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Article

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Abstract: Using the projection evolution (PEv) approach, time can be included in the quantum mechanics as an observable. Having the time operator, it is possible to explore the temporal structure of various quantum events. In the present paper we discuss the possibility of constructing a quantum clock, which advances in time during its quantum evolution, in each step having some probability to localize itself on the time axis in the new position. We propose a working two-state model as the simplest example of such a clock.

Keywords: foundation of quantum mechanics; quantum time; quantum clock

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1. Introduction

Time is one of the most important feature of our physical world. Its measurement is a fundamental procedure not only for physics. At every step of technical development there are projects of investigation leading to construction of better and better clocks which are able to measure extremaly small time intervals [1–4].

In the book [5] three types of time are discussed: time as a parameter, dynamical time and time as a quantum observable. The most consistent with quantum mechanics is the last concept, i.e., time as a quantum observable considered on the same footing as the other position operators. In physics, especially in the quantum regime, one requires more and more precise measurements of time to better understand and predict evolution of such systems. On the other hand, to measure very short time intervals one needs to construct a clock which, in fact, is a specific quantum system working according to quantum rules and, in addition, is often (weakly) coupled to the measured system.

For many years time was treated in physics as a universal parameter enumerating evolution of physical systems. The special and general relativity change this notion substantially. In quantum mechanics, however, especially after the publication of the Pauli theorem [6,7] time is still considered as an evolution parameter.

An overview of the role of time in quantum mechanics can be found, among others, in [8–41].

In this paper we treat time to be a component of a composite quantum observable of the spacetime position. To be consistent, we use the projection evolution model (PEv) which recent version can be found in [42,43]. These references present also more extended introduction to the problem and the corresponding bibliography.

According to the PEv approach quantum time and space positions should be considered on the same footing. However, one needs to realize that we have direct access to the space position observable but only indirect access to the time position observable on the time axis. The latter requires special measuring devices called clocks. A proper clock definition is vital for investigating the structure of the real spacetime.

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2. Projection Evolutsectionion

For readers' convenience we summarize the main points of the PEv approach.

The main assumption of the PEv evolution model is the **changes principle** which states that:

The evolution of a system is a random process caused by spontaneous changes in the Universe. These spontaneous changes are primary processes in the Universe.

In this approach the changes in the quantum state space happen according to a probability distribution which is dictated by the properties of the Universe and its subsystems. A very important attribute of our Universe is the spacetime itself, which should emerge from the quantum state space. In this article we consider the flat spacetime.

Every single step of the evolution describes the state of a physical subsystem (possibly the Universe as a whole) in this state space. It means that the evolution is not driven by time, which is a part of the system's description at a given evolution step, but it is driven by an extra parameter τ. This parameter is not time; it belongs to a linearly ordered set with no aditional structure required.

In the following we are using τ which can be represented by integers $\mathbb{Z}$. This allows to order quantum events and to have the notion of the predecessor and the successor in any set of physical events. As a consequence one may expect the existence of a kind of pseudo-causality based on the ordering of the quantum events, which leads to the causality principle in the case of macroscopic physical systems. An additional, very important feature of the PEv approach is that this idea does not need the spacetime as the background, it is background free. The spacetime is "generated" by the spacetime position observable.

In the projection evolution formalism we propose to use the generalized form of the Lüders [44] type of the projection postulate as an evolution principle. It allows to reproduce, as a special case, the standard unitary evolution represented by relativistic and non-relativistic evolution equations.

In the following we introduce the evolution operators which can characterize a given physical system. They are formally responsible for quantum evolution of this object. In this paper we are using the evolution operators represented by an orthogonal resolution of unity:

$$\begin{aligned} &\text{Ɇ}(\tau_n;\nu_n)^\dagger = \text{Ɇ}(\tau_n;\nu_n),\\ &\text{Ɇ}(\tau_n;\nu_n)\text{Ɇ}(\tau_n;\nu'_n) = \delta_{\nu_n\nu'_n}\text{Ɇ}(\tau_n;\nu_n),\\ &\sum_{\nu_n\in\mathcal{Q}_n}\text{Ɇ}(\tau_n;\nu_n) = \hat{\mathbf{1}}\,, \end{aligned} \tag{1}$$

where $\hat{\mathbf{1}}$ denotes the unit operator and $\nu_n \in \mathcal{Q}_n$ represents a set of quantum numbers describing quantum states. Different alternatives representing random choices of quantum states are described by different sets of quantum numbers ν_n. Such type of operators are able to describe a wide class of physical systems.

Assume now, that the vector $|\Phi(\tau_{n-1};\nu_{n-1})\rangle$ represents a given quantum state at the evolution step τ_{n-1}, where $\nu_{n-1} \in \mathcal{Q}_{n-1}$.

The changes principle implies that there exists in the Universe a physical mechanism, we call it the chooser, which chooses randomly the next state of the system from the set of states determined by the projection postulates,

$$|\Phi(\tau_n;\nu_n)\rangle = e^{i\alpha_n}\frac{\text{Ɇ}(\tau_n;\nu_n)|\Phi(\tau_{n-1};\nu_{n-1})\rangle}{\|\text{Ɇ}(\tau_n;\nu_n)\Phi(\tau_{n-1};\nu_{n-1})\|}\,, \tag{2}$$

where the global phase α_n can be chosen arbitrarily and $\|v\|$ denotes the norm of the vector v.

In other words, Eq.(2) determines the set of allowed states to which a physical system can randomly evolve from the state $|\Phi(\tau_{n-1};\nu_{n-1})\rangle$. To fully describe this stochastic process, one needs to know the probability distribution for getting a given state in the next step of the evolution. It is given by:

$$\text{pev}(\nu_{n-1}\to\nu_n) = \langle\Phi(\tau_{n-1};\nu_{n-1})|\text{Ɇ}(\tau_n;\nu_n)|\Phi(\tau_{n-1};\nu_{n-1})\rangle\,. \tag{3}$$

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It is also useful to introduce a tool which facilitates the construction of the evolution operators in terms of the projection operators. We have found that required evolution operators can be obtained from some operators $\mathbb{W}$ which we call generators of the projection evolution [42].

For a given evolution step τ the projection evolution generator $\mathbb{W}(\tau)$ is defined as a self-adjoint operator which spectral decomposition gives the orthogonal resolution of unity representing the required set of the evolution operators.

For example, assuming a discrete spectrum of an evolution generator $\mathbb{W}(\tau_n)$, the spectral theorem gives the following relation between $\mathbb{W}$ and the evolution operators $\mathbb{E}$: $\mathbb{W}(\tau_n) = \sum_\nu w_\nu \mathbb{E}(\tau_n; \nu)$, where w_ν are the eigenvalues of $\mathbb{W}$. In the case of a continuous spectrum one needs to use the integral form of the spectral theorem.

