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Abstract: Re-interpretation of the Fermat principle governing propagation of light in media within the Ramsey Theory is suggested. Complete bi-colored graphs corresponding to the light propagation in media are considered. The vertices of the graphs correspond to the points of the real physical space, in which light sources or sensors are placed. Red links in the graphs correspond to the actual optical paths, emerging from the Fermat principle. A variety of optical events: refraction, reflection, *etc.* may be involved in the light propagation. Green links, in turn, denote the trial/virtual optical paths, which actually do not occur. The Ramsey Theorem states that within the graph containing six points, inevitably, actual or virtual optical cycle will be necessarily present. Implementation of the Ramsey Theorem to the light propagation in metamaterials is discussed. The Fermat Principle states that in metamaterials a light ray in going from point *S* to point *P* must traverse an optical path length *L* that is stationary with respect to variations of this path. Thus, bi-colored graphs built of links corresponding to maxima or minima of optical paths become possible. Graphs, comprising six vertices, inevitably will demonstrate optical cycles, built of the mono-colored links corresponding to the maxima or minima of the optical path. The notion of the "inverse graph" is introduced and discussed. The total number of triangles in the "direct" (source) and "inverse" Ramsey optical graphs is the same, Optical interpretation of the infinite Ramsey Theorem is suggested.

Keywords: Fermat principle; optical path; metamaterials; left-handed materials; graphs; Ramsey Theory; Ramsey Theorem; optical cycle

1. Introduction

The paper presents synthesis of the Fermat principle and Ramsey theory and reports the optical interpretation of the Fermat principle, applied to left-handed metamaterials. Fermat principle (in its initial formulation, which was essentially corrected recently, as it will be discussed below) states that, the actual path between two points taken by a beam of light is the one as traversed in the least time [1]. This principle is one of the variational principles of physics, which play fundamental, basic role in its structure (including the Maupertuis and Hamilton principle and the Hilbert variational principle for the general relativity). The Fermat's principle was formulated in 1662 by Pierre de Fermat a French mathematician and lawyer; but anticipated nearly a millennium ago by the Arab scientist Ibn al-Haytham and inspired by the Greek polymath Hero of Alexandria's (Ηρων ο Αλεξανδρε΄υς) reflections on light almost two millennia ago [2,3]. The Fermat principle is more accurately and rigorously formulated with the use of the notion of the optical length/optical path length. The optical length/optical path length *L* between points P_1 and P_2 is defined according to Eq. 1:

$$L = \int_{P_1}^{P_2} n\,ds \qquad (1)$$

where *n* is the refraction index of the medium, taken as a function of distance along the optical path.

The Principle of Fermat asserts the optical path of actual optical ray between any two points P_1 and P_2 is shorter than an optical length of any other curve which join these points and which lies in a certain regular neighborhood of it [1,4]. Thus, the Fermat principle is often regarded as the principle of the least optical path. Considering Eq. 2:

$$n\,ds = \frac{c}{v}dt = c\,dt \qquad (2)$$

and substitution of Eq. 2 into Eq. 3 gives rise to Eq. 3

$$L = c \int_{P_1}^{P_2} dt, \tag{3}$$

which immediately results in the principle of the least time, as it was formulated by Pierre de Fermat, namely: the actual path between two points taken by a beam of light is the one that is traversed in the *least time*, which is weaker than the Principle of the Least Optical Length [1,4].

Development of metamaterials (or left-handed materials) led to the generalization of the Fermat Principle [5–8]. Vesselago predicted that electromagnetic plane waves in a medium having simultaneously negative electric permittivity ε and magnetic permeability μ would propagate in a direction opposite to that of the flow of energy [5–8]. This result follows not from the wave equation, which remains unchanged in the absence of sources, but rather from the individual Maxwell curl equations. In the left-handed materials vectors $\vec{k}$, $\vec{E}$, $\vec{H}$ form a left-handed set, while in the usual materials ($\varepsilon > 0, \mu > 0$) they form a right-handed set. Applications of left-handed include lenses, antennas and effective radio-cloaking [8,9].

