzhou, Jiangsu 215123, P R China
[2] Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, P R China
* Correspondence: czbai@hytc.edu.cn

Abstract: In this paper, we obtain the existence and uniqueness theorem for solutions of Caputo type fractional stochastic delay differential systems (FSDDSs) with Poisson jumps by utilizing delayed perturbation of Mittag-Leffler function. Moreover, by using Burkholder-Davis-Gundy's inequality, Doob's martingale inequality and Hölder inequality, we prove that the solution of the averaged FSDDSs converges to that of the standard FSDDSs in the sense of L^p. Some known results in the literature are extended.

Keywords: stochastic fractional delay differential systems; delayed mittag-leffler type matrix function; existence and uniqueness; averaging principle; L^p convergence

MSC: 34A08; 34F05; 60H10

1. Introduction

Fractional stochastic delay differential systems (FSDDs) are mathematical models that involve fractional derivatives, stochastic noise, and time delays. The fractional derivatives represent the memory effects and long-range dependence in the system, while the stochastic noise and delays account for the random fluctuations and time delays, respectively. FSDDs find applications in many fields, including physics, biology, finance, and engineering. They can be used to model systems with memory and randomness, such as anomalous diffusion processes, fractional-order control systems with stochastic disturbances, and biological systems with fractional-order kinetics and stochastic effects. They provide a powerful framework for understanding and predicting the behavior of complex systems with memory, randomness, and time delays. See for examples [1-6], and the references cited therein.

The averaging principle is a mathematical tool used to simplify the analysis of dynamical systems with fast and slow time scales. It provides an approximate description of the system's behavior. In 1968, Khasminskii [7] first used the average principle to prove that the solution of the average equation can converge to the solution of the complex system. In [8], the authors presented an averaging method for stochastic differential equations with non-Gaussian Lévy noise. With the development of fractional calculus, many works have emerged that apply the averaging principle to fractional stochastic differential equations (FSDEs). In [9], Xu, et.al. presents an averaging principle for Caputo FSDEs driven by Brown motion. In [10], Luo, et.al. established an averaging principle for the solution of the a class of FSDEs with time-delays. In [11], Ahmed and Zhu investigated the averaging principle for the Hilfer fractional stochastic delay differential equation with Poisson jumps in the sense of mean square. The periodic averaging method for impulsive conformable fractional stochastic differential

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

equations with Poisson jumps are discussed in [12] by Ahmed. In [13], Wang and Lin extended the averaging principle of the following FSDEs

$$\begin{cases} {}^{C}D_{0}^{\alpha}[x(t)-h(t,x(t)]=f(t,x(t))+g(t,x(t))\frac{dB_t}{dt}, \quad t\in J=[0,T], \\ x(0)=x_0, \end{cases} \tag{1}$$

in the sense of mean square (L^2 convergence) to L^p convergence ($p\geq 2$), which generated some works on the averaging principle for FSDES [9,10,14]. In [15], Yang, et.al. studied the averaging principle for a class of ψ-Capuo fractional stochastic delay differential equations with Poisson jumps.

Recently, Li and Wang in [16] studied the following Caputo type FSDDEs:

$$\begin{cases} ({}^{C}D_{0}^{\alpha}Y)(t)=AY(t)+BY(t-h)+f(t,Y(t))+\sigma(t,Y(t))\frac{dW(t)}{dt},\ t\in J, \\ Y(t)=\Phi(t), \quad -h\leq t\leq 0,\ h>0, \end{cases} \tag{2}$$

the existence, uniqueness and the averaging principle for (1.2) are established.

In the present paper, motivated by [11,13,16], we study the following Caputo FSDDEs with Poisson jumps

$$\begin{cases} ({}^{C}D_{0}^{\alpha}x)(t)=Ax(t)+Bx(t-\tau)+f(t,x(t),x(t-\tau))+\sigma(t,x(t),x(t-\tau))\frac{dW(t)}{dt}, \\ \qquad\qquad +\int_V g(t,x(t),x(t-\tau),v)\bar{N}(dt,dv), \quad t\in J, \\ x(t)=\phi(t), \quad -\tau\leq t\leq 0, \end{cases} \tag{3}$$

where ${}^{C}D_{0}^{\alpha}$ is the left Caputo fractional derivative with $\frac{1}{2}<\alpha<1$, $J=[0,T]$, $A,B\in\mathbb{R}^{n\times n}$ are two constant matrices, the state vector $x\in\mathbb{R}^n$ is a stochastic process, $f: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ and $g: J\times\mathbb{R}^n\times\mathbb{R}^n\times V\to\mathbb{R}^n$ are measurable continuous functions. Let $(\Omega,\mathcal{F},P)$ be a complete probability space equipped with some filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual condition, $W(t)$ is an m-dimensional Brownian motion on the probability space $(\Omega,\mathcal{F},P)$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(V,\Phi,\lambda(dv))$ be a σ-finite measurable space. Given stationary Poisson point process $(p_t)_{t\geq 0}$, which is defined on $(\Omega,\mathcal{F},P)$ with values in V and with characteristic measure λ. We denote by $N(t,dv)$ the counting measure of p_t such that $\bar{N}(t,\Theta):=\mathbb{E}(N(t,\Theta))=t\lambda(\Theta)$ for $\Theta\in\Phi$. Define $\bar{N}(t,dv):=N(t,dv)-t\lambda(dv)$, and the Poisson martingale measure generated by p_t.

In this paper, we first prove the existence and uniqueness of solutions of Caputo type FSDDEs (1.3) by using delayed perturbation of Mittag-Leffler function and Banach fixed point theorem; Secondly, we prove the averaging principle for Caputo FSDDEs (1.3) in the sense of L_p (pth moment) with inequality techniques. The main contributions and advantages of this paper are as follows:

(1) The solution of the averaged FSDDEs converges to that of the standard FSDDEs in the sense of L_p, which is a generalization of the existing result ($p=2$) of the averaging principle for FSDDEs,

(2) The fractional calculus, stochastic inequality and Hölder inequality are effectively used to establish our result.

(3) our work in this paper is novel and more technical. Our result extends the main results of [17].

This paper will be organized as follows. In Section 2, we will briefly recall some definitions and preliminaries. In Section 3, we prove the existence and uniqueness of solutions for Caputo FSDDEs (1.3) with Poisson jumps. In Section 4, we prove that the solution of the FSDDEs (1.3) converges to that of the standard one in L_p sense. In Section 5, an example is presented to illustrate our theoretical results. Finally, the paper is concluded in Section 6.

2. Preliminaries

In this section, we recall some basic definitions and lemmas which are used in the sequel.

Let $Y=\mathbb{L}^p(\Omega,\mathcal{F},\mathbb{P})$ denote the space of all $\mathcal{F}(t)$ measurable, p squqre integrable functions $x:\Omega\to\mathbb{R}^n$ with $\|x(t)\|_{ps}:=\left(\sum_{i=1}^{n}\mathbb{E}(|x_i(t)|^p)\right)^{1/p}$, and $\|x\|=\sum_{i=1}^{n}|x_i|$ and $\|A\|=\max_{1\leq i\leq n}\sum_{i=1}^{n}|a_{ij}|$ be

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

the vector norm and matrix norm, respectively. A process $x : [-\tau, T] \to \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}(t)$-adapted if $x(t) \in Y$.

Definition 2.1 [17]. Let $\alpha > 0$, and f be an integrable function defined on $[a, b]$. The left Riemann-Liouville fractional integral operator of order α of a function f is defined by

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a. \tag{4} $$

Definition 2.2 [17]. Let $n - 1 < \alpha < n$, and $f \in C^n([a, b])$. The left Caputo fractional derivative of order α of a function f is defined by

$$ {}_a^C D_t^\alpha f(t) = ({}_aI_t^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad t > a, \tag{5} $$

where $n = [\alpha] + 1$.

Definition 2.3 [18]. The coefficient matrices $Q_k(s)$, $k = 0, 1, 2, ...$, satisfy the following multivariate determining matrix equation

$$ Q_0(s) = Q_k(-\tau) = \Theta, \quad Q_1(0) = I, \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots, $$

$$ Q_{k+1}(s) = AQ_k(s) + BQ_k(s-\tau), \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots, $$

where I is an identity matrix and Θ is a zero matrix.

Definition 2.4 [18]. Delayed perturbation of two parameter Mittag-Leffler type matrix function $X_{\tau,\alpha,\beta}^{A,B}$ generated by A, B is defined by

$$ X_{\tau,\alpha,\beta}^{A,B}(t) := \begin{cases} \Theta, \quad t \in [-\tau, 0), \\ I, \quad t = 0, \\ \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{t^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} + \sum_{i=1}^{\infty} Q_{i+1}(\tau) \frac{(t-\tau)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\ \quad + \cdots + \sum_{i=0}^{\infty} Q_{i+1}(p\tau) \frac{(t-p\tau)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, \quad p\tau < t \le (p+1)\tau. \end{cases} \tag{6} $$

From [17], we can easily obtain the following definition.