The evolution generators allow to use the already known quantum operators to generate the appropriate evolution operators $\mathbb{E}(\tau_n, \nu)$.

3. Quantum Clock in the Structureless Flat Spacetime

In particle, atomic, molecular and some other branches of physics the clock can be considered to be independent of the physical system under consideration. In this case one can treat time as the so-called external time. However, it is only an approximation because the clock is, in fact, a part of this system. In models in which time is a quantum observable, it should be considered as a part of an observable describing position in spacetime. The position in the quantum spacetime is, in turn, one of the attributes of physical matter.

In this paper we consider the simplest, approximate model which leads to the spacetime based on the $L^2(\mathbb{R}^4, d^4x)$ quantum state space, i.e., the Hilbert space of square integrable functions with the scalar product (note the integration over time) $\langle \psi_2 | \psi_1 \rangle = \int_{\mathbb{R}^4} d^4x \, \psi_2(x)^\star \, \psi_1(x)$ [42,43].

In a fixed coordinate frame, a possible realization of the spacetime position operator can be given by the four-vector operator $\hat{x}^\mu : \hat{x}^\mu f(x) = x^\mu f(x)$, where $\mu = 0, 1, 2, 3$.

We understand the term four-vector as a four-component object which transforms with respect to a given group of the spacetime transformations. In our case we think about two groups: either the Galilean or the Lorentz group. It means that using the same denotations we may consider either a non-relativistic or relativistic four dimensional flat spacetime.

The canonically conjugated observable is the four-momentum operator $\hat{p}_\mu = i\frac{\partial}{\partial x^\mu}$. Because time is a quantum observable $\hat{t} \equiv \hat{x}^0$, the temporal momentum $\hat{p}_0$ is the corresponding counterpart. By analogy to the space components of the momentum operator one can think about $\hat{p}_0$ as an observable representing the measure of motion in the temporal dimension. The sign of $\hat{p}_0$ describes the arrow of time. The temporal momentum can be measured using different equations of motions which relate it to the spatial and other properties of the system. For example, the Schrödinger equation $(\hat{p}_0 - \hat{H})\psi(x, \zeta) = 0$, where the variables ζ represent some additional degrees of freedom of the quantum system, relates $\hat{p}_0$ to the Hamiltonian $\hat{H}$. One can say the same about the Klein-Gordon, Dirac and other equations of motion.

In principle, we need some kind of "detectors" measuring positions in spacetime. They can be represented by POV measures [45]. In the ideal case, this measure is represented by a sharp observable given by the projection operators $M_X(x^0, x^1, x^2, x^3) = |x^0, x^1, x^2, x^3\rangle\langle x^0, x^1, x^2, x^3|$, where the vectors $|x^0, x^1, x^2, x^3\rangle$ are eigenstates of the position operators $\hat{x}^\mu$.

In such models we come across the (1+3)-D position measurement. There is no problem with performing the 3D spatial measurement, however, we do not have devices which allow to see the whole time axis, i.e., the past, present and the future. It seems that our material world is, in most cases, rather well localized in time and that only a narrow temporal window moving along the time axis is available for us in the experiments. From this perspective a clock has to be a part of the quantum system, well localized in time, possibly coupled to other physical subsystems. Having a reference clock, one can construct other types of clocks as devices synchronized with this clock. This procedure

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allows to introduce the notion of the "external time", however only for the systems which are well decoupled from the reference clock.

We define a quantum reference clock as a kind of device, localized on the time axis and moving with a fixed sign of the average temporal momentum $\langle \hat{p}_0 \rangle_{\text{clock state}}$ in spacetime. The sign is conventional but it defines the arrow of time. The clock localizing itself in a time interval (temporal window) shows us, usually indirectly by some kind of an interface (measurement), its position in time. It means that the clock consists of two subsystems: the proper clock, evolving from one localization on the time axis to another, and the interface reading off the clock's time position.

In the PEv model [43] the average spacetime localization of a physical system can change only with the change of the evolution parameter τ_n. In the following we denote by τ_n, $n = 1, 2, 3, \ldots$ the evolution steps of the proper clock, and by τ'_n, where $\tau_n < \tau'_n < \tau_{n+1}$, the evolution steps at which the clock interface reads its temporal position. It implies that a given quantum clock at the evolution step τ_n is represented by a set of clock states $\Phi(\tau_n; \nu_n)$. The label ν_n represents a set of quantum numbers describing both the temporal and other properties of observables required for the construction of a quantum clock. For a good clock, the variance of the time operator

$$\hat{t} = \int_{\mathbb{R}} dx^0 M_T(x^0), \tag{4}$$

where

$$M_T(x^0) = \int_{\mathbb{R}^3} d\mathbf{x} M_X(x^0, \mathbf{x}) \tag{5}$$

projects onto the subspaces of simultaneous events, should be small. More precisely, the variance should fulfill the following inequality:

$$\text{var}(\hat{t}; \Phi(\tau_n; \nu_n)) = \langle \Phi(\tau_n; \nu_n) | (\hat{t} - \langle \Phi(\tau_n; \nu_n) | \hat{t} | \Phi(\tau_n; \nu_n) \rangle)^2 | \Phi(\tau_n; \nu_n) \rangle \leq \text{cr}\,, \tag{6}$$

where cr is a small number and denotes the clock resolution. The corresponding expectation value of the time operator within the clock states, $t = \langle \Phi(\tau_n; \nu_n) | \hat{t} | \Phi(\tau_n; \nu_n) \rangle$, gives the expected temporal position of the proper clock, i.e., it gives a parameter t representing the classical time. In the ideal case the clock states are eigenstates of the time operator with the variance (6) being equal to 0.

In the projection evolution model [43] an important element is the construction of the evolution operators for the subsequent steps of the evolution $\tau_{n-1} \to \tau'_{n-1} \to \tau_n$. It means that we have to describe the following process: for the evolution step τ_{n-1} the proper clock is localized at some time t_{n-1}, this position is read off by the interface at τ'_{n-1} and the proper clock is moving to the next temporal position at τ_n.

In this paper, we propose a set of unitary operations, driven by a random variable $\xi \in \mathbb{R}_+$, transforming the evolution generators, not states, from the previous to the next step of the evolution process:

$$U(\xi, \gamma) = \exp(i(\xi \hat{p}_0 - \gamma \hat{A}))\,, \tag{7}$$

where γ is a coupling constant. The temporal momentum operator $\hat{p}_0$ and the self adjoint reconfiguring operator $\hat{A}$ commute, i.e., $[\hat{p}_0, \hat{A}] = 0$. The temporal momentum operator $\hat{p}_0$ is responsible for the motion along the time axis. The values of the random variable ξ are related to the internal processes of the clock and to the influence of the environment on the clock. This leads effectively to a movement of the clock along the time axis. Direction of the clock motion along the time axis is determined by the expectation value of the temporal momentum operator $\hat{p}_0$, calculated in the clock states. The reconfiguring operator $\hat{A}$ changes some internal states of the clock and allows to define the required clock interface.