We demonstrate in our paper, how the Ramsey theory may be applied for the analysis of optical systems. The application of the Ramsey theory to optical systems exploits the Fermat principle, when the points of physical space are represented by the vertices of the graph, and the optical paths represent the edges/links of the graph. The Ramsey theory, introduced by the British mathematician and philosopher Frank Plumpton Ramsey, is the field of combinatorics/graph theory, which deals with the specific kind of mathematical structures, namely: complete graphs. A graph is a mathematical structure comprising a set of objects in which some pairs of the objects are in some sense "related" [10,11]. A complete graph is a graph in which each pair of graph vertices is connected by an edge/link. The typical problem considered by the Ramsey theory (as it was stated by Frank Ramsey) is the so-called "party problem", which predicts the minimum number $R(m,n)$ of participants gathered together in the room (each of whom is either a friend or a stranger to the others) must be invited so that at least *m* of the participants will be acquainted with each other, or at least *n* of them will not be familiar with each other [12–16]. In this case $R(m,n)$ is known as a Ramsey number [12–16]. Consider the particular formulation of the party problem: "What is the smallest number of people in a gathering, every two of whom are either friends or strangers, that will guarantee that there are either three mutual friends or three mutual strangers in the gathering"? In this particular case $R(3,3) = 6$. A classical result in Ramsey theory states that if some mathematical structure/graph is separated into finitely many sub-parts, then one of the sub-parts necessarily must contain a substructure/graph of the given type. Aphoristically speaking the Ramsey theory is the study of preservation of properties under the set partitions [16].

The rigorous mathematical statement of the Ramsey Theorem is formulated as follows: For any $k + 1 \geq 3$ positive integers t, $n_1, n_2, \dots n_k$ there exists a positive integer N such that if each of the t-element subsets of the set $\{1,2,\dots N\}$ is colored with one of the k colors 1, 2, . . ., k, then for some integer i with $1 \leq i \leq k$ there is a subset S of $\{1,2,\dots N\}$ containing n_i elements such that every t-element subset of S is colored I [16]. Frank Plumpton Ramsey, Paul Erdős and Ronald Lewis Graham crucially contributed into development of the Ramsey Theory [15,17,18].

Somewhat surprisingly, the physical and engineering applications of the Ramsey Theory are still scarce. For example, the Ramsey approach was successfully applied for the theory of communication and decision making [19]. A Ramsey theory of financial distortions was reported recently [20]. Dynamic Ramsey Theory of mechanical systems forming complete graphs in its relation to analysis of vibrations of cyclic compounds was addressed [21]. Successful application of the Ramsey theory for the formulation of axiomatic thermodynamics was reported [22]. We demonstrate for the first time the application of the Ramsey theory for optical systems. We also demonstrate, how the Ramsey approach may be re-shaped for optical metamaterials [5–9,23–26].

2. Results

2.1. Ramsey approach to optical systems: from real optical system to a graph

Ramsey Theory considers complete graphs, in which vertices are connected with at least two kinds of edges/links (multi-colored Ramsey graphs were also addressed [27]). Let us explain in detail, how the Ramsey Theory is introduced for optical systems. Consider the simplest system built of the source, located in point S, mirror and the sensor located in point P, as depicted in **Figure 1A**. The mirror is supposed to be ideal; thus, the usual law of light reflection works [1], and the angle of reflection equals the angle of incidence $\theta_1 = \theta_2$ as shown in **Figure 1A**.
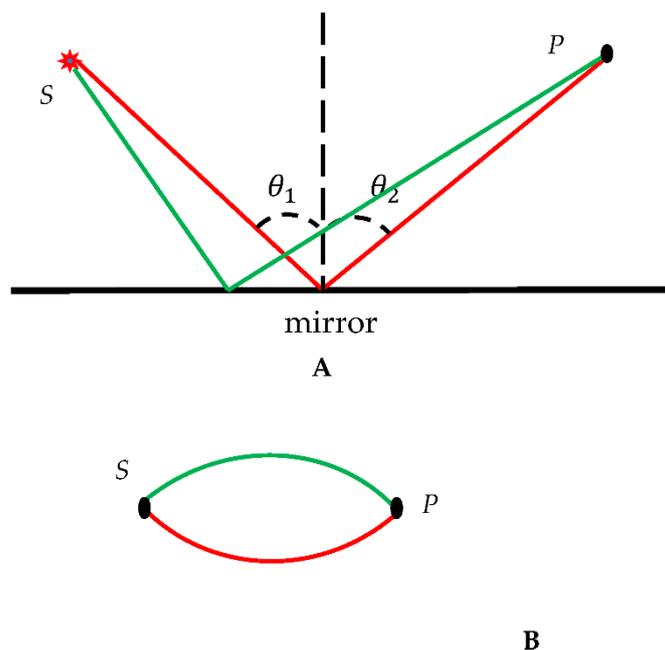

Figure 1. Converting of the reflection experiment into the bi-color graph is depicted. **A**. The system is built from the source, placed in point S, sensor located in point P and ideal mirror; thus, $\theta_1 = \theta_2$ takes place. Two kinds of optical paths are depicted. The red trajectory corresponds to the actual optical path; the green trajectory corresponds to the trial/virtual optical path. **B**. Bi-colored sketch, corresponding to the reflection experiment is shown.