Definition 2.5. A $\mathbb{R}^n$-value stochastic process $\{x(t) : -\tau \le t \le T\}$ is called a solution of (1.3) if $x(t)$ satisfies the integral equation of the following form:

$$ x(t) = \begin{cases} X_{\tau,\alpha,1}^{A,B}(t+\tau)\phi(-\tau) + \int_{-\tau}^0 X_{\tau,\alpha,\alpha}^{A,B}(t-s)[({}^C D_{-\tau^+}^\alpha \phi)(s) - A\phi(s)]ds \\ \quad + \int_0^t X_{\tau,\alpha,\alpha}^{A,B}(t-s) f(s, x(s), x(s-\tau))ds \\ \quad + \int_0^t X_{\tau,\alpha,\alpha}^{A,B}(t-s)\sigma(s, x(s), x(s-\tau))dW(s) \\ \quad + \int_0^t X_{\tau,\alpha,\alpha}^{A,B}(t-s) \int_V g(s, x(s), x(s-\tau), v)\tilde{N}(ds, dv), \quad t \in J, \\ \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{7} $$

where $x(t)$ is $\mathcal{F}(t)$-adapted and $\mathbb{E}(\int_{-\tau}^T \|x(t)\|^p dt) < \infty$.

Lemma 2.1 ([19]). *For any $t \ge 0$, $0 < \alpha < 1$, $0 < \beta \le 1$ and $\alpha + \beta \ge 1$, we have*

$$ \|X_{\tau,\alpha,\beta}^{A,B}(t)\| \le t^{\beta-1} E_{\alpha,\beta}((\|A\| + \|B\|)t^\alpha), \tag{8} $$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\beta)}$, $z \in \mathbb{R}$ is the Mittag-Leffler function.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

Lemma 2.2 *For any* $p \geq 2$, $\alpha \in \left(1 - \frac{1}{p}, 1\right)$ *and* $\gamma > 0$, *one has*

$$\int_0^t (t-s)^{p\alpha - p} E_{p\alpha - p+1,1}(\gamma s^{p\alpha - p+1}) ds \leq \frac{\Gamma(p\alpha - p + 1)}{\gamma} E_{p\alpha - p+1,1}(\gamma t^{p\alpha - p+1}), \tag{9}$$

where $\Gamma(\alpha) := \int_0^{+\infty} s^{\alpha - 1} e^{-s} ds$ *is the Gamma function.*

Proof. Let $\gamma > 0$ be arbitrary. Consifer the corresponding linear Caputo fractional differential equation of the following form

$$^C D_{0^+}^{p\alpha - p+1} x(t) = \gamma x(t). \tag{10}$$

From [20], it is easy to know that the Mittag-Leffler function $E_{p\alpha - p+1,1}(\gamma t^{p\alpha - p+1})$ is a solution of (2.7). So, the following equality holds:

$$E_{p\alpha - p+1,1}(\gamma t^{p\alpha - p+1}) = 1 + \frac{\gamma}{\Gamma(p\alpha - p+1)} \int_0^t (t-s)^{p\alpha - p} E_{p\alpha - p+1,1}(\gamma s^{p\alpha - p+1}) ds,$$

which completes the proof.

Lemma 2.3 ([21, 22]). *Let* $\phi : R_+ \times V \to R^n$ *and assume that*

$$\int_0^t \int_V |\phi(s,v)|^p \lambda(dv) ds < \infty, \quad p \geq 2.$$

Then there exists $D_p > 0$ *such that*

$$\begin{aligned} &\mathbb{E}\left(\sup_{0 \leq t \leq u} \left| \int_0^t \int_V \phi(s,v) \tilde{N}(ds,dv) \right|^p \right) \\ &\leq D_p \left\{ \mathbb{E}\left(\int_0^u \int_V |\phi(s,v)|^2 \lambda(dv) ds \right)^{\frac{p}{2}} + \mathbb{E} \int_0^u \int_V |\phi(s,v)|^p \lambda(dv) ds \right\}. \end{aligned} \tag{11}$$

Lemma 2.4 ([23]). *Let* u, v *be two integrable functions and* g *be continuous defined on domain* $[a,b]$. *Moreover, assume that*

(1) u *and* v *are nonnegative, and* v *is nondecreasing;*

(2) g *is nonnegative and nondecreasing.*

If

$$u(t) \leq v(t) + g(t) \int_a^t (t-\tau)^{\alpha - 1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) E_\alpha(g(t)\Gamma(\alpha)(t-a)^\alpha), \quad \forall t \in [a,b],$$

where $E_\alpha(\cdot)$ *is the Mittag-Leffler function.*

To study the qualitative properties of solution for (1.3), we impose the following conditions on data of the problem.

(H1) For any $x_1, x_2, y_1, y_2 \in R^n$ and $t \in J$, there exist two constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &\|f(t,x_1,y_1) - f(t,x_2,y_2)\|^p \vee \|\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)\|^p \\ &\vee \int_V \|g(t,x_1,y_1,v) - g(t,x_2,y_2,v)\|^p \lambda(dv) \leq C_1^p (\|x_1 - x_2\|^p + \|y_1 - y_2\|^p), \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

where $\|\cdot\|$ is the norm of $\mathbb{R}^n$, $x \vee y = max\{x, y\}$.

(H2) Let $\sigma(\cdot,0,0)$ and $g(\cdot,0,0)$ are essentially bounded, i.e.

$$\|\sigma(\cdot,0,0)\|_\infty := ess \sup_{t\in[0,\infty)} \|\sigma(t,0,0)\| < +\infty, \quad \|g(\cdot,0,0)\|_\infty := ess \sup_{t\in[0,\infty)} \|g(t,0,0)\| < +\infty,$$

and $f(\cdot,0,0)$ is $\mathbb{L}^p$ integrable, i.e.

$$\|f\|_{\mathbb{L}^p} = \int_0^T \|f(t,0,0)\|^p dt < +\infty.$$

3. Existence and uniqueness result

Let $\mathbb{H}^p([0,T])$ be the space of all the processes x which are measurable, $\mathcal{F}(t)$-adapted, and satisfied that $\|x\|_{\mathbb{H}^p} := \sup_{0\le t\le T} \|x(t)\|_{ps} < \infty$. Obviously, $(\mathbb{H}^p([0,T]), \|\cdot\|_{\mathbb{H}^p})$ is a Banach space. Set $\mu = \|A\| + \|B\|$. For any $t \in [-\tau, T]$ and $\phi \in C([-\tau,0],\mathbb{R}^n)$, we define an operator $\mathcal{T} : \mathbb{H}^p([0,T]) \to \mathbb{H}^p([0,T])$ as follows :

$$\begin{aligned}(\mathcal{T}x)(t) = {} & X^{A,B}_{\tau,\alpha,1}(t+\tau)\phi(-\tau) + \int_{-\tau}^0 X^{A,B}_{\tau,\alpha,\alpha}(t-s)[{}^C D^\alpha_{-\tau^+}\phi)(s) - A\phi(s)]ds \\ & + \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s) f(s,x(s),x(s-\tau))ds \\ & + \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\ & + \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V g(s,x(s),x(s-\tau),v)\tilde{N}(ds,dv).\end{aligned} \tag{12}$$

Lemma 3.1 *Let $1-\frac{1}{p} < \alpha < 1$. Assume that (H1) and (H2) hold. Then the operator $\mathcal{T}$ is well-defined.*

Proof. For any $x \in \mathbb{H}^p([0,T])$, by (3.1) and the following elementary inequality

$$\left\|\sum_{i=1}^m a_i\right\|^p \le m^{p-1}\sum_{i=1}^5 \|a_i\|^p, \quad a_i \in \mathbb{R}^n, \ i = 1,2,...,m. \tag{13}$$

we have

$$\begin{aligned}\|(\mathcal{T}x)(t)\|^p_{ps} \le {} & 5^{p-1}\mathbb{E}(\|X^{A,B}_{\tau,\alpha,1}(t+\tau)\phi(-\tau)\|^p) \\ & + 5^{p-1}\mathbb{E}\left(\left\|\int_{-\tau}^0 X^{A,B}_{\tau,\alpha,\alpha}(t-s)[({}^C D^\alpha_{-\tau^+}\phi)(s) - A\phi(s)]ds\right\|^p\right) \\ & + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)f(s,x(s),x(s-\tau))ds\right\|^p\right) \\ & + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\right\|^p\right) \\ & + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V g(s,x(s),x(s-\tau),v)\tilde{N}(ds,dv)\right\|^p\right) \\ := {} & I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{14}$$

