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Reachable Set Estimation and Controller Design for Linear Time-Delayed Control System with Disturbances

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Abstract: This paper investigates reachable set estimation and state-feedback controller design for linear time-delay control system with bounded disturbances. First, by constructing an appropriate Lyapunov-Krasovskii functional, we obtained a delay-dependent condition, which determed the admissible bounding ellipsoid for the reachable set of the system we considered. Then, a sufficient condition in form of liner matrix inequalities is given to solve the problem of controller design with reachable set estimation. Finally, by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. A comparative numerical example is given to verify the usefulness of our result.

Keywords: time-delay; ellipsoid; Lyapunov-Krasovskii functional; reachable set; linear matrix inequalities

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1. Introduction

The reachable set estimation of dynamic systems is an important research topic in control theory, since it has a large number of applications in control systems with actuator saturation([12,14,16]), peak-to-peak gain minimization [1] and aircraft collision avoidance[13]. The reachable set of a dynamic system with bounded peak input is defined as the set of system state vectors in the presence of all allowed input disturbances. Reachable set bounding was first considered in the late 1960s in the context of state estimation and it has later received a lot of attention in parameter estimation [4]. Boyd et al. researched the problem of reachable set estimation of linear systems without time-delay and got an LMI condition for an ellipsoid that bounds the reachable set[2].

It is well kown the existence of time delay is extremely common in practice, such as aircraft, chemical processes, long pipeline supply, belt transmission and extremely complex online analyzers in various industrial systems. Usually the occurrence of time delay may lead to instability or performance degradation of dynamic systems [9,13,19,26,29,30]. Therefore, extensive reseachers are devoted to research reachable set estimation issue of dynamic systems with delay. During the past few decades, there have been some excellent results related to the reachability set estimation of time-delay systems [3,5,6,17,18,21,25,27–29,31–37].

In [5], based on the Lyapunov-Razumikhin method, Fridman and Shaked firstly investigated the reachable set estimation of a linear system with time-varying delay and obtained a LMI criteria of an ellipsoid bounding the set of reachable states. Kim applied Lyapunov-Krasovskii functional to get an improved ellipsoidal bound of reachable set [17]. Nam and Pathirana employed the delay decomposition technique to get a smaller reachable set bound [21]. Zuo et al. got a non-ellipsoidal bound of a reachable set of linear time-delayed systems through the maximal Lyapunov functionals and the Razumikhin method [35]. More recently, Zhang et al. investigated the reachable set estimation

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for uncertain nonlinear systems with time delay.[36] However, there are few works on the estimation of reachable set for linear time-delayed control system with disturbances, thus which motivated us to write this paper.

In this paper, we intend to design state feedback controller so that the reachable set of the resulting closed-loop system is contained in an ellipsoid, and the admissible ellipsoid should be as small as possible. The rest of this article is organized as follows. In section 2, in order to obtain main result facilitately, some useful lemmas and preliminary knowledge are given. In section 3, First, by constructing an appropriate Lyapunov-Krasovskii functional, we obtained a condition related to delay-dependent, which determed the admissible bounding ellipsoid for the reachable set of the system we considered. Then, a sufficient condition in form of liner matrix inequalities is given to solve the problem of controller design with reachable set estimation.finally,by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. In section 4, a comparative numerical example is given to verify the usefulness of the proposed methods. The paper ends up with conclusions and references.

Notation

Throughout this paper, the notations are standard. $\mathbb{R}^n$ is the vector of real numbers, $\mathbb{R}^{n\times m}$ is the $n\times m$ real matrix, I is the identity matrix, 0 is the zero matrix,and A^T presents the transpose of A. For a matrix P, $P>0$ denotes P is a symmetric positive definite matrix, also, $x_t(\theta)=x(t+\theta),\theta\in[-h,0]$, and $(\star)$ in a matrix presents the symmetric part.