Two kinds of optical paths of the beam generated by the source, located in point S are possible; namely, the actual path, depicted with the red line, in **Figure 1A**, for which the reflection law $\theta_1 = \theta_2$ is true, and the trial/test/virtual path depicted with the green line in **Figure 1A**. The actual/red optical path emerges from the Fermat Principle [1]. The use of the trial pathways is the usual procedure exploited within the variational principles of physics, such as the Hamilton principle and the Fermat principle [1]. We propose now the following mathematical procedure converting the real optical experiment depicted in **Figure 1A** into the scheme shown in **Figure 1B**. Vertices in the graph, presented in **Figure 1B** corresponds to the actual points S and P in the source/mirror experiment, shown in **Figure 1A**, red and green edges/links correspond to the actual/trial optical paths, presented in **Figure 1A**. According to principle of reversibility of light, if the path of the light is reversed after suffering a number of reflections and refractions, then it retraces its path [1]. Thus, vertices S and P may be replaced. It should be emphasized that the coloring scheme, presented in **Figure 1B** is not a graph; in a graph two vertices are connected by the unique edge/link. The introduced coloring scheme omits the real physical processes occurring within the physical system, and represents only actual and trial optical paths of the light propagation.

Now we apply the introduced procedure of the converting of real optical event into the optical scheme emerging from the refraction experiment. When the light forming a ray moves from one medium to another – say from air to the glass slab – the incident ray changes the direction at the boundary between the media; the ray is said to undergo – refraction (see **Figure 2**). Let the index of refraction of the medium with the incident ray be n_1 and that of the medium to be with the refracted ray be n_2. In optics, very roughly speaking, the refractive index (or refraction index) of an optical medium is a dimensionless number that gives the indication of the light bending ability of that medium. The angles, that the incident and refracted rays make with the line normal to the boundary between the media are denoted θ_1 and θ_2 (the angles are shown in **Figure 2**, the normal is depicted with the dashed line). The interrelation between the angles and the refraction indices is given by Eq. 4:

$$n_1 sin\theta_1 = n_2 sin\theta_2 \qquad\qquad (4)$$

This result found by Willibord Snell in 1621 is known as a Snell' law [1]. Restoring historical justice demands to say that refraction and reflection of light were first studied by Ḥasan Ibn al-Haytham (965-1040) a medieval mathematician, astronomer, and physicist of the Islamic Golden Age from present-day Iraq [28]. Ibn al-Haytham was the first to correctly explain the theory of vision, and to argue that vision occurs in the brain, pointing to observations that it is subjective and affected by personal experience [28]. He also stated the principle of least time for refraction, which would later become the Fermat's principle addressed in the Introduction Section. Coverting of the refraction experiment into the bi-colored graph is illustrated with **Figure 2**. The source is located in point S, the sensor is placed in point P. Two kinds of optical paths of the beam generated by the source, located in point S are possible; namely, the actual path, depicted with the red line, in **Figure 2A**, for which the Snell/Ibn al-Haytham is true, namely $n_1 sin\theta_1 = n_2 sin\theta_2$ takes place; and the trial/test/virtual path depicted with the green line in **Figure 2A**.