For I_1, from Lemma 2.1, one has

$$\begin{aligned}I_1 = 5^{p-1}\mathbb{E}(\|X^{A,B}_{\tau,\alpha,1}(t+\tau)\phi(-\tau)\|^p) &\le 5^{p-1}\mathbb{E}(\|X^{A,B}_{\tau,\alpha,1}(t+\tau)\|^p\|\phi(-\tau)\|^p) \\ &\le 5^{p-1}\|\phi(-\tau)\|^p(E_{\alpha,1}(\mu(T+\tau))^\alpha))^p.\end{aligned} \tag{15}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

For I_2, by Lemma 2.1, Hölder inequality and $\alpha > 1 - \frac{1}{p}$, we obtain

$$\begin{aligned} I_2 &= 5^{p-1}\mathbb{E}\left(\left\|\int_{-\tau}^{0} X^{A,B}_{\tau,\alpha,\alpha}(t-s)[({}^{C}D^{\alpha}_{-\tau^+}\phi)(s) - A\phi(s)]ds\right\|^p\right) \\ &\leq 5^{p-1}\int_{-\tau}^{0}\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p ds \cdot \mathbb{E}\left(\int_{-\tau}^{0}\|({}^{C}D^{\alpha}_{-\tau^+}\phi)(s) - A\phi(s)\|^q ds\right)^{p-1} \\ &\leq 5^{p-1}\Xi\frac{(T+\tau)^{p\alpha-p+1}}{p\alpha-p+1}(E_{\alpha,\alpha}(\mu(T+\tau)^{\alpha})))^p, \end{aligned} \tag{16}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Xi = \left(\int_{-\tau}^{0}\|({}^{C}D^{\alpha}_{-\tau^+}\phi)(s) - A\phi(s)\|^q ds\right)^{p-1} < \infty$.

For I_3, applying (H1), (H2), Hölder inequality, Lemma 2.1 and Jensen inequality, one has

$$\begin{aligned} I_3 &= 5^{p-1}\mathbb{E}\left(\left\|\int_{0}^{t} X^{A,B}_{\tau,\alpha,\alpha}(t-s)f(s,x(s),x(s-\tau))ds\right\|^p\right) \\ &\leq 5^{p-1}\left(\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^q ds\right)^{\frac{p}{q}} \cdot \mathbb{E}\left(\int_0^t \|f(s,x(s),x(s-\tau)) - f(s,0,0) + f(s,0,0)\|^p ds\right) \\ &\leq 5^{p-1}\left(\int_0^t t^{q(\alpha-1)}E_{\alpha,\alpha}(\mu(t-s)^{\alpha})^q ds\right)^{\frac{p}{q}} \\ &\quad \cdot 2^{p-1}\mathbb{E}\left(\int_0^t \|f(s,x(s),x(s-\tau)) - f(s,0,0)\|^p ds + \int_0^t \|f(s,0,0)\|^p ds\right) \\ &\leq 10^{p-1}E_{\alpha,\alpha}(\mu T^{\alpha})^p\left(\frac{T^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}}\mathbb{E}\left(\int_0^t C_1^p(\|x(s)\|^p + \|x(s-\tau)\|^p)ds + \int_0^t \|f(s,0,0)\|^p ds\right) \\ &\leq 10^{p-1}E_{\alpha,\alpha}(\mu T^{\alpha})^p\left(\frac{T^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}}\left(TC_1^p(2\|x\|^p_{\mathbb{H}^p} + \|\phi\|^q) + \|f\|_{\mathbb{L}^p}\right), \end{aligned} \tag{17}$$

since

$$\begin{aligned} \sup_{0\leq t\leq T}\mathbb{E}\|x(t-\tau)\|^q &\leq \max\left\{\sup_{-\tau\leq t\leq 0}\mathbb{E}\|\phi(t)\|^q,\ \sup_{0\leq t\leq T}\mathbb{E}\|x(t)\|^q\right\} \\ &= \max\left\{\|\phi\|^q,\ \|x\|^q_{\mathbb{H}^q}\right\} \leq \|\phi\|^q + \|x\|^q_{\mathbb{H}^q}. \end{aligned}$$

For I_4, by using (H1), (H2), Cauchy-Schwarz inequality, Ito's isometry, Lemma 2.1 and Jensen inequality, we have

$$\begin{aligned} I_4 &= 5^{p-1}\mathbb{E}\left(\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\right\|^2\right)^{\frac{p}{2}}\right) \\ &\leq 5^{p-1}\mathbb{E}\left(\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^2\|\sigma(s,x(s),x(s-\tau))\|^2 ds\right)^{\frac{p}{2}} \\ &\leq 5^{p-1}\mathbb{E}\left(\left(\int_0^t 1^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}} \cdot \left(\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|\sigma(s,x(s),x(s-\tau))\|^p ds\right)^{\frac{2}{p}}\right)^{\frac{p}{2}} \\ &\leq 5^{p-1}T^{\frac{p}{2}-1}\mathbb{E}\left(\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|\sigma(s,x(s),x(s-\tau))\|^p ds\right) \\ &\leq 5^{p-1}T^{\frac{p}{2}-1}2^{p-1}E_{\alpha,\alpha}(\mu T^{\alpha})^p\mathbb{E}\left(\int_0^t (t-s)^{p\alpha-p}[C_1^p(\|x(s)\|^p + \|x(s-\tau)\|^p) + \|\sigma(s,0,0)\|^p]ds\right) \\ &\leq \frac{10^{p-1}T^{p\alpha-\frac{p}{2}}}{p\alpha-p+1}E_{\alpha,\alpha}(\mu T^{\alpha})^p(C_1^p(2\|x\|^p_{\mathbb{H}^p} + \|\phi\|^p) + \|\sigma(\cdot,0,0)\|^p_{\infty}). \end{aligned} \tag{18}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

For I_5, by using (H1), (H2), Lemmas 2.1, 2.3 and Jensen inequality, we obtain

$$\begin{aligned} I_5 &= 5^{p-1}\mathbb{E}\left(\left\|\int_0^t \int_V X^{A,B}_{\tau,\alpha,\alpha}(t-s)g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv)\right\|^p\right) \\ &\le 5^{p-1}D_p\mathbb{E}\left(\int_0^t \int_V |X^{A,B}_{\tau,\alpha,\alpha}(t-s)|^2 g^2(s,x(s),x(s-\tau),v)\lambda(dv)ds\right)^{\frac{p}{2}} \\ &\quad + 5^{p-1}D_p\mathbb{E}\left(\int_0^t \int_V |X^{A,B}_{\tau,\alpha,\alpha}(t-s)|^p g^p(s,x(s),x(s-\tau),v)\lambda(dv)ds\right) \\ &\le 5^{p-1}D_p(T^{\frac{p}{2}-1}+1)\mathbb{E}\left(\int_0^t |X^{A,B}_{\tau,\alpha,\alpha}(t-s)|^p \int_V g^p(s,x(s),x(s-\tau),v)\lambda(dv)ds\right) \\ &\le 5^{p-1}D_p(T^{\frac{p}{2}-1}+1)2^{p-1}E_{\alpha,\alpha}(\mu T^\alpha)^p \\ &\quad \cdot \mathbb{E}\left(\int_0^t (t-s)^{p\alpha-p}[C_1^p(\|x(s)\|^p + \|x(s-\tau)\|^p) + \|g(s,0,0,0)\|^p]ds\right) \\ &\le \frac{10^{p-1}D_p(T^{\frac{p}{2}-1}+1)T^{p\alpha-p+1}}{p\alpha-p+1}E_{\alpha,\alpha}(\mu T^\alpha)^p(C_1^p(2\|x\|^p_{\mathbb{H}^p} + \|\phi\|^p) + \|g(\cdot,0,0,0)\|^p_\infty). \end{aligned} \tag{19}$$

Submitting (3.4)-(3.8) into (3.3), which implies that $\|\mathcal{T}x\|_{\mathbb{H}^p} < \infty$. Thus, the operator $\mathcal{T}$ is well-defined.

Theorem 3.1 *Let* $1-\frac{1}{p} < \alpha < 1$. *Assume that* $(H1)$ *and* $(H2)$ *hold, then (1.3) has a unique solution* $x \in \mathbb{H}^p([0,T])$.

Proof. For $T > 0$, we choosing and fix a constant $\gamma > 0$ such that

$$\gamma > 2\cdot 3^{p-1}C_1^p E_{\alpha,\alpha}(\mu T^\alpha)^p(T^{\frac{p}{q}} + (D_p+1)T^{\frac{p}{2}-1}+1)\Gamma(p\alpha-p+1). \tag{20}$$

On the space $\mathbb{H}^p([0,T])$, we define a weighted norm $\|\cdot\|_\gamma$ as below

$$\|x\|_\gamma := \sup_{t\in[0,T]}\left(\frac{\mathbb{E}(\|x(t)\|^p)}{E_{p\alpha-p+1,1}(\gamma t^{p\alpha-p+1})}\right)^{\frac{1}{p}}, \quad \forall x \in \mathbb{H}^p([0,T]).$$

Similar to the Theorem 1 in [18], It is easy to know that the norms $\|\cdot\|_{\mathbb{H}^p}$ and $\|\cdot\|_\gamma$ are equivalent. Hence, $(\mathbb{H}^p([0,T]), \|\cdot\|_\gamma)$ is a Banach space. We can easily prove that $\mathcal{T} : \mathbb{H}^p([0,T]) \to \mathbb{H}^p([0,T])$ defined in (3.1) is uniformly bounded operator by Lemma 3.1. Next, we only check that $\mathcal{T}$ is a contraction operator.