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Let us denote by $\mathbb{W}(\tau_n)$ the evolution generator of the proper clock at the evolution step τ_n. The unitary transformations (7) form a two-parameter group, $U(\xi+\xi', (k+k')\gamma) = U(\xi, k\gamma)U(\xi', k'\gamma)$, where $k, k' \in \mathbb{Z}$. This feature allows to relate the evolution generator $\mathbb{W}(\tau_n)$ to its initial form $\mathbb{W}(\tau_0)$:

$$\mathbb{W}(\tau_n) = U(\xi_n, \gamma)\, \mathbb{W}(\tau_{n-1})\, U(\xi_n, \gamma)^\dagger = U(\beta_n, n\gamma)\, \mathbb{W}(\tau_0)\, U(\beta_n, n\gamma)^\dagger\,, \tag{8}$$

where $\beta_0 = 0$, $\beta_n = \sum_{k=1}^{n} \xi_k$ and $\xi_1, \xi_2, \ldots \xi_n$ are subsequent values of the random variable ξ.

To construct the evolution operators $\mathbb{E}(\tau_n; \nu_n)$ from the evolution generator $\mathbb{W}(\tau_n)$ one needs to find its spectral decomposition. Assuming discrete spectrum of the generator $\mathbb{W}(\tau_n)$ one has to solve the following eigenequation

$$\mathbb{W}(\tau_n)|\psi_{bc}^{(n)}\rangle = w_b^{(n)}|\psi_{bc}^{(n)}\rangle\,, \tag{9}$$

where c represents the possible degeneration of the spectrum. The transformation property (8) implies that it is sufficient to solve the eigenequation (9) only for τ_0 to get all solutions for every τ_n. If $|\psi_{bc}^{(0)}\rangle \equiv |\psi_{bc}\rangle$ are eigenvectors of $\mathbb{W}(\tau_0)$, the states

$$|\psi_{bc}^{(n)}\rangle = U(\beta_n, n\gamma)|\psi_{bc}\rangle \tag{10}$$

are eigenstates of $\mathbb{W}(\tau_n)$. The eigenvalues $w_b^{(n)} = w_b^{(0)} \equiv w_b$ are independent of n.

The distribution of ξ should have a pronounced maximum very close to zero[1]. It implies, that the next instant of classical time during the evolution should be very close to the previous one. The probability distribution for ξ is a parameter of the clock. It seems that this distribution is strongly related to both the construction of the clock, and the structure of spacetime.

The second component of the clock is an interface allowing to read the clock. The interface is therefore a measuring device and its full evolution operator has to contain the appropriate projection operators.

The projection evolution generators simplify the construction of the required evolution operators for a quantum reference clock. This method will be used in the next section.

3.1. The Proper Clock

For a given observer the quantum spacetime splits into time and 3D position space. In the following, to present the clock idea we choose a non-covariant description of a proper clock and its interface.

Because we treat time on the same footing as the other coordinates one expects that every interaction $V_{int}(x-y)$, where $x = (t_x, \mathbf{x}), y = (t_y, \mathbf{y})$ are spacetime points, depends not only on the positions in space but also on positions in time [42]. One can say the same about the effective potentials $V(t, \mathbf{x})$. According to observations, in a wide range of its density, the physical matter is well localized in time. This supports the parametric time as a good approximation. On the other hand, what is even more important, this feature also suggests that the interaction in the temporal dimension has a very short range. In the normal density matter the temporal parts of interactions among many particles lead to a temporal mean-field effective potential operator $V_T(\hat{t})$. One can find similar effect in nuclear physics, where the mean-field approach to a short range nuclear interaction is a good approximation.

In the following, we present a nonrelativistic proper clock described by a modified Schrödinger type of quantum motion in the structureless flat spacetime [42]. Relativistic versions of the clock can be constructed in a similar way.

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[1] In this work we assume $\xi > 0$ but the general condition is, that $\xi \in \mathbb{R}$ has a distribution around zero, assymetric towards $\xi > 0$ values. This will result in the net time flow in the positive direction of the time axis. We will discuss this problem in a subsequent paper.

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Following the discussion about the evolution generators described in [42], the initial projection evolution generator $\mathbb{W}(\tau_0)$ for a proper clock is postulated in the following form:

$$\mathbb{W}(\tau_0) = \hat{p}_0 - \hat{H} - \frac{\hat{p}_0^2}{2m_T} - V_T(\hat{t}), \tag{11}$$

where the temporal momentum operator is $\hat{p}_0 = i\frac{\partial}{\partial t}$, the Hamiltonian $\hat{H}$ acts on functions of the position variables $\mathbf{x} \in \mathbb{R}^3$ and potentially on some intrinsic variables, i.e., it does not depend on time. The signs in front of the temporal part of the generator (11) are chosen to keep the coefficient m_T positive. By analogy to the spatial kinetic term in the Hamiltonian $\hat{H}$ the term m_T can be called "the temporal inertia".

The proper clock is determined by the generator, which describes the physical system localized in time, and the Hamiltonian $\hat{H}$ which allows to build an appropriate interface in its state space. The proper clock is localized on the time axis by the effective temporal potential operator $V_T(\hat{t})$.

Using the unitary operator (7) one gets the evolution generator for any arbitrary evolution step τ_n:

$$\mathbb{W}(\tau_n) = U(\beta_n, n\gamma)\, \mathbb{W}(\tau_0)\, U(\beta_n, n\gamma)^\dagger = \hat{p}_0 - \hat{H}^{(n)} - \frac{\hat{p}_0^2}{2m_T} - V_T^{(n)}(\hat{t})\,, \tag{12}$$

where the potential localizing the clock in time is given by

$$V_T^{(n)}(\hat{t}) = V_T\left(\hat{t} - \beta_n \hat{\mathbf{1}}\right)\,. \tag{13}$$

The modified Hamiltonian has the following form:

$$\hat{H}^{(n)} = \exp\left(-in\gamma\,\hat{A}\right)\hat{H}\exp\left(+in\gamma\,\hat{A}\right)\,. \tag{14}$$

According to the definition of the projection evolution generators we have to construct the spectral decomposition of $\mathbb{W}(\tau_n)$. For this purpose one needs to look for the eigenfunctions of $\mathbb{W}(\tau_0)$. In our case we can find them by separating the variables,

$$\psi_{\lambda\kappa\nu\mu}(t, \mathbf{x}) = \chi_{\lambda\kappa}(t)\phi_{\nu\mu}(\mathbf{x})\,, \tag{15}$$

where $\chi_{\lambda\kappa}(t) := e^{-im_T t} f_{\lambda\kappa}(t)$. The indices κ and μ indicate a possible degeneration of the solutions. In the following we assume that the temporal functions $\chi_{\lambda\kappa}(t)$ represent the states which, on average, move in the positive direction of time. This means that the temporal functions $\chi_{\lambda\kappa}(t)$ are the states for which the expectation value of the temporal momentum operator,

$$\langle\chi_{\lambda\kappa}|\hat{p}_0|\chi_{\lambda\kappa}\rangle = m_T + \int_R dt\, f_{\lambda\kappa}(t)^\star \hat{p}_0 f_{\lambda\kappa}(t), \tag{16}$$

is a positive number.