Again, according to principle of reversibility of light, the vertices S and P may be mutually replaced [1].
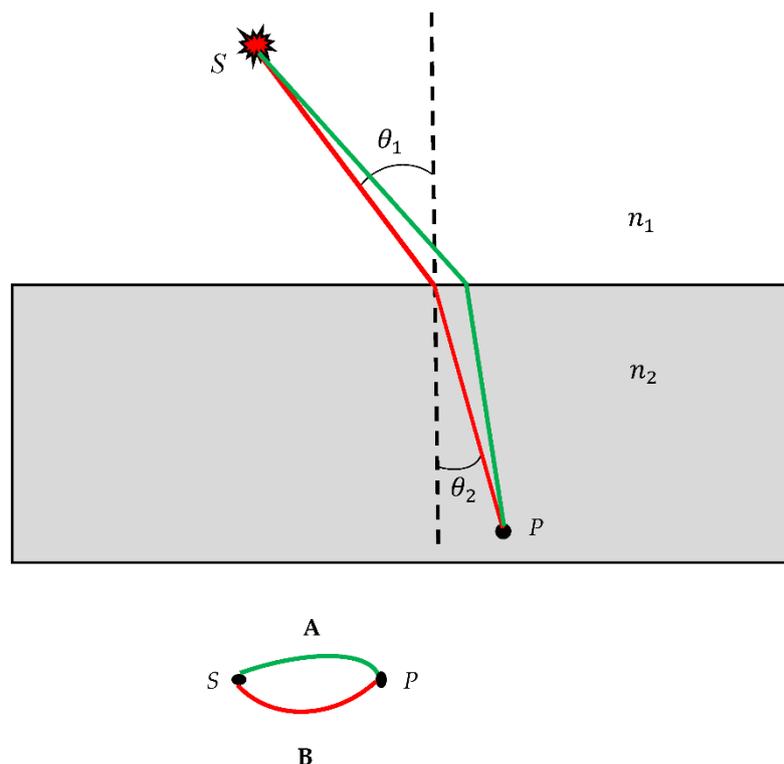

Figure 2. Bi-color sketch emerging from the refraction experiment is depicted. **A**. Light beam passes from the medium with the refraction index n_1 to the medium n_2. The interrelation between the

angles θ_1 and θ_2 is given by the Snell Law: $n_1 sin\theta_1 = n_2 sin\theta_2$; Two kinds of optical paths are depicted. The red trajectory corresponds to the actual optical path; the green trajectory corresponds to the trial/virtual optical path. **B**. Bi-colored scheme, corresponding to the refraction experiment is shown.

Now we consider the optical experiment in which light rays are generated by a pair of sources, located in points "1" and "2", as shown in **Figure 3A**. Light is refracted at the inteface separating media n_1 and n_2 (see **Figure 3A**). Light sources are placed in points "1" and "2"; the sensors are placed in points "3" and "4". Again we consider two kinds of optical paths: actual paths governed by the Snell law, colored with red, and the trial virtual paths, which actually do not take place, shown with green links in **Figure 3**A. Every pair of vertices is connected by a single link. Thus, complete bi-colored graph, depicted in **Figure 3B** emerges (and it should be emphasized that **Figure 3B** depicts a graph). We call this graph the "optical graph". This graph contains no mono-colored (fully red or green) triangle; in other words no opical cycle is recognized in the graph. Indeed, according to the Ramsey Theorem it is possible to create complete bi-colored graph, in which no mono-colored triangle appear, and this is to the $R(3,3) = 6$. Again, it should be emphasized, that the optical graph depicted in **Figure 3B** is a complete, bi-colored, non-directed one [10–14].
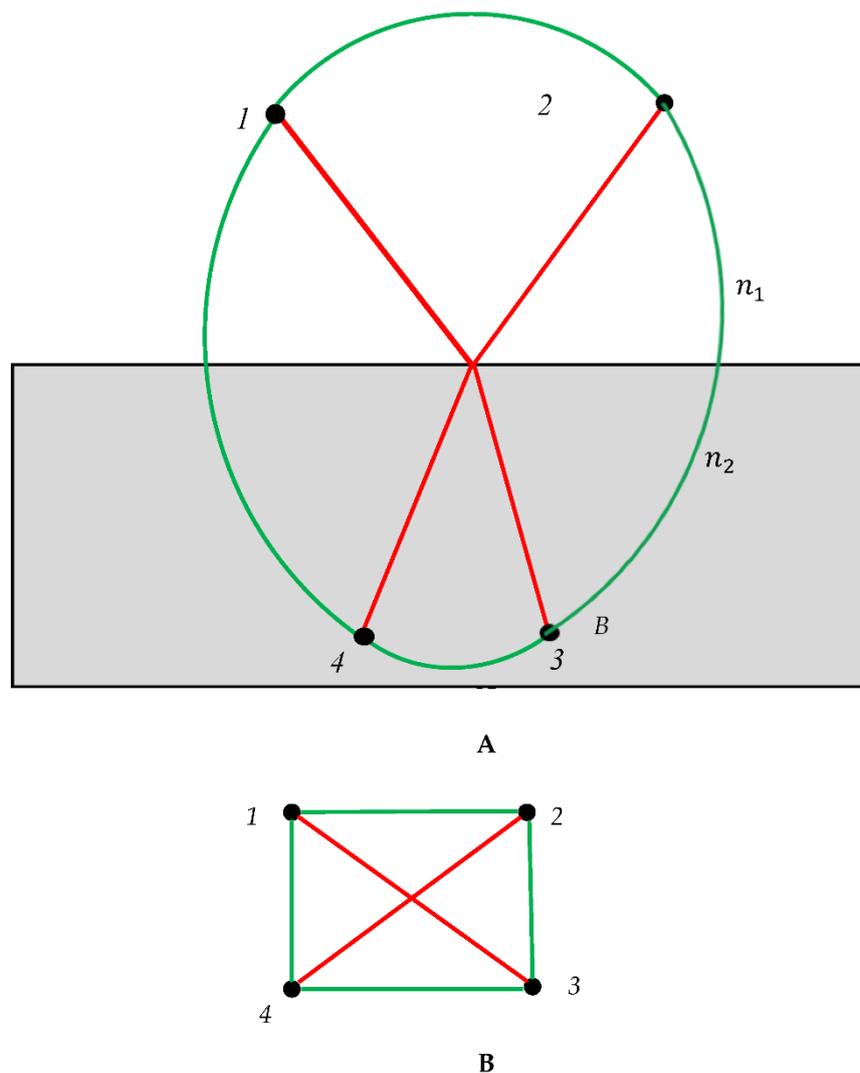