Firstly, by using Hölder inequality, (H1) and Lemma 2.1, we obtain

$$\begin{aligned} &\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)(f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau)))ds\right\|^p \\ &\le \left(\int_0^t 1^q ds\right)^{\frac{p}{q}} \cdot \int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))\|^p ds \\ &\le t^{\frac{p}{q}}\int_0^t (t-s)^{p(\alpha-1)}E_{\alpha,\alpha}(\mu(t-s)^\alpha)^p\|f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))\|^p ds \\ &\le T^{\frac{p}{q}}E_{\alpha,\alpha}(\mu T^\alpha)^p C_1^p\int_0^t (t-s)^{p(\alpha-1)}(\|x(s)-y(s)\|^p + \|x(s-\tau)-y(s-\tau)\|^p)ds. \end{aligned} \tag{21}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

Secondly, similar to the proof of (3.7), one has

$$
\begin{aligned}
&\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)(\sigma(s,x(s),x(s-\tau))-\sigma(s,y(s),y(s-\tau)))dW(s)\right\|^p\\
&\leq T^{\frac{p}{2}-1}\int_0^t\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|\sigma(s,x(s),x(s-\tau))-\sigma(s,y(s),y(s-\tau))\|^p ds\\
&\leq T^{\frac{p}{2}-1}E_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p\alpha-p}[C_1^p(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau)\|^p)ds\\
&\leq T^{\frac{p}{2}-1}C_1^pE_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p\alpha-p}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau)\|^p)ds.
\end{aligned}
\tag{22}
$$

Thirdly, similar to the proof of (3.8), we obtain

$$
\begin{aligned}
&\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V(g(s,x(s),x(s-\tau),v)-g(s,y(s),y(s-\tau),v))\tilde{N}(ds,dv)\right\|\\
&\leq D_p\left(\int_0^t\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^2\int_V\|g(s,x(s),x(s-\tau),v)-g(s,y(s),y(s-\tau),v)\|^2\lambda(dv)ds\right)^{\frac{p}{2}}\\
&\quad+D_p\int_0^t|X^{A,B}_{\tau,\alpha,\alpha}(t-s)|^p\int_V|g(s,x(s),x(s-\tau),v)-g(s,y(s),y(s-\tau),v)|^p\lambda(dv)ds\\
&\leq D_p(T^{\frac{p}{2}-1}+1)\int_0^t\|X^{A,B}_{\tau,\alpha,\alpha}(T-s)\|^p\int_V\|g(s,x(s),x(s-\tau),v)-g(s,y(s),y(s-\tau),v)\|^p\lambda(dv)ds\\
&\leq D_pC_1^p(T^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p\alpha-p}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau\|^p)ds.
\end{aligned}
\tag{3.12}
$$

For each $x,y\in\mathbb{H}^p([0,T])$, from (3.1), (3.2), and (3.10)-(3.12), we have

$$
\begin{aligned}
&\mathbb{E}(\|\mathcal{T}x(t)-\mathcal{T}y(t)\|^p)\\
&\leq 3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)(f(s,x(s),x(s-\tau))-f(s,y(s),y(s-\tau)))ds\right\|^p\right)\\
&\quad+3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)(\sigma(s,x(s),x(s-\tau))-\sigma(s,y(s),y(s-\tau)))dW(s)\right\|^p\right)\\
&\quad+3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V(g(s,x(s),x(s-\tau),v)-g(s,y(s),y(s-\tau),v))\tilde{N}(ds,dv)\right\|^p\right)\\
&\leq 3^{p-1}T^{\frac{p}{q}}C_1^pE_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p(\alpha-1)}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau\|^p)ds\\
&\quad+3^{p-1}T^{\frac{p}{2}-1}C_1^pE_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau\|^p)ds\\
&\quad+3^{p-1}C_1^pD_p(T^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\mu T^\alpha)^p\int_0^t(t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau\|^p)ds\\
&=\omega\int_0^t(t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\tau)-y(s-\tau\|^p)ds,
\end{aligned}
\tag{3.13}
$$

where

$$\omega:=3^{p-1}C_1^pE_{\alpha,\alpha}(\mu T^\alpha)^p(T^{\frac{p}{q}}+(D_p+1)T^{\frac{p}{2}-1}+D_p).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

For $t > \tau$, one has

$$\int_0^t (t-s)^{p(\alpha-1)}\|x(s-\tau)-y(s-\tau)\|^p ds = \int_0^\tau + \int_\tau^t (t-s)^{p(\alpha-1)}\|x(s-\tau)-y(s-\tau)\|^p ds$$
$$= \int_\tau^t (t-s)^{p(\alpha-1)}\|x(s-\tau)-y(s-\tau)\|^p ds$$
$$= \int_0^{t-\tau} (t-\tau-u)^{p(\alpha-1)}\|x(u)-y(u)\|^p du. \tag{3.14}$$

From Lemma 2.2, combining (3.13) and (3.14), for each $t \in [0,T]$, we get

$$\frac{\mathbb{E}(\|\mathcal{T}x(t)-\mathcal{T}y(t)\|^p)}{E_{p\alpha-p+1,1}(\gamma t^{p\alpha-p+1})}$$
$$\leq \frac{\omega}{E_{p\alpha-p+1,1}(\gamma t^{p\alpha-p+1})}\int_0^t (t-s)^{p\alpha-p}E_{p\alpha-p+1,1}(\gamma s^{p\alpha-p+1})ds\|x-y\|_\gamma^p$$
$$+ \frac{\omega}{E_{p\alpha-p+1,1}(\gamma (t-\tau)^{p\alpha-p+1})}\int_0^{t-\tau} (t-\tau-u)^{p\alpha-p}E_{p\alpha-p+1,1}(\gamma u^{p\alpha-p+1})du\|x-y\|_\gamma^p$$
$$\leq \frac{2\omega\Gamma(p\alpha-p+1)}{\gamma}\|x-y\|_\gamma^p,$$

which implies that

$$\|\mathcal{T}x-\mathcal{T}y\|_\gamma \leq \rho\|x-y\|_\gamma,$$

where $\rho = \left(\frac{2\omega\Gamma(p\alpha-p+1)}{\gamma}\right)^{\frac{1}{p}}$.

Based on (3.9), one can obtain $\rho < 1$ and the operator $\mathcal{T}$ is a contractive. Thus, there exists a unique solution of (1.3) by using of the Banach fixed point theorem. The proof of this theorem is complete.

4. An averaging principle

In this section, we shall investigate the averaging principle for Caputo type FSDDEs. For any $t \in J$, we consider the following standard form of (1.3)

$$\begin{aligned} x_\epsilon(t) = {} & X^{A,B}_{\tau,\alpha,1}(t+\tau)\phi(-\tau) + \int_{-\tau}^0 X^{A,B}_{\tau,\alpha,\alpha}(t-s)[({}^C D^\alpha_{-\tau^+}\phi)(s) - A\phi(s)]ds \\ & + \epsilon\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)f(s,x(s),x(s-\tau))ds \\ & + \sqrt{\epsilon}\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\ & + \sqrt{\epsilon}\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv), \end{aligned} \tag{23}$$

where $\epsilon \in (0,\epsilon_0]$ is a positive small parameter with ϵ_0 being a fixed number.