2. Problem statement and preliminaries

Consider the following linear time-delay control system with bounded disturbances:

$$\begin{cases} \dot{x}(t)=Ax(t)+Dx(t-d(t))+Bu(t)+Ew(t); \\ x(t)\equiv 0, t\in[-h,0] \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)$ is the control vector, $A\in\mathbb{R}^{n\times n}, D\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times n}$ and $E\in\mathbb{R}^{n\times m}$, A, D, B and E are constant matrices ; $w(t)\in\mathbb{R}^m$ is the disturbance satsfying

$$w^T(t)w(t)\leq w_m^2 \tag{2}$$

and $d(t)$ is time-varying delay satisfying

$$0\leq d(t)\leq h, \qquad |\dot{d}(t)|\leq u\leq 1 \tag{3}$$

where w_m, d and u are constants.

In this paper, based on the modify Lyapunov-Krasovskii functional, which is used for exponential stability analysis in[20,24], we intend to design state feedback controller K, G, that is $u(t)=Kx(t)+Gx(t-d(t))$, such that the reachable set of closed-loop system

$$\dot{x}(t)=(A+BK)x(t)+(D+BG)x(t-d(t))+Ew(t) \tag{4}$$

is bounds by an ellipsoid $\varepsilon(P,1)$:

$$\varepsilon(P,1)=\left\{x\in\mathbb{R}^n : x^TPx\leq 1; P>0\right\} \tag{5}$$

The reachable set of system (4) is denoted as follows:

$$R_x=\{x(t)\,|x(t)\ and\ w(t)\ satisfy\ (2)\ and\ (3), t\geq 0\}$$

The following three useful lemmas are given to obtain main result facilitately.

Lemma 2.1. *[22]The following relation is known as the Leibniz rule*

$$\frac{d}{dt}\int_{b(t)}^{a(t)} f(t,s)\,ds = \dot{a}(t)f[t,a(t)] - \dot{b}(t)f[t,b(t)] + \int_{b(t)}^{a(t)} \frac{\delta}{\delta t} f(t,s)\,ds$$

Lemma 2.2. *[7] For any constant matrix $Q = Q^T > 0$, we have*

$$d(t)\int_{t-d(t)}^{t} f^T(s)Qf(s)\,ds \geq [\int_{t-d(t)}^{t} f(s)\,ds]^T Q[\int_{t-d(t)}^{t} f(s)\,ds]$$

Lemma 2.3. *[2] Let Q be a symmetric positive definite matrix. For any matrices P, S with appropriate dimensions, where $P = P^T$, then*

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0$$

if and only if $P - SQ^{-1}S^T > 0$.

Lemma 2.4. *[2] Let $V(x(0)) = 0$ and $w^T(t)w(t) \leq w_m^2$, if $\dot{V}(x_t) + \alpha V(x_t) - \beta w^T(t)w(t) \leq 0, \alpha > 0, \beta > 0$, then we have $V(x_t) \leq \frac{\beta}{\alpha} w_m^2, \forall t > 0$.*

3. Main results

Theorem 3.1. *For given scalars $h, u > 0$, if there exist matrices $L, H \in \mathbb{R}^{1\times n}, \overline{M}, \tilde{P}, \tilde{R}, \tilde{S}, \tilde{W}, \tilde{X}, \tilde{Y}, \tilde{Z}, \hat{R}, \hat{Z}, \check{Z} \in \mathbb{R}^{n\times n}$ with $\overline{M}, \tilde{R}, \tilde{S}, \tilde{W}, \tilde{X} > 0$ and a scalar $\alpha > 0$ such that they satisfy the following matrix inequalities:*

$$\begin{bmatrix} \overline{\Phi}_{11} & D\overline{M} + BH - \tilde{Y} & \overline{M}A^T + L^T B^T + \check{Z} + \alpha\overline{M} & E & \tilde{Y} \\ \star & -(1-u)e^{-\alpha h}\tilde{S} + u^2\tilde{W} & \overline{M}D + H^T B^T - \check{Z} & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}\hat{R} + \alpha\hat{Z} & E & \check{Z}^T \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -\tilde{W} \end{bmatrix} \leq 0 \quad (6)$$

$$\begin{bmatrix} \overline{M} - \tilde{P} & \tilde{Y} \\ \star & \tilde{Z} + \frac{1}{h}e^{-\alpha h}\tilde{S} \end{bmatrix} \geq 0 \quad (7)$$

where $\overline{\Phi}_{11} = A\overline{M} + BL + \overline{M}A^T + L^T B^T + \alpha\overline{M} + \tilde{Y} + \tilde{Y}^T + \tilde{S} + h\tilde{R}$.