Figure 3. Formation of the graph emerging from the ligth refraction. **A**. Refraction of the light generated by the sources located in points "1" and "2" is depicted. sensors are placed into points "3"

and "4". Red paths coresspond to the actual optical paths; green paths coresspond to the trial optical paths which actually do not occur. **B**. Converting of the optical experiment into the bi-colored, non-directed optical graph, built of 4 vertices and 6 links is depicted. No mono-colored triangle/cycle is reconnized in the graph. This is possible due to the Ramsey Theorem, $R(3,3) = 6$.

Now consider the optical experiment in which we have three sources placed in the points "1", "2" and "3; sensors are placed in points "4", "5" and "6". Again we consider two kinds of optical paths: actual paths (we do not specify the physics of the light propagation), colored with red, and the trial virtual paths, which actually do not take place, shown with green links in **Figure 4**A. Every pair of verices is connected with a unique link/optical path. Thus, the optical complete, non-directed, bi-colored graph, depicted in **Figure 4A** emerges.

It should be emphasized that that the introduced coloring of the links is non transitive. Indeed, consider propagation of the light according to red/actual paths $1 \rightarrow 2$, followed by actual propagation $2 \rightarrow 3$. In this case points "1" and "3" may be conneccted by red/actual or green/virtual paths/links. This fact is very important ia view of application of the Ramsey Theorem for the analysis of complete, bi-colored graphs.
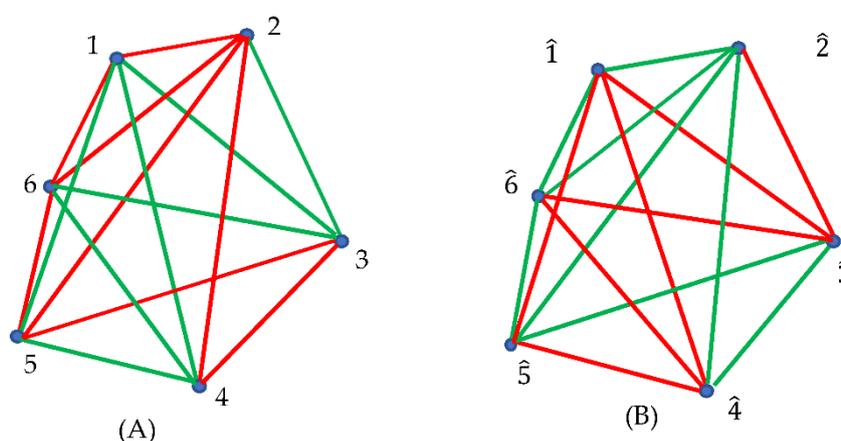

(A)   (B)

Figure 4. **A**. Complete, bi-colored graph corresponding to the optical experiment, in which light sources are located in the points labeled "1", "2" and "3". Red paths coresspond to the actual optical paths; green paths coresspond to the trial optical paths which actually do not occur. The triangle "126" and "256" are mono-colored and built of red edges only. The mono-colored triangles represent actual optical cycles. **B**. The inverse graph emerging from the graph, shown in **Figure 4A** is depicted; triangles $(\hat{1}\hat{2}\hat{6})$ and $(\hat{2}\hat{5}\hat{6})$ are monochromatic and represent the optical cycles which do not occur.