Consider the averaged form which corresponds to the standard form (4.1) as follows :

 doi:10.20944/preprints202311.1668.v1

$$
\begin{aligned}
y_\epsilon(t) &= X^{A,B}_{\tau,\alpha,1}(t+\tau)\phi(-\tau) + \int_{-\tau}^{0} X^{A,B}_{\tau,\alpha,\alpha}(t-s)[{}^{C}D^{\alpha}_{-\tau^+}\phi)(s) - A\phi(s)]ds \\
&+ \epsilon \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\hat{f}(s,y(s),y(s-\tau))ds \\
&+ \sqrt{\epsilon} \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\hat{\sigma}(s,y(s),y(s-\tau))dW(s) \\
&+ \sqrt{\epsilon} \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s) \int_V \hat{g}(s,y(s),y(s-\tau),v)\tilde{N}(ds,dv),
\end{aligned}
\tag{24}
$$

where $\hat{f} : R^n \times R^n \to R^n$, $\hat{\sigma} : R^n \times R^n \to R^{n\times m}$, and $\hat{g} : R^n \times R^n \times V \to R^n$ satisfying the following averaging condition :

(H3) For any $t \in J$, $x, y \in \mathbb{R}^n$, and $p \geq 2$, there exists a positive bounded function $\varphi_i(\cdot)$, $i = 1,2,3$ such that

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \|f(s,x,y) - \hat{f}(x,y)\|^p ds \leq \varphi_1(t)(1 + \|x\|^p + \|y\|^p), \\
&\frac{1}{t}\int_0^t \|(t-s)^{\alpha-1}(\sigma(s,x,y) - \hat{\sigma}(x,y))\|^p ds \leq \varphi_2(t)(1 + \|x\|^p + \|y\|^p), \\
&\frac{1}{t}\int_0^t \left(\int_V \|(t-s)^{\alpha-1}(g(s,x,y,v) - \hat{g}(x,y,v))\|^p \lambda(dv)\right) ds \leq \varphi_3(t)(1 + \|x\|^p + \|y\|^p),
\end{aligned}
$$

where $\lim_{t\to\infty} \varphi_i(t) = 0$, $i = 1,2,3$.

Theorem 4.1. *Assume that (H1)-(H3) are satisfied. Then for a given arbitrary small number $\delta > 0$, $p \geq 2$ with $1 - \frac{1}{p} < \alpha < 1$, there exist $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\beta \in (0,1)$ such that*

$$
\mathbb{E}\left(\sup_{t\in[-\tau, L\epsilon^{-\beta}]} |x_\epsilon(t) - y_\epsilon(t)|^p\right) \leq \delta, \tag{25}
$$

for all $\epsilon \in (0, \epsilon_1]$.

Proof. If $p = 2$, it is easy to prove that (4.3) holds by using the similar method as in [20]. In the following, we will only consider the case of $p > 2$. From Eqs. (4.2), (4.3), and inequality (3.2), we obtain

$$
\begin{aligned}
\|x_\epsilon(t) - y_\epsilon(t)\|^p &\leq 3^{p-1}\epsilon^p \left\| \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau)) - \hat{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds \right\|^p \\
&+ 3^{p-1}\epsilon^{\frac{p}{2}} \left\| \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau)) - \hat{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dW(s) \right\|^p \\
&+ 3^{p-1}\epsilon^{\frac{p}{2}} \left\| \int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s) \int_V [g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)) - \hat{g}(x_\epsilon(s),x_\epsilon(s-\tau),v))]\tilde{N}(ds,dv) \right\|^p .
\end{aligned}
\tag{26}
$$

For any $t \in [0, u] \subset [0, T]$, taking the expectation on both sides Eq. (4.4), we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le u}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right)\\
&\le 3^{p-1}\epsilon^p\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-\hat{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds\right\|^p\right)\\
&\quad+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\hat{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dW(s)\right\|^p\right)\\
&\quad+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)\int_V[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v))-\hat{g}(x_\epsilon(s),x_\epsilon(s-\tau),v))]\tilde{N}(ds,dv)\right\|^p\right)\\
&=I_1+I_2+I_3.
\end{aligned} \tag{27}$$

Applying Jensen's inequality, we get

$$\begin{aligned}
I_1 &\le 6^{p-1}\epsilon^p\cdot\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[f(s,x_\epsilon(s),x_\epsilon(s-\tau))-f(s,y_\epsilon(s),y_\epsilon(s-\tau))]ds\right\|^p\right)\\
&\quad+6^{p-1}\epsilon^p\cdot\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[f(s,y_\epsilon(s),y_\epsilon(s-\tau))-\hat{f}(y_\epsilon(s),y_\epsilon(s-\tau))]ds\right\|^p\right)\\
&=I_{11}+I_{12}.
\end{aligned} \tag{28}$$

Thanks to Hölder inequality and (H2), we obtain

$$\begin{aligned}
I_{11} &\le 6^{p-1}\epsilon^p\left(\int_0^u 1^q ds\right)^{\frac{p}{q}}\\
&\qquad\cdot\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|f(s,x_\epsilon(s),x_\epsilon(s-\tau))-f(s,y_\epsilon(s),y_\epsilon(s-\tau))\|^p ds\right)\\
&\le 6^{p-1}\epsilon^p u^{p-1}C_1^p E_{\alpha,\alpha}(\mu u^\alpha)^p\\
&\qquad\cdot\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t(t-s)^{p(\alpha-1)}[\|x_\epsilon(s)-y_\epsilon(s)\|^p+\|x_\epsilon(s-\tau))-y_\epsilon(s-\tau))\|^p]ds\right)\\
&\le 6^{p-1}\epsilon^p u^{p-1}C_1^p E_{\alpha,\alpha}(\mu u^\alpha)^p\\
&\qquad\cdot\int_0^u(u-s)^{p(\alpha-1)}\left[\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)\|^p\right)\right]ds\\
&\le 2\cdot 6^{p-1}\epsilon^p u^{p-1}C_1^p E_{\alpha,\alpha}(\mu u^\alpha)^p\int_0^u(t-s)^{p(\alpha-1)}\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds,
\end{aligned} \tag{29}$$

since

$$\sup_{0\le\theta\le s}\|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)\|^p\le\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

Applying Hölder inequality, we obtain

$$\begin{aligned}
I_{12} &\leq 6^{p-1}\epsilon^p \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\left(\left(\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^q ds\right)^{\frac{p}{q}}\right.\right.\\
&\qquad \left.\left.\cdot \int_0^t \|f(s,y_\epsilon(s),y_\epsilon(s-\tau)) - \hat{f}(y_\epsilon(s),y_\epsilon(s-\tau))\|^p ds\right)\right)\\
&\leq 6^{p-1}\epsilon^p E_{\alpha,\alpha}(\mu u^\alpha)^p \left(\frac{u^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}} \\
&\qquad \cdot u\|\varphi_1\|_\infty \left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t)\|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t-\tau)\|^p\right)\right]\\
&= 6^{p-1}\|\varphi_1\|_\infty M_1(q\alpha-q+1)^{-(p-1)}\epsilon^p E_{\alpha,\alpha}(\mu u^\alpha)^p u^{p\alpha},
\end{aligned} \tag{30}$$

here $\|\varphi_1\|_\infty = \sup_{t\in[0,u]}|\varphi_1(t)|$, $M_1 = 1+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t)\|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t-\tau)\|^p\right)$.

For the second term I_2, we have

$$\begin{aligned}
I_2 &\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))]dW(s)\right\|^p\right)\\
&\quad + 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left\|\int_0^t X^{A,B}_{\tau,\alpha,\alpha}(t-s)[\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))-\hat{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))]dW(s)\right\|^p\right)\\
&= I_{21}+I_{22}.
\end{aligned} \tag{31}$$

In view of the Burkholder-Davis-Gundy's inequality, Hölder's inequality and Doob's martingale inequality, and (H1), one has

$$\begin{aligned}
I_{21} &\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^2\|\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))\|^2 ds\right)^{\frac{p}{2}}\\
&\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left(\left(\int_0^t 1^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}\cdot\frac{p}{2}}\right.\right.\\
&\qquad \left.\left.\cdot\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|\sigma(s,x_\epsilon(s),x_\epsilon(s-\tau))-\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))\|^p ds\right)\right)\\
&\leq 6^{p-1}\epsilon^{\frac{p}{2}}C_1^p u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\mu u^\alpha)\\
&\qquad \cdot\int_0^u (u-s)^{p\alpha-p}\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)\|^p\right)\right]ds\\
&\leq 2\cdot 6^{p-1}\epsilon^{\frac{p}{2}}C_1^p u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\mu u^\alpha)\int_0^u (u-s)^{p\alpha-p}\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds.
\end{aligned} \tag{32}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

Applying (H3) and an estimation method similar to Eq. (4.10), we get

$$\begin{aligned} I_{22} &\le 6^{p-1}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}\mathbb{E}\left(\sup_{0\le\theta\le u}\int_0^t \|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|\sigma(s,y_\epsilon(s),y_\epsilon(s-\tau))-\hat{\sigma}(y_\epsilon(s),y_\epsilon(s-\tau))\|^p ds\right) \\ &\le 6^{p-1}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\mu u^\alpha)^p u\|\varphi_2\|_\infty\left[1+\mathbb{E}\left(\sup_{0\le t\le u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\le t\le u}|y_\epsilon(t-\tau)|^p\right)\right] \\ &= 6^{p-1}M_1\|\varphi_2\|_\infty\epsilon^{\frac{p}{2}}E_{\alpha,\alpha}(\mu u^\alpha)^p u^{\frac{p}{2}}. \end{aligned} \tag{33}$$