The functions $\phi_{\nu\mu}(\mathbf{x})$ denote the eigenvectors of the Hamiltonian $\hat{H}$,

$$\hat{H}\phi_{\nu\mu}(\mathbf{x}) = E_\nu \phi_{\nu\mu}(\mathbf{x})\,, \tag{17}$$

where E_ν are the eigenvalues of $\hat{H}$.

The function $f_{\lambda\kappa}(t)$ solves the equation

$$\left(\frac{\hat{p}_0^2}{2m_T} + V_T(\hat{t})\right) f_{\lambda\kappa}(t) = \left(\frac{m_T}{2} - \epsilon_\lambda^{(T)}\right) f_{\lambda\kappa}(t), \tag{18}$$

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while together with the exponential function it constitutes the full solution of the equation

$$\left(\hat{p}_0 - \frac{\hat{p}_0^2}{2m_T} - V_T(\hat{t})\right)\chi_{\lambda\kappa}(t) = \epsilon_\lambda^{(T)}\chi_{\lambda\kappa}(t)\,. \tag{19}$$

This allows one to write the eigenequation for $\mathbb{W}(\tau_0)$ as

$$\mathbb{W}(\tau_0)\psi_{\lambda\kappa\nu\mu}(t,\mathbf{x}) = w_{b(\lambda,\nu)}\psi_{\lambda\kappa\nu\mu}(t,\mathbf{x}), \tag{20}$$

with $\psi_{\lambda\kappa\nu\mu}$ given by Equation (15) and

$$w_{b(\lambda,\nu)} = \epsilon_\lambda^{(T)} - E_\nu\,, \tag{21}$$

where the function labelling the eigenvalues of $\mathbb{W}(\tau_0)$ fulfil the condition: $b(\lambda',\nu') = b(\lambda,\nu) \Leftrightarrow w_{b(\lambda',\nu')} = w_{b(\lambda,\nu)}$.

Collecting the above partial solutions the eigenstates of $\mathbb{W}(\tau_n)$ for a given evolution step τ_n are given by the vectors

$$\psi_{\lambda\kappa\nu\mu}^{(n)}(t,\mathbf{x}) = U(\beta_n, n\gamma)\psi_{\lambda\kappa\nu\mu}(t,\mathbf{x}) = \chi_{\lambda\kappa}^{(n)}(t)\phi_{\nu\mu}^{(n)}(\mathbf{x})\,, \tag{22}$$

where $\chi_{\lambda\kappa}^{(n)}(t) := \chi_{\lambda\kappa}(t-\beta_n)$ and $\phi_{\nu\mu}^{(n)}(\mathbf{x}) := \exp(-in\gamma\hat{A})\phi_{\nu\mu}(\mathbf{x})$.

The evolution operators generated by $\mathbb{W}(\tau_n)$ are the projections onto eigenspaces of the evolution generator,

$$\mathbb{E}(\tau_n; w) = \sum_{\lambda\nu}\delta(w = \epsilon_\lambda^{(T)} - E_\nu)\sum_{\kappa\mu}|\psi_{\lambda\kappa\nu\mu}^{(n)}\rangle\langle\psi_{\lambda\kappa\nu\mu}^{(n)}| = U(\beta_n, n\gamma)\mathbb{E}(\tau_0; w)U(\beta_n, n\gamma)^\dagger\,. \tag{23}$$

The Kronecker's type function is defined as: $\delta(w = \epsilon_\lambda^{(T)} - E_\nu) = 1$ if $w = \epsilon_\lambda^{(T)} - E_\nu$, otherwise it is equal to 0. The sums will be substituted by some integrals in the case of a continuous spectrum.

3.2. The Clock Interface

To read the change of the internal state of the proper clock, we introduce the interface evolution operators denoted by $\mathbb{E}_I(\tau_n', \nu)$, where $\tau_n < \tau_n' < \tau_{n+1}$ describes an intermediate event between τ_n and τ_{n+1}. They form an orthogonal resolution of unity projecting onto eigenstates of the Hamiltonian $\hat{H}$ showing in which eigenstate of $\hat{H}$ the clock actually is. Changes of the states of the Hamiltonian along the time axis represent the clock ticks. The clock interface should disturb neither the proper clock localization in time nor its movement along the time axis but it should show in which eigenstate of the Hamiltonian the proper clock is.

Let us denote an orthonormal basis $g_{s\nu\mu}(t,x) = e_s(t)\phi_{\nu\mu}(\mathbf{x})$ in the state space. We obtain the required properties by assuming that

$$\mathbb{E}_I(\tau_n', \nu) = \sum_\mu |\phi_{\nu\mu}\rangle\left(\sum_s |e_s\rangle\langle e_s|\right)\langle\phi_{\nu\mu}| \equiv \mathbb{E}_I(\nu)\,, \tag{24}$$

where $(\sum_s |e_s\rangle\langle e_s|) = \hat{\mathbf{1}}_T$ is the unit operator in the time domain. The projection operators (24) are independent of τ_n', i.e., we keep the same interface for every evolution step.

3.3. The Clock

The evolution operators described above allow to construct different reference clocks using different forms of the operators: $\hat{A}, \hat{H}$ and $V_T(\hat{t})$.

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The clock starts from any arbitrary quantum state $|\Phi_0\rangle$. The evolution operator $\mathbb{\Phi}(\tau_0;u_0)$ prepares the initial state of the clock

$$\Phi(\tau_0;u_0;t,\mathbf{x}) = \frac{\mathbb{\Phi}(\tau_0;u_0)\Phi_0(t,\mathbf{x})}{\|\mathbb{\Phi}(\tau_0;u_0)\Phi_0\|}\,, \tag{25}$$

where u_0 represents, according to the PEv formalism, a randomly chosen eigenvalue $w_{b(\lambda,\nu)}$ of the evolution generator $\mathbb{W}(\tau_0)$.