According to the Ramsey theorem, at least one monochromic triangle/cycle should necessarily appear within the graph, shown in **Figure 4A** due to the fact that the Ramsey number is $R(3,3) = 6$. Indeed, the triangles "126" and "256" are built of red edges, and corresponding to actual optical cycles (see **Figure 4A).** And this result will be true for any optical experiment, represented by the bi-colored, complete, undirected graph comprising six verices; namely, at least green or red optical cycle will necessarily appear. Thus, we recognize two so-called "optical cycles", appearing in **Figure 4A**. Optical cycles attracted an essential attention of invesigators in the fields of photonics and nano-photonics recently [29–31]. Regrettably, it is impossible to predict what kind of mono-colored triangle: red/actual or green/virtual one will appear in the graphs; the Ramsey Theory has no tools for such a prediction, and it it is impossible to predict: what kind of cycles (actual or virtual) will be necessarily present in the graph. This is, of course, a weak point of the Ramsey approach.

Now we introduce the notion of the inverse bi-color Ramsey graphs, generated by the source graph; namely we replace red links appearing in the source graph with red ones, and *vice versa*, as shown in inset (**B**) of **Figure 4**. In other words, actual optical paths are replaced with the trial, virtual

ones, and *vice versa*. The inverse optical graph is a complete graph. The vertices of this optical graph are denoted $(\hat{1}, ... \hat{6})$ in inset (**B**) of **Figure 4**. We call such a Ramsey network the "inverse graph". Obviously, introducing an inverse Ramsey network is possible for any complete source graph, and in particular for the graphs, representing optical experiments. According to the Ramsey theorem, both the source and inverse optical graphs, arising from six vertices, contain at least one monochromatic triangle. Thus, actual or virtual optical cycles will be necessarily present in the both of source and inverse optical graphs. Indeed, we recognize red monochromatic triangles (126) and (256) in inset (A), and, correspondingly green triangles $(\hat{1}\hat{2}\hat{6})$ and $(\hat{2}\hat{5}\hat{6})$ in inset (B) of **Figure 4**. Green triangles represent virtual optical cycles, which actually do not occur.

It is noteworthy, that the total number of triangles in the "direct" (source) and "inverse" Ramsey optical graphs is the same, thus, yielding the conservation law:

$$\zeta = t_r + t_g = \hat{t}_r + \hat{t}_g, \tag{5}$$

where t_r and t_g are the numbers of red and green triangles in the source graph; $\hat{t}_r$ and $\hat{t}_g$ are correspondingly the numbers of red and green triangles in the inverse graph. Eq. 5 represents the "conservation law" for the Ramsey complete networks built of six elements. It is noteworthy that direct and inverse graphs form the Abelian (commutative group), when the inversion of the color of the link is taken as an operation.

2.2. Ramsey approach to metamaterials

The alternative application of the Ramsey approach is possible for the light propagating in metamaterials (left-handed media [5,6]). We already mentioned that in the metamaterials the Fermat principle should be generalized as follows: a light ray in going from point S to point P must traverse an optical path length L that is stationary with respect to variations of this path. By a stationary value of the function $L(s)$ (see Eq. 1), we mean one for which the slope of $L(s)$ versus s is zero or equivalently where the function $L(s)$ has a maximum, minimum or a point of inflection with a horizontal tangent [1,6]. The kind of extremum (maximum, minimum or inflection point) depends on the actual values of the refraction index of the medium [1,6]. Thus, one more optical interpretation of the Ramsey theory becomes possible in meta-materials. For a sake of simplicity, we exclude the exotic situation, when the optical path corresponds to the inflection point of the function $L(s)$. Thus, bi-colored, complete graphs, similar to the depicted in **Figure 5** become possible.
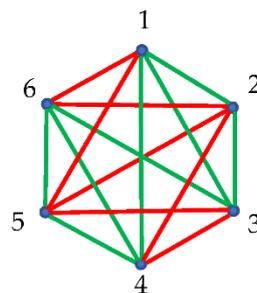

Figure 5. Complete optical graph corresponding to propagation of light in the meta-material. The vertices of the graph correspond to the point of the physical space in which sources or sensors are placed. Links corresspond to the actual optical rays. Red links correspond to the optical paths for the function $L(s)$ demonstrates maximum; green links corresspond to the optical paths for the function $L(s)$ demonstrates minimum. Triangles "456" and "123" are mono-colored (green).