For the third term I_3, we have

$$\begin{aligned} I_3 &\le 3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\left\|\int_V X^{A,B}_{\tau,\alpha,\alpha}(t-s)[g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)]\bar{N}(ds,dv)\right\|^p\right) \\ &+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\left\|\int_V X^{A,B}_{\tau,\alpha,\alpha}(t-s)[g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\hat{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)]\bar{N}(ds,dv)\right\|^p\right) \\ &= I_{31}+I_{32}. \end{aligned} \tag{34}$$

From Lemma 2.3, similar to the proof of (3.8), one has

$$\begin{aligned} I_{31} &\le 3^{p-1}\epsilon^{\frac{p}{2}}D_p\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^2\|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)\|^2\lambda(dv)ds\right)^{\frac{p}{2}} \\ &+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)\|^p\lambda(dv)ds\right) \\ &\le 3^{p-1}\epsilon^{\frac{p}{2}}(D_pu^{\frac{p}{2}-1}+1) \\ &\quad\cdot\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|g(s,x_\epsilon(s),x_\epsilon(s-\tau),v)-g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)\|^p\lambda(dv)ds\right) \\ &\le 3^{p-1}\epsilon^{\frac{p}{2}}(D_pu^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\mu u^\alpha)^pC_1^p \\ &\quad\cdot\int_0^u(u-s)^{p\alpha-p}\left[\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)\|^p\right)\right]ds \\ &\le 2\cdot3^{p-1}C_1^p\epsilon^{\frac{p}{2}}(D_pu^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\mu u^\alpha)^p\int_0^u(t-s)^{p\alpha-p}\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds. \end{aligned} \tag{35}$$

Moreover, by (H3), we also have

$$\begin{aligned} I_{32} &\le 3^{p-1}\epsilon^{\frac{p}{2}}D_p\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^2\|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\hat{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)\|^2\lambda(dv)ds\right)^{\frac{p}{2}} \\ &+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\hat{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)\|^p\lambda(dv)ds\right) \\ &\le 3^{p-1}\epsilon^{\frac{p}{2}}(D_pu^{\frac{p}{2}-1}+1) \\ &\quad\cdot\mathbb{E}\left(\sup_{0\le t\le u}\int_0^t\int_V\|X^{A,B}_{\tau,\alpha,\alpha}(t-s)\|^p\|g(s,y_\epsilon(s),y_\epsilon(s-\tau),v)-\hat{g}(y_\epsilon(s),y_\epsilon(s-\tau),v)\|^p\lambda(dv)ds\right) \\ &\le 3^{p-1}\epsilon^{\frac{p}{2}}(D_pu^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\mu u^\alpha)^pu\|\varphi_3\|_\infty\left[1+\mathbb{E}\left(\sup_{0\le t\le u}\|y_\epsilon(t)\|^p\right)+\mathbb{E}\left(\sup_{0\le t\le u}\|y_\epsilon(t-\tau)\|^p\right)\right] \\ &\le 3^{p-1}M_1\|\varphi_3\|_\infty\epsilon^{\frac{p}{2}}E_{\alpha,\alpha}(\mu u^\alpha)^p(D_pu^{\frac{p}{2}}+u). \end{aligned} \tag{4.14}$$

From (4.5)-(4.14), for $u \in (0, T]$, we obtain

$$\mathbb{E}\left(\sup_{0\le t\le u}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right) \le A(u) + B(u)\int_0^u (u-s)^{p\alpha-p}\mathbb{E}\left(\sup_{0\le\theta\le s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds, \tag{4.15}$$

where

$$\begin{aligned} A(u) &= 6^{p-1}\|\varphi_1\|_\infty M_1(q\alpha-q+1)^{-(p-1)}\epsilon^p E_{\alpha,\alpha}(\mu u^\alpha)^p u^{p\alpha} \\ &\quad + 6^{p-1}M_1\|\varphi_2\|_\infty \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\mu u^\alpha)^p u^{\frac{p}{2}} \\ &\quad + 3^{p-1}M_1\|\varphi_3\|_\infty \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\mu u^\alpha)^p (D_p u^{\frac{p}{2}} + u), \end{aligned}$$

and

$$\begin{aligned} B(u) &= 2\cdot 6^{p-1}C_1^p\epsilon^p E_{\alpha,\alpha}(\mu u^\alpha)^p u^{p-1} + 2\cdot 6^{p-1}C_1^p\epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\mu u^\alpha)^p u^{\frac{p}{2}-1} \\ &\quad + 2\cdot 3^{p-1}C_1^p\epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\mu u^\alpha)^p (D_p u^{\frac{p}{2}-1} + 1). \end{aligned}$$

By using of Lemma 2.4, we get

$$\mathbb{E}\left(\sup_{0\le t\le u}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right) \le A(u)E_{p(\alpha-1)+1}\left(B(u)\Gamma(p(\alpha-1)+1)u^{p(\alpha-1)+1}\right).$$

Choose $L > 0$ and $\beta \in (0,1)$ such that for all $t \in (0, L\epsilon^{-\beta}] \subset (0, T]$ satisfies the following

$$\mathbb{E}\left(\sup_{0< t\le L\epsilon^{-\beta}}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right) \le \bar{A}(\epsilon)E_{p(\alpha-1)+1}(\bar{B}(\epsilon)\Gamma(p(\alpha-1)+1))\epsilon^{1-\beta},$$

where

$$\begin{aligned} \bar{A}(\epsilon) &= 6^{p-1}E_{\alpha,\alpha}(\mu T^\alpha)^p\|\varphi_1\|_\infty M_1(q\alpha-q+1)^{-(p-1)}L^{p\alpha}\epsilon^{p(1-\alpha\beta)} \\ &\quad + 6^{p-1}M_1\|\varphi_2\|_\infty E_{\alpha,\alpha}(\mu T^\alpha)^p L^{\frac{p}{2}}\epsilon^{\frac{p}{2}(1-\beta)} \\ &\quad + 3^{p-1}M_1\|\varphi_3\|_\infty E_{\alpha,\alpha}(\mu T^\alpha)^p (D_p L^{\frac{p}{2}}\epsilon^{\frac{p}{2}(1-\beta)} + L\epsilon^{\frac{p}{2}-\beta}), \end{aligned}$$

and

$$\begin{aligned} \bar{B}(\epsilon) &= 2\cdot 6^{p-1}C_1^p E_{\alpha,\alpha}(\mu T^\alpha)^p L^{p-1}\epsilon^{p-(p-1)\beta} \\ &\quad + 2\cdot 6^{p-1}C_1^p E_{\alpha,\alpha}(\mu T^\alpha)^p L^{\frac{p}{2}-1}\epsilon^{\frac{p}{2}-(\frac{p}{2}-1)\beta} \\ &\quad + 2\cdot 3^{p-1}C_1^p E_{\alpha,\alpha}(\mu T^\alpha)^p (D_p L^{\frac{p}{2}-1}\epsilon^{\frac{p}{2}-(\frac{p}{2}-1)\beta} + \epsilon^{\frac{p}{2}}). \end{aligned}$$

are two constants. Thus, for any given number $\delta > 0$, there exists $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and $t \in [0, L\epsilon^{-\beta}] \subset J$,

$$\mathbb{E}\left(\sup_{t\in[0,L\epsilon^{-\beta}]}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right) \le \delta.$$

Remark 4.1. If $p = 2$ and $g \equiv 0$, then FSDDEs (1.3) reduces to FSDEs (1.1) in [14]. Therefore, Theorem 3.1 generalizes the main result of [14].

By using Theorem 4.1 and Chebyshev-Markov inequality, we can obtain the following Corollary.

Corollary 4.1. *Assume that (H1)-(H3) are satisfied. Then for a given arbitrary small number* $\delta > 0$, $p \ge 2$ *with* $1 - \frac{1}{p} < \alpha < 1$, *then for arbitrarily number* $\bar{\delta} > 0$ *such that for* $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ *and* $\beta \in (0,1)$ *satisfying for all* $\epsilon \in (0, \epsilon_1]$

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,L\epsilon^{-\beta}]}\|x_\epsilon(t)-y_\epsilon\|^p > \bar{\delta}\right) = 0.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