The state (25) is read by the interface

$$\Phi(\tau_0';\sigma_0,u_0;t,\mathbf{x}) = \frac{\mathbb{\Phi}_I(\sigma_0)\Phi(\tau_0;u_0;t,\mathbf{x})}{\|\mathbb{\Phi}_I(\sigma_0)\Phi(\tau_0;u_0)\|}\,, \tag{26}$$

where σ_0 represents a randomly chosen eigenvalue E_{σ_0} of the Hamiltonian $\hat{H}$. The simplest choice of the initial state is any eigenstate of the evolution generator (15),

$$\Phi_0(t,\mathbf{x}) = \psi_{\lambda_0\kappa_0\nu_0\mu_0}(t,\mathbf{x}) \equiv \chi_{\lambda_0\kappa_0}(t)\phi_{\nu_0\mu_0}(\mathbf{x})\,. \tag{27}$$

Then $\Phi(\tau_0';\sigma_0,u_0;t,\mathbf{x}) = \Phi_0(t,\mathbf{x})$, where $\sigma_0 = \nu_0$ and $u_0 = \epsilon^T_{\lambda_0} - E_{\nu_0}$.

To simplify the notation let us denote by $(\sigma,u)_n$ the sequence of quantum numbers which gives a possible evolution path

$$(\sigma,u)_n = (\sigma_n,u_n,\sigma_{n-1},u_{n-1},\ldots,\sigma_0,u_0) = (\sigma_n,u_n,(\sigma,u)_{n-1}).$$

The subsequent cycles of the clock are described by the following recurrence relations

$$\Phi(\tau_n;u_n,(\sigma,u)_{n-1};t,x) = \frac{\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1};t,x)}{\left\|\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1})\right\|}\,, \tag{28}$$

and

$$\begin{aligned}\Phi(\tau'_n;(\sigma,u)_n;t,x) &= \frac{\mathbb{\Phi}_I(\sigma_n)\Phi(\tau_n;u_n,(\sigma,u)_{n-1};t,x)}{\|\mathbb{\Phi}_I(\sigma_n)\Phi(\tau_n;u_n,(\sigma,u)_{n-1})\|}\\ &= \frac{\mathbb{\Phi}_I(\sigma_n)\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1};t,x)}{\left\|\mathbb{\Phi}_I(\sigma_n)\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1})\right\|}\,,\end{aligned} \tag{29}$$

where the probabilities of choosing next states are given by (3), i.e., by the denominators of (28) and (29), respectively.

The action of the evolution operator $\mathbb{\Phi}_I(\sigma_n)\mathbb{\Phi}(\tau_n;u_n)$ on any function $\Phi(\tau'_{n-1};(\sigma,u)_{n-1};t,x)$ can be written as

$$\mathbb{\Phi}_I(\sigma_n)\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1};t,x) = \mathcal{G}((\sigma,u)_n;\lambda\kappa\mu')\chi^{(n)}_{\lambda\kappa}(t)\phi_{\sigma_n\mu'}(\mathbf{x})\,, \tag{30}$$

where the expansion coefficients, taking into account (23) and (24), read

$$\mathcal{G}((\sigma,u)_n;\lambda\kappa\mu') = \sum_{\nu\mu}\delta(u_n = \epsilon^{(T)}_\lambda - E_\nu)\,\langle\phi_{\sigma_n\mu'}|\phi^{(n)}_{\nu\mu}\rangle\,\langle\psi^{(n)}_{\lambda\kappa\nu\mu}|\Phi(\tau'_{n-1};(\sigma,u)_{n-1})\rangle\,. \tag{31}$$

Because the functions $\chi^{(n)}_{\lambda\kappa}(t)\phi_{\sigma_n\mu'}(\mathbf{x})$ and $\chi^{(n)}_{\lambda'\kappa'}(t)\phi_{\sigma'_n\mu''}(\mathbf{x})$ are orthonormal, the coefficient $N(\tau_n;(\sigma,u)_n)$ which normalizes Equation (30) can be expressed by the coefficients (31)

$$N(\tau_n;(\sigma,u)_n)^2 = \left\|\mathbb{\Phi}_I(\sigma_n)\mathbb{\Phi}(\tau_n;u_n)\Phi(\tau'_{n-1};(\sigma,u)_{n-1})\right\|^2 = \sum_{\lambda\kappa\mu'}|\mathcal{G}((\sigma,u)_n;\lambda\kappa\mu')|^2\,. \tag{32}$$

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Using the coefficients $\mathcal{G}$ the clock state (29) can be rewriten as

$$\Phi(\tau_n';(\sigma,u)_n;t,x) = N(\tau_n;(\sigma,u)_n)^{-1}\sum_{\lambda\kappa\mu'}\mathcal{G}((\sigma,u)_n;\lambda\kappa\mu')\chi^{(n)}_{\lambda\kappa}(t)\phi_{\sigma_n\mu'}(\mathbf{x})\,. \tag{33}$$

The definition (31) and Eqs. (32) and (33) lead to the recurrence relation for the $\mathcal{G}$-coefficients

$$\begin{aligned}\mathcal{G}((\sigma,u)_n;\lambda\kappa\mu') &= N(\tau_{n-1};(\sigma,u)_{n-1})\sum_{\nu_1\mu_1}\delta(u_n=\epsilon^{(T)}_\lambda - E_{\nu_1})\langle\phi_{\sigma_n\mu'}|\phi^{(n)}_{\nu_1\mu_1}\rangle\\ &\sum_{\lambda_2\kappa_2\mu_2'}\mathcal{G}((\sigma,u)_{n-1};\lambda_2\kappa_2\mu_2')\langle\chi^{(n)}_{\lambda\kappa}|\chi^{(n-1)}_{\lambda_2\kappa_2}\rangle\langle\phi^{(n)}_{\nu_1\mu_1}|\phi_{\sigma_{n-1}\mu_2'}\rangle.\end{aligned} \tag{34}$$

Having the $\mathcal{G}$-coefficients, one can calculate the transition probability between the subsequent readings off the interface. The square of the normalization factor represents this probability:

$$\begin{aligned}&\mathrm{pev}(\tau_{n-1}'\to\tau_n\to\tau_n')\\ &= |\langle\Phi(\tau_n';(\sigma,u)_n)|\Phi(\tau_n;u_n,(\sigma,u)_{n-1})\rangle\langle\Phi(\tau_n;u_n,(\sigma,u)_{n-1})|\Phi(\tau_{n-1}';(\sigma,u)_{n-1})\rangle|^2\\ &= \left\|\mathbb{E}_I(\sigma_n)\mathbb{E}(\tau_n;u_n)\Phi(\tau_{n-1}';(\sigma,u)_{n-1})\right\|^2 = N(\tau_n;(\sigma,u)_n)^2\,.\end{aligned} \tag{35}$$