Then the reachable sets of the system (4) is bounded by an ellipsoid $\varepsilon(P, 1)$ defined in (5). At this point, the state feedback gain is $K = L\overline{M}^{-1}, G = H\overline{M}^{-1}$.

Proof. To prove this theorem, let us consider the following Lyapunov-Krasovskii function candidate:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \quad (8)$$

where

$$V_1(x_t) = x^T(t)Px(t)$$

$$V_2(x_t) = \int_{t-d(t)}^{t} e^{-\alpha(s-t)}[x^T(s)Sx(s) + (h - t + s)x^T(s)Rx(s)]\,ds$$

$$V_3(x_t) = \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}$$

$$\eta(t) = \int_{t-d(t)}^{t} x(s)\,ds$$

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and P, S, R, X, Y, Z are symmetric matrices with appropriate dimensions. First, we prove that $V(x_t)$ in (8) is a good Lyapunov-Krasovskii functional candidate. For $t - d(t) \leq s \leq t$ and $0 \leq d(t) \leq h$, we can get $e^{-h} \leq e^{-d(t)} \leq e^{s-t} \leq 1$ and $0 \leq h - d(t) \leq h - t + s \leq h$. In the light of the Lamma 2.2, we have

$$V_2(x_t) \geq \int_{t-d(t)}^{t} e^{-\alpha(s-t)} x^T(s)Sx(s)\,ds \geq \frac{1}{h} e^{-\alpha h} \eta^T(t) S \eta(t)$$

therefore

$$V_2(x_t) + V_3(x_t) \geq \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}$$

If

$$\begin{bmatrix} X & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \geq 0 \tag{9}$$

then we have $V_2(x_t) + V_3(x_t) \geq 0$.
hence

$$\begin{cases} V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \geq V_1(x_t) = x^T(t)Px(t), \\ V(x_t) = 0, \ when \ \ x(s) = 0, \forall s \in [t - d(t), t]. \end{cases} \tag{10}$$

which shows $V(x_t)$ in (8) is a good L-K functional.
Next, according to Lemma 2.1, we obtain the following time-derivatives:

$$\frac{d}{dt}V_1(x_t) = 2x^T(t)P[(A + BK)x(t) + (D + BG)x(t - d(t)) + Ew(t)] \tag{11}$$

$$\begin{aligned}
\frac{d}{dt}V_2(x_t) &= x^T(t)(S + hR)x(t) - (1 - \dot{d}(t))e^{-\alpha d(t)} x^T(t - d(t))Sx(t - d(t)) \\
&- (1 - \dot{d}(t))e^{-\alpha d(t)}(h - d(t))x^T(t - d(t))Rx(t - d(t)) \\
&- \int_{t-d(t)}^{t} e^{-\alpha(s-t)} x^T(s)Rx(s)\,ds - \alpha V_2(x_t) \\
&\leq x^T(t)(S + hR)x(t) - (1 - u)e^{-\alpha h} x^T(t - d(t))Sx(t - d(t)) \\
&- \frac{1}{h}e^{-\alpha h}\eta^T(t)R\eta(t) - \alpha V_2(x_t)
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{d}{dt}V_3(x_t) &= 2\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ (x(t) - x(t - d(t)) \end{bmatrix} \\
&+ 2\dot{d}\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} x(t - d(t)) \\
&\leq 2\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \begin{bmatrix} (A + BK)x(t) + (D + BG)x(t - d(t)) + Ew(t) \\ x(t) - x(t - d(t)) \end{bmatrix} \\
&+ \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} W^{-1} \begin{bmatrix} Y^T & Z^T \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + u^2 x^T(t - d(t))Wx(t - d(t))
\end{aligned} \tag{13}$$