Consider the graph built of six vertices, corresponding to the points of the physical space, in which sources or sensors are placed. All of the links of the graph correspond to the actual optical pathways. We assume that red links correspond to the optical paths for the function $L(s)$ demonstrates maximum; green links, in turn, correspond to the optical paths for the function

$L(s)$ demonstrates minimum. Thus, the complete, undirected, optical graph, such depicted in **Figure 5** emerges. According to the Ramsey Theorem this complete graph inevitably contains at least one mono-colored triangle. Indeed, triangles "456" and "123" are mono-colored (green). Thus, two optical cycles built of optical pathways "456" and "123", for which $L(s)$ is minimal will be observed in the addressed optical experiment *[29–31]*. And now, these pathways are actual optical paths.

Again, the inverse optical graph, may be defined according to the procedure introduced in Section 2.1, and the conservation law resumed by Eq. 5 will take place.

2.3. Optical Interpretation of the Infinite Ramsey Theorem

Until now, we consider finite graphs. In this Section we will address infinite complete graphs, Infinite Ramsey Theory and their optical interpretation, giving rise to infinite optical graphs. The Infinite Ramsey Theory states: if the complete graph of a countably infinite set is colored with finitely many colors, there is an infinitely monochromatic clique. A clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent. Let us illustrate the Infinite Ramsey Theorem with an understandable example **[11–15]**: let us think about all the positive integers, and imagine joining every pair with a line, as shown in Figure 6. Every pair of positive integers is joined by a line/link. Let us denote the emerging graph as K_∞. Now we color each link either red or green; thus, we build the complete two-colored graph. The infinite two-color Ramsey Theorem States that: no matter how we two-color the edges in K_∞, it will always be possible to find infinitely many points that are all connected by the same color. In other words: consider K_∞ which is the complete graph whose vertex set is countably infinite, every 2-colored K_∞ must contain a countably infinite monochromatic complete graph. The formulation of the infinite Ramsey theorem for multi-colored graphs, states that if we color each edge of an infinite, complete graph, with one out of finitely many prescribed colors, then there is an infinite, complete monochromatic subgraph. That is, an infinite set of vertices such that all edges among them have the same color.
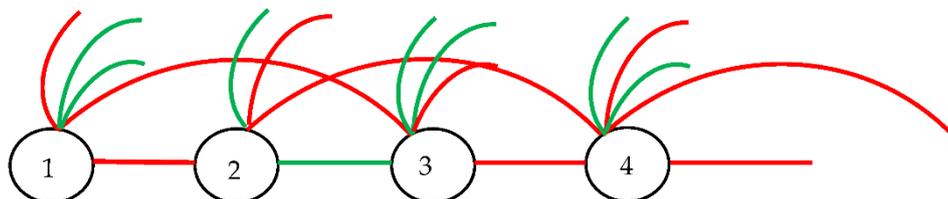

Figure 6. Infinite Ramsey Theorem is illustrated for twin-colored infinite graphs: any two-colored K_N must contain a red or green monochromatic countably infinite complete subgraph.

The more general formulation of the Infinite Ramsey Theorem states, that if we split an infinite object with a certain regularity property (such a set containing arbitrary long arithmetical progressions) into two parts, one infinite part will exhibit this property too [12].

Now let us supply the optical interpretation of the Infinite Ramsey Theory. Consider the infinite number of the points of the physical space, numbered "1", "2", "3"… Sources or sensors are located in the points. Every point is joint by a link corresponding to the optical event/optical path, namely red link corresponds to actual optical paths coressponding to the light propagation between the connected points; a green link, in turn, corresponds to the trial/virtual optical path, which does not actually occurs. Two points are connected with a single link. Thus, the complete, undirected optical graph, similar to that, depicted in **Figure 6** emerges. According to the Infinite Ramsey Theorem, infinite, complete monochromatic subgraph (either green or red) will necessarily appear in the graph, in other words an infinite monochromatic clique will be present in the graph. Thus, the actual or trial/virtual optical path connecting all of the points will be necessarily present in the graph. According to principle of reversibility of light the addressed infinite bi-colored graph will be non-directional. The generalization of the infinite Ramsey Theorem for metamaterials is straightforward

(see Section 2.2), if we assume that red links correspond to the optical paths for the function $L(s)$ demonstrates maximum; green links, in turn, correspond to the optical paths for the function $L(s)$ demonstrates minimum.