In our model, the clock shifts subsequently on the time axis about the random variable ξ. However, our knowledge about this shift is supplied only by the clock interferface. It is important to calculate the expectation value of the time operator within the states corresponding to reading off the interface. For this purpose, let us denote by t_n the average value of the time operator in the state (29), i.e., $t_n = \langle\Phi(\tau_n';(\sigma,u)_n)|\hat{t}|\Phi(\tau_n';(\sigma,u)_n)\rangle$. Using (29) and the following matrix elements:

$$\begin{aligned}\langle\chi^{(n)}_{\lambda_1\kappa_1}|\hat{t}|\chi^{(n)}_{\lambda_2\kappa_2}\rangle &= \int_{\mathbb{R}}dt\, f_{\lambda_1\kappa_1}(t-\beta_n)^\star t f_{\lambda_2\kappa_2}(t-\beta_n)\\ &= \delta_{\lambda_1\lambda_2}\delta_{\kappa_1\kappa_2}\beta_n + \int_{\mathbb{R}}dt\, f_{\lambda_1\kappa_1}(t)^\star\, t\, f_{\lambda_2\kappa_2}(t)\\ &= \delta_{\lambda_1\lambda_2}\delta_{\kappa_1\kappa_2}\beta_n + \langle\chi_{\lambda_1\kappa_1}|\hat{t}|\chi_{\lambda_2\kappa_2}\rangle,\end{aligned} \tag{36}$$

the required expectation value reads

$$\begin{aligned}t_n = \beta_n + \left(\frac{1}{N(\tau_n;(\sigma,u)_n)}\right)^2\Bigg[&\sum_{\lambda_1\kappa_1}\sum_{\lambda_2\kappa_2}\\ &\sum_{\mu'}\mathcal{G}((\sigma,u)_n;\lambda_1\kappa_1,\mu')^\star\mathcal{G}((\sigma,u)_n;\lambda_2\kappa_2,\mu')\Bigg]\langle\chi_{\lambda_1\kappa_1}|\hat{t}|\chi_{\lambda_2\kappa_2}\rangle\,.\end{aligned} \tag{37}$$

The ideal clock should show the value $t_n = \beta_n$, i.e., the place where the clock is localized on the time axis. The interface, however, is also a quantum device and acts randomly. There is therefore a finite probability that the proper clock moves to the next position on the time axis, but the interface does not change its state, i.e., the "hand of the clock" does not advance forward. This implies that for a good clock the second term in (37) should be always close to zero.

4. A Two-State Clock

As an example, we build a schematic clock for which there is no summation over λ, μ, κ and ν in Equation (23).

This is often the case when both (17) and (18) have discrete nondegenerate spectra, which allows to rewrite the evolution operators (23) in a simpler form:

$$\mathbb{E}(\tau_n;\lambda\nu) = |\psi^{(n)}_{\lambda\nu}\rangle\langle\psi^{(n)}_{\lambda\nu}|. \tag{38}$$

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We express the clock Hamiltonian describing the clock interface by making use of the spectral theorem,

$$\hat{H} = \sum_{\nu=0}^{\infty} E_\nu |\phi_\nu\rangle\langle\phi_\nu| \,. \tag{39}$$

The reconfiguration operator is taken in the following form:

$$\hat{A} = |\phi_0\rangle\langle\phi_1| + |\phi_1\rangle\langle\phi_0| \,. \tag{40}$$

This means that our clock interface is oscillating between two states, $|\phi_0\rangle$ and $|\phi_1\rangle$, and the corresponding evolution operators, reading the actual state of the interface, are given by

$$\mathbb{E}_I(\sigma) = |\phi_\sigma\rangle\langle\phi_\sigma|, \quad \sigma = 0, 1. \tag{41}$$

The state of the proper clock and its interface at the evolution step τ'_{n-1} is $|\Phi(\tau'_{n-1}; (\sigma, \lambda, \nu)_{n-1})\rangle$. To get to the next step of the evolution one needs to calculate

$$\mathbb{E}(\tau_n; \lambda_n, \nu_n)|\Phi(\tau'_{n-1}; (\sigma, \lambda, \nu)_{n-1})\rangle = \langle\psi^{(n)}_{\lambda_n\nu_n}|\Phi(\tau'_{n-1}; (\sigma, \lambda, \nu)_{n-1})\rangle|\psi^{(n)}_{\lambda_n\nu_n}\rangle \tag{42}$$

and

$$\begin{aligned}&\mathbb{E}_I(\sigma_n)\mathbb{E}(\tau_n; \lambda_n, \nu_n)|\Phi(\tau'_{n-1}; (\sigma, \lambda, \nu)_{n-1})\rangle = \\ &\langle\phi_{\sigma_n}|\phi^{(n)}_{\nu_n}\rangle\langle\psi^{(n)}_{\lambda_n\nu_n}|\Phi(\tau'_{n-1}; (\sigma, \lambda, \nu)_{n-1})\rangle|\chi^{(n)}_{\lambda_n}\rangle|\phi_{\sigma_n}\rangle \,.\end{aligned} \tag{43}$$

Because in the formulas (28) and (29) the common multiplication factors in the nominators and denominators can be simplified, and overall phases are unimportant, the resulting clock states, after simplification of notation, are represented by the vectors

$$|\Phi(\tau_n; (\sigma, \lambda, \nu)_n)\rangle = |\Phi(\tau_n; \lambda_n\nu_n)\rangle \equiv |\psi^{(n)}_{\lambda_n\nu_n}\rangle, \tag{44}$$

$$|\Phi(\tau'_n; (\sigma, \lambda, \nu)_n)\rangle = |\Phi(\tau'_n; \sigma_n, \lambda_n)\rangle \equiv |\chi^{(n)}_{\lambda_n}\rangle|\phi_{\sigma_n}\rangle \,. \tag{45}$$

The clock states (44) and (45) allow to calculate the transition probability between the states representing two subsequent readings from the clock interface. The probability of passing the evolution path

$$(\tau', \sigma, \lambda, \nu)_{n-1} \to (\tau_n, \lambda_n, \nu_n, (\tau, \sigma, \lambda, \nu)_{n-1}) \to (\tau', \sigma, \lambda, \nu)_n$$

according to the expression (35) reads

$$\mathrm{pev}' = |\langle\Phi(\tau'_n; \sigma_n, \lambda_n)|\Phi(\tau_n; \lambda_n, \nu_n)\rangle\langle\Phi(\tau_n; \lambda_n, \nu_n)|\Phi(\tau'_{n-1}; \sigma_{n-1}, \lambda_{n-1})\rangle|^2 \,. \tag{46}$$

Using relations (42) and (43), one can cast (46) in the form

$$\mathrm{pev}' = |\langle\chi^{(n)}_{\lambda_n}|\chi^{(n-1)}_{\lambda_{n-1}}\rangle|^2|\langle\phi_{\sigma_n}|\phi^{(n)}_{\nu_n}\rangle\langle\phi^{(n)}_{\nu_n}|\phi_{\sigma_{n-1}}\rangle|^2. \tag{47}$$