5. An Example

Example 5.1. Consider the following Caputo fractional stochastic delay differential equation (FSDDEs) with Poisson jumps :

$$\begin{cases} ({}^{C}D_0^{0.7}x)t) = Ax(t) + Bx(t-0.4) + f(t,x(t),x(t-0.4)) + \sigma(t,x(t),x(t-0.4))\frac{dW(t)}{dt} \\ \qquad\qquad + \int_V g(t,x(t),x(t-0.4),v)\tilde{N}(dt,dv), \quad t \in J, \\ x(t) = \phi(t), \quad -0.4 \le t \le 0, \end{cases} \tag{36}$$

where $\alpha = 0.9$, $\tau = 0.4$, $J = [0,4]$, $x(t) = (x_1(t), x_2(t))^T$, and

$$A = \begin{pmatrix} 0.3 & 0.1 \\ 0.15 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix},$$

and

$$f(t,x(t),x(t-0.4)) = \begin{pmatrix} \frac{1}{3}e^{-2t}\sin(x_1(t)) + \frac{1}{4}e^{-t}\sin^3 t \arctan(x_1(t-0.4)) + \frac{1}{7} \\ \frac{1}{3}e^{-2t}\cos(x_2(t)) + \frac{1}{4}e^{-t}\cos^3 t \arctan(x_2(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$\sigma(t,x(t),x(t-0.4)) = \begin{pmatrix} \frac{1}{4}e^{-t}\arctan(x_1(t)) + \frac{1}{3}e^{-2t}\cos^2 t \sin(x_1(t-0.4)) + \frac{1}{3} \\ \frac{1}{4}e^{-t}\sin(x_2(t)) + \frac{1}{3}e^{-t}\sin^2 t \arctan(x_2(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$g(t,x(t),x(t-0.4),v) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

For each $x(t), y(t) \in \mathrm{Y}$ and $t \in [0,T]$, we have

$$\begin{aligned} &\|f(t,x(t),x(t-0.4)) - f(t,y(t),y(t-0.4))\| \\ &\quad\le \frac{1}{3}|x_1(t) - y_1(t)| + \frac{1}{4}|x_1(t-0.4) - y_1(t-0.4)| + \frac{1}{3}|x_2(t) - y_2(t)| + \frac{1}{4}|x_2(t-0.4) - y_2(t-0.4)| \\ &\quad\le \frac{1}{3}(\|x(t) - y(t)\| + \|x(t-0.4) - y(t-0.4)\|). \end{aligned}$$

Thus

$$\|f(t,x(t),x(t-0.4)) - f(t,y(t),y(t-0.4))\|^3 \le \frac{2^2}{3^3}(\|x(t) - y(t)\|^3 + \|x(t-0.4) - y(t-0.4)\|^3),$$

which implies that the function f satisfy the assumption (H1) and (H2). Similarly, we can obtain that the functions σ and g satisfy the assumption (H1) and (H2).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

Let $p = 3$. By calculation, we have $\mu = \|A\| + \|B\| = 0.8$, $\|f\|_{\mathbb{L}^p} = \int_0^4 \|f(t,0,0)\|^3 dt = 0.0651$, $\|\sigma(\cdot,0,0)\|_\infty = \frac{1}{2}$, $\|g(\cdot,0,0)\|_\infty = \frac{5}{6}$, $C_1 = \frac{4}{27}$ and

$$\begin{aligned}
\Xi &= \left(\int_{-0.4}^{0} \|({}^C D_{-0.4^+}^{0.9}\phi)(s) - A\phi(s)\|^{\frac{3}{2}} ds\right)^2 \\
&\le \left(\sqrt{2}\int_{-0.4}^{0} (\|({}^C D_{-0.4^+}^{0.9}\phi)(s)\|^{\frac{3}{2}} + \|A\phi(s)\|^{\frac{3}{2}}) ds\right)^2 \\
&\le \left(\sqrt{2}\int_{-0.4}^{0}\left(\left\|\begin{pmatrix} \frac{1}{\Gamma(0.1)}\int_{-0.4}^{s}(s-t)^{-0.9}dt \\ \frac{1}{2\Gamma(0.1)}\int_{-0.4}^{s}(s-t)^{-0.9}dt \end{pmatrix}\right\|^{\frac{3}{2}} + \left\|\begin{pmatrix} 0.35 \\ 0.25 \end{pmatrix}\right\|^{\frac{3}{2}}\right)\right)^2 \\
&\le 2\left(\int_{-0.4}^{0}\left(\left(\frac{3}{2\Gamma(1.1)}\right)^{\frac{3}{2}} (s+0.4)^{0.15} + 0.6^{1.5}\right) ds\right)^2 = 1.5722.
\end{aligned}$$

Hence, we may choose a suitable value $\gamma > 0$ such that

$$2 \cdot 3^2 C_1^3 E_{0.9,0.9}(0.8 \cdot 4^{0.9})^3 (4^2 + (D_3+1)2 + 1)\Gamma(0.7) < \gamma.$$

By Theorem 3.1, FSDDEs (5.1) has a unique solution $x \in \mathbb{H}^3([0,4])$.

In the following, we consider the standard form (4.1) as follows

$$\begin{cases} ({}^C D_0^{0.7} x_\epsilon)t) = Ax_\epsilon(t) + Bx_\epsilon(t-0.4) + \epsilon f(t, x_\epsilon(t), x_\epsilon(t-0.4)) + \sqrt{\epsilon}\sigma(t, x_\epsilon(t), x_\epsilon(t-0.4))\frac{dW(t)}{dt} \\ \qquad\qquad + \sqrt{\epsilon}\int_V g(t, x_\epsilon(t), x_\epsilon(t-0.4), v)\bar{N}(dt, dv), \quad t \in J, \\ x_\epsilon(t) = \phi(t), \quad -0.4 \le t \le 0, \end{cases} \tag{37}$$

where $x_\epsilon(t) = (x_{1,\epsilon}(t), x_{2,\epsilon}(t))^T$, and

$$f(t, x_\epsilon(t), x_\epsilon(t-0.4)) = \begin{pmatrix} \frac{1}{3}e^{-2t}\sin(x_{1,\epsilon}(t)) + \frac{1}{4}e^{-t}\sin^3 t \arctan(x_{1,\epsilon}(t-0.4)) + \frac{1}{7} \\ \frac{1}{3}e^{-2t}\cos(x_{2,\epsilon}(t)) + \frac{1}{4}e^{-t}\cos^3 t \arctan(x_{2,\epsilon}(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$\sigma(t, x_\epsilon(t), x_\epsilon(t-0.4)) = \begin{pmatrix} \frac{1}{4}e^{-t}\arctan(x_{1,\epsilon}(t)) + \frac{1}{3}e^{-2t}\cos^2 t \sin(x_{1,\epsilon}(t-0.4)) + \frac{1}{3} \\ \frac{1}{4}e^{-t}\sin(x_{2,\epsilon}(t)) + \frac{1}{3}e^{-2t}\sin^2 t \arctan(x_{2,\epsilon}(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$g(t, x_\epsilon(t), x_\epsilon(t-0.4), v) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

Under conditions (H1) and (H2), by Theorem 3.1, FSDDEs (5.2) has a unique solution x_ϵ given by

$$\begin{aligned}
x_\epsilon(t) \quad &= X_{0.4,0.9,1}^{A,B}(t+0.4)\phi(-0.4) + \int_{-0.4}^{0} X_{0.4,0.9,0.9}^{A,B}(t-s)[{}^C D_{-0.4^+}^{0.9}\phi)(s) - A\phi(s)]ds \\
&+\epsilon\int_0^t X_{0.4,0.9,0.9}^{A,B}(t-s) f(s, x_\epsilon(s), x_\epsilon(s-0.4))ds \\
&+\sqrt{\epsilon}\int_0^t X_{0.4,0.9,0.9}^{A,B}(t-s)\sigma(s, x_\epsilon(s), x_\epsilon(s-0.4))dW(s) \\
&+\sqrt{\epsilon}\int_0^t X_{0.4,0.9,0.9}^{A,B}(t-s)\int_V g(s, x_\epsilon(s), x_\epsilon(s-0.4), v)\bar{N}(ds, dv).
\end{aligned} \tag{38}$$

By calculation, one has

$$\bar{f}(x_\varepsilon(t), x_\varepsilon(t-\tau)) = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(s, x_\epsilon(s), x_\epsilon(s-\tau))ds = \begin{pmatrix} \frac{1}{7} \\ \frac{1}{6} \end{pmatrix},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 November 2023 doi:10.20944/preprints202311.1668.v1

$$\bar{\sigma}(x_\varepsilon(t), x_\varepsilon(t-\tau)) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma(s, x_\varepsilon(s), x_\varepsilon(s-\tau))ds = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \end{pmatrix},$$

$$\bar{g}(x_\varepsilon(t), x_\varepsilon(t-\tau), v) = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(s, x_\varepsilon(s), x_\varepsilon(s-\tau), v)ds = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