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where we used the relation that $2a^T b \leq a^T W^{-1} a + b^T W b, W > 0$ and the constraints (2) and (3) in the derivation of the inequality (13).
Through (9) and (11)-(13), we obtain

$$\begin{aligned}&\dot{V}(x_t) + \alpha V(x_t) - \beta w^T(t) w(t)\\ &\leq \xi_t^T \left(\Omega + \begin{bmatrix} Y^T \\ 0 \\ Z^T \\ 0 \end{bmatrix} W^{-1} \begin{bmatrix} Y^T & 0 & Z^T & 0 \end{bmatrix} \right) \xi_t\\ &:= \xi_t^T \Psi \xi_t\end{aligned}$$

where

$$\xi_t^T = [x^T(t) \;\; x^T(t-d(t)) \;\; \eta^T(t) \;\; w^T(t)]$$

$$\Omega = \begin{bmatrix} \phi_{11} & (P+X)(D+BG) - Y & (A+BK)^T Y + Z + \alpha Y & (P+X)E \\ \star & -(1-u)e^{-\alpha h} S + u^2 W & (D+BG)^T Y - Z & 0 \\ \star & \star & -\frac{1}{h} e^{-\alpha h} + \alpha Z & Y^T E \\ \star & \star & \star & \frac{-\alpha}{w_m^2} \end{bmatrix}$$

and $\phi_{11} = (P+X)(A+BK) + (A+BK)^T(P+X) + \alpha(P+X) + Y + Y^T + S + hR$. If $\Psi \leq 0$, by applying the Lamma 2.3, we get

$$\begin{bmatrix} \Phi_{11} & M(D+BG) - Y & (A+BK)^T Y + Z + \alpha Y & ME & Y \\ \star & -(1-u)e^{-\alpha h} S + u^2 W & (D+BG)^T Y - Z & 0 & 0 \\ \star & \star & -\frac{1}{h} e^{-\alpha h} + \alpha Z & Y^T E & Z \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -W \end{bmatrix} \leq 0 \tag{14}$$

where $M = P + X$, $\Phi_{11} = M(A+BK) + (A+BK)^T M + \alpha M + Y + Y^T + S + hR$. Define $N_1 = diag(M^{-1}; M^{-1}; Y^{-1}; I; M^{-1})$, Pre- and post-multiplying the inequality (14) by N_1 and N_1^T, and defining $\overline{M} = M^{-1}, L = KM^{-1}, H = GM^{-1}, \tilde{X} = M^{-1} X M^{-1}, \tilde{Y} = M^{-1} Y M^{-1}, \tilde{R} = M^{-1} R M^{-1}, \tilde{Z} = M^{-1} Z M^{-1}, \tilde{S} = M^{-1} S M^{-1}, \tilde{W} = M^{-1} W M^{-1}, \check{Z} = M^{-1} Z Y^{-1}, \hat{R} = Y^{-1} R Y^{-1}, \hat{Z} = Y^{-1} Z Y^{-1}$, the following inequality is derived:

$$\begin{bmatrix} \overline{\Phi}_{11} & D\overline{M} + BH - \tilde{Y} & \overline{M} A^T + L^T B^T + \check{Z} + \alpha \overline{M} & E & \tilde{Y} \\ \star & -(1-u)e^{-\alpha h} \tilde{S} + u^2 \tilde{W} & \overline{M} D + H^T B^T - \check{Z} & 0 & 0 \\ \star & \star & -\frac{1}{h} e^{-\alpha h} \hat{R} + \alpha \hat{Z} & E & \check{Z}^T \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -\tilde{W} \end{bmatrix} \leq 0 \tag{15}$$