4. Discussion

The paper presents the synthesis of the Fermat principle and Ramsey Theory. The variational principles of physics and in particular the Fermat Principle of the least optical path length classify all of possible paths connecting the points of physical space as actual or virtual. Actual paths correspond to trajectories of the light propagation; whereas, virtual or trial paths correspond to light ray trajectories which do not occur. We define, that physical points are "acquainted one with another" when are connected with an actual optical path, connecting sources and sensors; and they are "strangers" when they are connected with the virtual/trial optical paths. Thus, introduction of the Ramsey approach for the analysis of the light propagation in complex media becomes possible. Within this approach we are completely distracted from the peculiarities of optical events taking place under propagation of the light ray from one point to another; we consider only the demands of the Fermat Principle, which may be fulfilled or violated. Thus we conclude that in the "optical graph" containing six vertices, appearance of at least one actual or virtual optical cycle is inevitable, due to the fact that $R(3,3) = 6$.

5. Conclusions

Ramsey Theory supplies an abstract framework for the analysis of a broad diversity of events/facts/states interconnected by various relations, forming a complete graph. In the simplest case of bi-colored graphs, the vertices may be considered as "friends" or "strangers". Vertices may represent interacting particles or thermodynamic states [22], and the links of the graph represent at least two kinds of interrelations between the vertices (such as attraction or repulsion between interacting particles [32]). We demonstrate, how "optical Ramsey graphs" may be introduced for convential optical media and meta-materials. For this purpose we involve the Fermat principle of the least optical path lengths, and its generalization for meta-materials. This principle divides all of possible paths connecting two points, in which light sources and sensors are located, into two classes, namely: the actual paths/"friends" and the trial paths/"strangers". The verices/points are connected with a single link/optical path. This classification gives rise to the bi-colored, complete, non-directional graph, representing real optical systems. Optical pathways in this graph connect the veritces, which represent the points of a real physical space. According to principle of reversibility of light the suggested complete bi-colored graph will be non-directional. Thus, application of the Ramsey theorem becomes possible. For example, in the graph built of six vertices, at least one monochromatic optical cycle (actual or trial) will be necessarily present. Regrettably, the Ramsey theory does not predict, what kind of monochromatic cycle: actual or trial will appear in the graph, and this is a weak point of the Ramsey approach. We introduce the notion of the "inverse optical graph", generated by the original, source optical graph. Consider the bi-colored, complete optical graph. Let us replace the actual optical paths (red links) by the trial optical pathways (green links) and *vice versa*; this procedure gives rise to the inverse optical graph. The total number of triangles in the "direct" (source) and "inverse" Ramsey optical graphs ζ is the same; thus, yielding the conservation law: $\zeta = t_r + t_g = \hat{t}_r + \hat{t}_g$, where t_r and t_g are the numbers of red and green triangles in the source optical graph; $\hat{t}_r$ and $\hat{t}_g$ are correspondingly the numbers of red and green triangles in the inverse optical graph.

The alternative interpretation of the Ramsey approach becomes possible for the light propagating in meta-materials (left-handed media). In the metamaterials, the Fermat principle is generalized as follows: a light ray in going from point S to point P must traverse an optical path length $L(s)$ that is stationary with respect to variations of this path s. We assume now that red links correspond to the optical paths for the function $L(s)$ demonstrates maximum; green links, in turn, correspond to the optical paths for which the function $L(s)$ demonstrates minimum (we assume, that the function $L(s)$ does not demonstrate inflection points; this assumption keeps the binary paradigm

of analysis). Thus, the complete bi-colored optical graph, emerges. According to the Ramsey Theorem the complete bi-colored graph built of six vertices inevitably contains at least one mono-colored triangle. Thus, minimal, 3-vertices optical cycles will be necessarily observed in the addressed optical experiment. And now, these pathways are actual optical paths.

We introduce the notion of the infinite optical graph built of the infinite number of the points of the physical space, numbered "1", "2", "3"… Every point is joint to another point by a link corresponding to the optical event, namely red link corresponds to actual optical paths representing the actual light propagation between the connected points; a green link in turn corresponds to the trial/virtual optical path, which does not actually occurs. Two points are connected with a single link. Thus, the complete infinite, non-directional optical graph emerges. According to the Infinite Ramsey Theorem, infinite, complete monochromatic subgraph (either green or red) will necessarily appear in the graph. Thus, the actual or trial/virtual optical path connecting all of the points will be necessarily present in the graph.

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