For the reconfiguration operator (40), the $\hat{A}$-dependent part of the $U(\xi, \gamma)$ operator (7) is given by

$$\exp(-i\gamma\hat{A}) = \hat{1} - i\sin\gamma\hat{A} + (\cos\gamma - 1)\,\hat{A}^2, \tag{48}$$

where $\hat{1}$ is the unit operator in the full state space, while $\hat{A}$ acts in its two-dimensional subspace only. It follows that the scalar product reads

$$\langle\phi_{\sigma_n}|\phi^{(n)}_{\nu_n}\rangle = \langle\phi_{\sigma_n}|\exp(-i\gamma\hat{A})|\phi_{\nu_n}\rangle = \delta_{\sigma_n\nu_n}\cos\gamma - i(\delta_{\sigma_n 1}\delta_{\nu_n 0} + \delta_{\sigma_n 0}\delta_{\nu_n 1})\sin\gamma, \tag{49}$$

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which allows to calculate the transition amplitudes:

$$\langle\phi_1|\phi_1^{(n)}\rangle\langle\phi_1^{(n)}|\phi_0\rangle = \frac{i}{2}\sin(2\gamma), \tag{50}$$

$$\langle\phi_1|\phi_0^{(n)}\rangle\langle\phi_0^{(n)}|\phi_0\rangle = -\frac{i}{2}\sin(2\gamma), \tag{51}$$

$$\langle\phi_0|\phi_1^{(n)}\rangle\langle\phi_1^{(n)}|\phi_0\rangle = \sin^2\gamma, \tag{52}$$

$$\langle\phi_0|\phi_0^{(n)}\rangle\langle\phi_0^{(n)}|\phi_0\rangle = \cos^2\gamma, \tag{53}$$

$$\langle\phi_0|\phi_1^{(n)}\rangle\langle\phi_1^{(n)}|\phi_1\rangle = -\frac{i}{2}\sin(2\gamma), \tag{54}$$

$$\langle\phi_0|\phi_0^{(n)}\rangle\langle\phi_0^{(n)}|\phi_1\rangle = \frac{i}{2}\sin(2\gamma), \tag{55}$$

$$\langle\phi_1|\phi_1^{(n)}\rangle\langle\phi_1^{(n)}|\phi_1\rangle = \cos^2\gamma, \tag{56}$$

$$\langle\phi_1|\phi_0^{(n)}\rangle\langle\phi_0^{(n)}|\phi_1\rangle = \sin^2\gamma. \tag{57}$$

To register the time passing is equivalent for the clock to change the state from $\sigma_n = 1$ to $\sigma_n = 0$ or from 0 to 1, i.e., the clock "clicks". Using Eq. (47) and the appropriate amplitudes (50)–(57) gives the probabilities that the clock "clicks" in the form

$$\begin{aligned} &p(\lambda_n) := \mathrm{Prob}(\sigma_{n-1} = 1 \to \sigma_n = 0) = \mathrm{Prob}(\sigma_{n-1} = 0 \to \sigma_n = 1) \\ &= \frac{1}{2}\sin^2(2\gamma)|\langle\chi_{\lambda_n}^{(n)}|\chi_{\lambda_{n-1}}^{(n-1)}\rangle|^2 \leq \frac{1}{2}\,. \end{aligned} \tag{58}$$

The probability of the opposite event, i.e., that the clock changes its state without the "click" is $1 - p(\lambda_n)$.

The probability that the clock "clicks" exactly in the ℓ-th step is given by

$$p(\lambda_\ell)\prod_{k=1}^{\ell-1}(1 - p(\lambda_k))\,. \tag{59}$$

The probability amplitude $\langle\chi_{\lambda_n}^{(n)}|\chi_{\lambda_{n-1}}^{(n-1)}\rangle$ can be expressed as

$$\langle\chi_{\lambda_n}^{(n)}|\chi_{\lambda_{n-1}}^{(n-1)}\rangle = e^{-im_T\xi_n}\int_R dt\, f_{\lambda_n}(t - \xi_n)^\star f_{\lambda_{n-1}}(t)\,. \tag{60}$$

The elementary step of movement along the time axis should be extremely small, i.e., the expectation value of $\langle\xi_k\rangle \approx 0$, for every step k of the clock evolution. This implies that $|\langle\chi_{\lambda_n}^{(n)}|\chi_{\lambda_{n-1}}^{(n-1)}\rangle|^2 = 1$ with a very good approximation and one can write $p(\lambda_\ell) \approx p = \frac{1}{2}\sin^2(2\gamma)$.

Let us calculate the average value of the time operator $\hat{t}$. It is given by

$$\langle\Phi(\tau_n'; \sigma_n, \lambda_n)|\hat{t}|\Phi(\tau_n'; \sigma_n, \lambda_n)\rangle = \beta_n + \langle\chi_\lambda|\hat{t}|\chi_\lambda\rangle. \tag{61}$$

Since $\chi_\lambda = e^{-i\, m_T t} f_\lambda(t)$, we have

$$\langle\chi_\lambda|\hat{t}|\chi_\lambda\rangle = \int_{\mathbb{R}} dt\; t|f_\lambda(t)|^2. \tag{62}$$

A good clock is characterized by the term (62) being as small as possible. If $f_\lambda(-t) = e^{i\alpha} f_\lambda(t)$ this term vanishes. It follows, that after n steps of evolution, the localization of the clock on the time axis is close to β_n.

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In a similar way one may obtain the average value of the temporal momentum operator,

$$\langle \Phi(\tau_n'; \sigma_n, \lambda_n) | \hat{p}_0 | \Phi(\tau_n'; \sigma_n, \lambda_n) \rangle = m_T + \int_{\mathbb{R}} dt f_\lambda(t)^\star \hat{p_0} f_\lambda(t). \tag{63}$$

Since $\hat{p}_0 = i\frac{\partial}{\partial t}$, the condition $f_\lambda(-t) = e^{i\alpha} f_\lambda(t)$ implies that the second term in Equation (63) vanishes, which results in a constant arrow of time:

$$\langle \Phi(\tau_n'; \sigma_n, \lambda_n) | \hat{p}_0 | \Phi(\tau_n'; \sigma_n, \lambda_n) \rangle = m_T > 0. \tag{64}$$

To estimate how long, on average, do we have to wait until a "click" appears, we calculate the variance of ℓ. The normalized probability that a single "click" takes place at any of the steps, during an ℓ-step evolution, is given by

$$\text{Prob}(\text{click}, \ell) = \frac{\ell p (1-p)^{\ell-1}}{\sum_{\ell=1}^{\infty} \ell p (1-p)^{\ell-1}} = \ell p^2 (1-p)^{\ell-1}. \tag{65}$$