We now check that the condition (H3) is satisfied. In fact, one has

$$\begin{aligned}
&\frac{1}{t}\int_0^t \|f(s, x_\varepsilon(s), x_\varepsilon(s-\tau)) - \bar{f}(x_\varepsilon(s), x_\varepsilon(s-\tau))\|^p ds \\
&= \frac{1}{t}\int_0^t \left\| \begin{pmatrix} \frac{1}{3}e^{-2s}\sin(x_{1,\epsilon}(s)) + \frac{1}{4}e^{-s}\sin^3 s \arctan(x_{1,\epsilon}(s-0.4)) \\ \frac{1}{3}e^{-2s}\cos(x_{2,\epsilon}(s)) + \frac{1}{4}e^{-s}\cos^3 s \arctan(x_{2,\epsilon}(s-0.4)) \end{pmatrix} \right\|^p ds \\
&= \frac{1}{3^p t}\int_0^t (e^{-s}(|x_{1,\epsilon}(s)| + |x_{2,\epsilon}(s)|) + e^{-s}(|x_{1,\epsilon}(s-0.4)| + |x_{2,\epsilon}(s-0.4)|))^p ds \\
&= \frac{1}{3^p t}\int_0^t (e^{-ps}(\|x_\epsilon(s)\| + \|x_\epsilon(s-0.4)\|)^p ds \\
&\le \frac{2^{p-1}}{3^p t}(\|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p)\int_0^t e^{-ps}ds \\
&= \frac{2^{p-1}(1-e^{-pt})}{3^p pt}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{t}\int_0^t \|(t-s)^{\alpha-1}(\sigma(s, x_\epsilon(s), x_\epsilon(s-\tau)) - \bar{\sigma}(x_\epsilon(s), x_\epsilon(s-\tau)))\|^p ds \\
&= \frac{1}{t}\int_0^t \left\| \begin{pmatrix} \frac{1}{4}(t-s)^{\alpha-1}e^{-s}\arctan(x_{1,\epsilon}(s)) + \frac{1}{3}(t-s)^{\alpha-1}e^{-2s}\cos^2 s \sin(x_{1,\epsilon}(s-0.4)) \\ \frac{1}{4}(t-s)^{\alpha-1}e^{-s}\sin(x_{2,\epsilon}(s)) + \frac{1}{3}(t-s)^{\alpha-1}e^{-s}\sin^2 s \arctan(x_{2,\epsilon}(s-0.4)) \end{pmatrix} \right\|^p ds \\
&\le \frac{2^{p-1}}{3^p t}(\|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p)\int_0^t (t-s)^{p\alpha-p}ds \\
&= \frac{2^{p-1}}{(p\alpha-p+1)3^p}t^{p\alpha-p}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{t}\int_0^t \left(\int_V \|(t-s)^{\alpha-1}(g(s, x_\epsilon(s), x_\epsilon(s-0.4), v) - \hat{g}(x_\epsilon(s), x_\epsilon(s-0.4), v)\|^p \lambda(dv) \right) ds \\
&= \frac{1}{t}\int_0^t \int_V \left\| \begin{pmatrix} \frac{1}{2}(t-s)^{\alpha-1} \\ \frac{1}{3}(t-s)^{\alpha-1} \end{pmatrix} \right\|^p \lambda(dv)ds \\
&= \frac{1}{t}\left(\frac{5}{6}\right)^p \lambda(V)\int_0^t (t-s)^{p(\alpha-1)}ds \\
&\le \frac{5^p\lambda(V)}{(p\alpha-p+1)6^p}t^{p\alpha-p}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p).
\end{aligned}$$

Thus, (H3) is satisfied with

$$\varphi_1(t) = \frac{2^{p-1}(1-e^{-pt})}{3^p pt}, \quad \varphi_2(t) = \frac{2^{p-1}}{(p\alpha-p+1)3^p}t^{p\alpha-p}, \quad \text{and } \varphi_3(t) = \frac{5^p|\Omega|}{(p\alpha-p+1)6^p}t^{p\alpha-p}.$$

It is easy to check that the conditions of Theorem 4.1 and Corollary 4.1 are satisfied. So, as $\epsilon \to 0$, the original solution $x_\epsilon(\cdot) \to y_\epsilon(\cdot)$ in the sense of p square ($p = 3$) and in the probability, where

$$\begin{aligned} y_\epsilon(t) &= X^{A,B}_{0.4,0.9,1}(t+0.4)\phi(-0.4) + \int_{-0.4}^{0} X^{A,B}_{0.4,0.9,0.9}(t-s)[{}^{C}D^{0.9}_{-0.4^+}\phi)(s) - A\phi(s)]ds \\ &+\epsilon \int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\hat{f}(y_\epsilon(s), y_\epsilon(s-0.4))ds \\ &+\sqrt{\epsilon}\int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\hat{\sigma}(y_\epsilon(s), y_\epsilon(s-0.4))dW(s) \\ &+\sqrt{\epsilon}\int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\int_V \hat{g}(y_\epsilon(s), y_\epsilon(s-0.4), v)\tilde{N}(ds,dv). \end{aligned} \tag{39}$$

6. Conclusion

In this article, we established and proved the existence and uniqueness theorem for solutions of Caputo type fractional stochastic delay differential systems (FSDDSs) with Poisson jumps. By utilizing Hölders inequality, Jensen's inequality, Burkholder-Davis-Gundys inequality, Doobs martingale inequality and fractional Gronwall's inequality, we proved the averaging principle for FSDDEs in the sense of L^p. Our results generalize the cases of $p = 2$ and enriched the field of fractional stochastic delay differential equations. Finally, we provided an example to show the usefulness of our results.

Author Contributions: Conceptualization, Z.B. and C.B.; formal analysis, Z.B.; investigation, Z.B., C.B.; and writing—review and editing, Z.B. and C.B. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by Natural Science Foundation of China (11571136).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that there are no conflicts of interest.

References

1. Zhu, Q. Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control. *IEEE T. Automat. Contr.* **2019**, 64, 3764-3771.
2. Zhu, Q.; Huang, T. Stability analysis for a class of stochastic delay nonlinear systems driven by G-Brownian motion. *Systems Control Lett.* **2020**, 140, 104699.
3. Arthi, G.; Suganya, K. Controllability of higher order stochastic fractional control delay systems involving damping behavior. *Appl. Math. Comput.* **2021**, 410, 126439.
4. Chen, Z.; Wang, B. Existence, exponential mixing and convergence of periodic measures of fractional stochastic delay reaction-diffusion equations on $\mathbb{R}^n$. *J. Differ. Equations* **2022**, 336, 505-564.
5. Xu, L.; Li, Z. Stochastic fractional evolution equations with fractional brownian motion and infinite delay. *Appl. Math. Comput.* **2018**, 336, 36-46.
6. Li, M.; Niu, Y.; Zou, J. A result regarding finite-time stability for Hilfer fractional stochastic differential equations with delay. *Fractal Fract.* **2023**, 7, 622.
7. Khasminskii, R.Z. On the principle of averaging the Itö stochastic differential equations. *Kibernetika* **1968**, 4, 260-279.
8. Xu, Y.; Duan, J.Q.; Xu, W. An averaging principle for stochastic dynamical systems with Lévy noise. *Physica D.* **2011**, 240, 1395-1401.
9. Xu, W.; Xu, W.; Zhang, S. The averaging principle for stochastic differential equations with Caputo fractional derivative. *Appl. Math. Lett.* **2019**, 93, 79-84.
10. Luo, D.; Zhu, Q.; Luo, Z. An averaging principle for stochastic fractional differential equations with time-delays. *Appl. Math. Lett.* **2020**, 105, 106290.
11. Ahmed, H.M.; Zhu, Q. The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps. *Appl. Math. Lett.* **2021**, 112, 106755.
12. Ahmed, H.M. Impulsive conformable fractional stochastic differential equations with poisson jumps. *Evolution Equations and Control Theory* **2022**, 11, 2073-2080.
13. Wang, Z.; Lin, P. Averaging principle for fractional stochastic differential equations with L^p convergence. *Appl. Math. Lett.* **2022**, 130, 108024.

 doi:10.20944/preprints202311.1668.v1

14. Xu, W.; Xu, W.; Lu, K. An averaging principle for stochastic differential equations of fractional order $0 < \alpha < 1$. *Fract. Calc. Appl. Anal.* **2020**, 23, 908-919.
15. Yang, D.; Wang, J.; Bai, C. Averaging principle for ψ-Capuo fractional stochastic delay differential equations with Poisson jumps. *Symmetry* **2023**, 15, 1346.
16. Li, M.; Wang, J. The existence and averaging principle for Caputo fractional stochastic delay differential systems. *Fract. Calc. Appl. Anal.* **2023**, 26, 893-912.
17. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier, Amsterdam, 2006.
18. Mahmudov, N.I. Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations. *Math. Meth. Appl. Sci.* **2019**, 42, 5489-5497.
19. You, Z.; Fečkan, M.; Wang, J. Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions. *J. Comput. Appl. Math.* **2020**, 378, 112939.
20. Son, D.; Huong, P.; Kloeden, P.; Tuan, H. Asymptotic separation between solutions of Caputo fractional stochastic differential equations. *Stoch. Anal. Appl.* **2018**, 36, 654-664.
21. Applebaum, D. *Lévy Process and Stochastic Calculus*; Cambridge University Press : Cambridge, UK, 2009.
22. Kunita, H. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In Real and Stochastic Analysis, New Perspectives, Birkhauser: Basel, Switzerland, 2004, pp. 305-373.
23. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, 328, 1075-1081.