This results in the average ℓ,

$$\langle \ell \rangle = \sum_{\ell=1}^{\infty} \ell \, \text{Prob}(\text{click}, \ell) = \sum_{\ell=1}^{\infty} \ell p^2 (1-p)^{\ell-1} = \frac{2-p}{p}, \tag{66}$$

and the average ℓ^2,

$$\langle \ell^2 \rangle = \sum_{\ell=1}^{\infty} \ell^2 \, \text{Prob}(\text{click}, \ell) = \sum_{\ell=1}^{\infty} \ell^2 p^2 (1-p)^{\ell-1} = \frac{p^2 - 6p + 6}{p^2}. \tag{67}$$

Thus the variance $\text{var}(\ell)$ is given by

$$\text{var}(\ell) = \langle \ell^2 \rangle - \langle \ell \rangle^2 = \frac{2-2p}{p^2}. \tag{68}$$

The probability Equation (65) is presented on Figure 1 as a function of the number of steps ℓ. We notice that for higher probabilities p the number of required evolution steps ℓ decreases. The exact position of the maxima is given by the relation

$$\ell_{\max} = -\frac{1}{\ln(1-p)}, \tag{69}$$

which for $p = 0.1$ gives $\ell_{\max} = 9.49$, for $p = 0.25$ is $\ell_{\max} = 3.48$, and for $p = 0.5$ drops to $\ell_{\max} = 1.44$.

The variance of ℓ is presented on Figure 2. For the ideal case of $p = 0.5$, the average ℓ from Equation (66) is $\langle \ell \rangle = 3$ while the standard deviation reads $\sigma = \sqrt{\text{var}(\ell)} = 2$. This gives the average number of the required evolution steps 3 ± 2.

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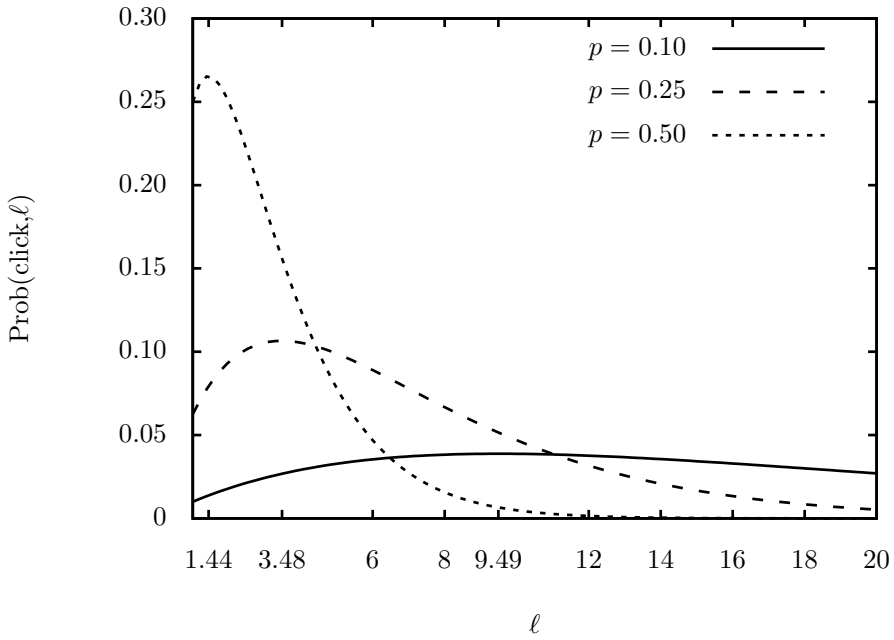

Figure 1. The probability (65) of a single click during ℓ steps of the clock's evolution. The values of p are 0.1, 0.25, and 0.5.

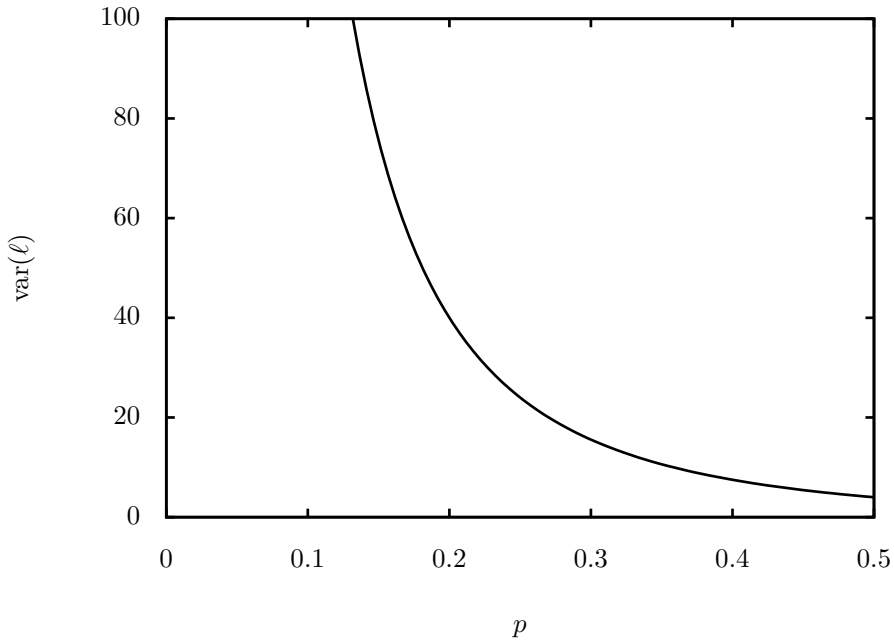

Figure 2. The variance (68) of the number of steps ℓ as a function of p.

5. Final Remarks

One may ask, how to compare the discussed quantum clock with the currently used atomic clocks? The time unit measured by an atomic clock is derived from the frequency of the photon emitted by an atom during its de-excitation. From the perspective of our model, the stochastic process of the atomic de-excitation plays the role of the proper clock, while the interface can be found in the clock's system which detects the photon.

The stability of the chosen atomic transitions is so high, that modern atomic clocks [1–4] achieve systematic uncertainty on the level of 10^{-18}s, where this number is measured with respect to an external laboratory clock. Since each evolution step advances the quantum clock on the time axis by ξ

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and we are required to wait between 1 and 5 steps to register the click, the value of ξ must be small enough to assure the already achieved precission. If we allow ξ to be a random variable, with different values in each step of the clock's evolution, the average $\langle\xi\rangle$ should be small. In that case one may take the statistics from as many as needed evolution steps, which will lower the value of var(ℓ) and increase the clock's accuracy to the required level.

We notice also that every clock, even theoreticaly considered as a quantum system, is influenced by spacetime and physical fields. Observed changes in the clock can give information about the temporal structure of these objects. This is an open problem for future investigations.

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