where $\overline{M} = (P+X)^{-1}, \overline{\Phi}_{11} = A\overline{M} + BL + \overline{M} A^T + L^T B^T + \alpha \overline{M} + \tilde{Y} + \tilde{Y}^T + \tilde{S} + h\tilde{R}$. Thus, if inequality (15) holds, we have

$$\dot{V}(x_t) + \alpha V(x_t) - \frac{\alpha}{w_m^2} w^T(t) w(t) \leq 0$$

which means, by the Lemma 2.4, that $V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \leq 1$. since $V_2(x_t) + V_3(x_t) \geq 0$ from inequality (9), therefore, we get $V_1(x_t) = x^T(t) P x(t) \leq 1$. Following a similar line, we need to convert conditon (9) into

$$\begin{bmatrix} M - P & Y \\ \star & Z + \frac{1}{h} e^{-\alpha h} S \end{bmatrix} \geq 0 \tag{16}$$

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Define $N_2 = diag(M^{-1}; M^{-1})$, Pre- and post-multiplying the inequality (16) by N_2 and N_2^T, the following inequality is derived:

$$\begin{bmatrix} \overline{M} - \tilde{P} & \tilde{Y} \\ \star & \tilde{Z} + \frac{1}{h}e^{-\alpha h}\tilde{S} \end{bmatrix} \geq 0 \tag{17}$$

This completes the proof. This implies that the reachable sets of closed-loop system in (4) is bounded by the ellipsoid $\varepsilon(P, 1)$ defined in (5), and the desired state-feedback controller can be obtained as $K = L\overline{M}^{-1}, G = H\overline{M}^{-1}$. □

Remark 3.1 In order to get the 'smallest' possible bound for the reachable set, we introduce the method in [5,17]. That is, maximise δ subject to $\delta I \leq P$, which can be transformed to the following optimisation problem for a scalar $\delta > 0$:

$$\begin{cases} min & \bar{\delta}\ ,\ (\bar{\delta} = \frac{1}{\delta}) \\ s.t & \begin{bmatrix} \bar{\delta} & I \\ I & P \end{bmatrix} \geq 0 \end{cases} \tag{18}$$

Then ,define $N_3 = diag(I; \overline{M})$, Pre- and post-multiplying inequality (18) by N_3 and N_3^T ,and defining $\tilde{P} = \overline{M}P\overline{M}$, the following inequality is derived:

$$\begin{cases} min & \bar{\delta}\ ,\ (\bar{\delta} = \frac{1}{\delta}) \\ s.t & \begin{bmatrix} \bar{\delta} & \overline{M} \\ \overline{M} & \tilde{P} \end{bmatrix} \geq 0 \end{cases} \tag{19}$$

Therefore, we get the 'smallest' possible bound for the reachable set of system (4), by solving the following optimisation problem for a scalar $\delta > 0$:

$$\begin{aligned} & min\ \ \bar{\delta}\ ,\ (\bar{\delta} = \tfrac{1}{\delta}) \\ & s.t \begin{cases} (a) & \begin{bmatrix} \bar{\delta} & \overline{M} \\ \overline{M} & \tilde{P} \end{bmatrix} \geq 0 \\ (b) & (6), (7) or (15), (17) \end{cases} \end{aligned} \tag{20}$$

Remark 3.2 In [17],for given scalars $h, u > 0$, if there exist matrices $P, S, R, W, X, Y, Z \in \mathbb{R}^{n\times n}$ with $P, S, R, W > 0$ and a scalar $\alpha > 0$ such that they satisfy the following matrix inequalities:

$$\begin{bmatrix} \Phi_{11} & (P+X)D - Y & A^TY + Z + \alpha Y & (P+X)E & Y \\ \star & -(1-u)e^{-\alpha h}S + u^2W & D^TY - Z & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h} + \alpha Z & Y^TE & Z \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -W \end{bmatrix} \leq 0 \tag{21}$$

$$\begin{bmatrix} X & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \geq 0 \tag{22}$$

where $\Phi_{11} = (P+X)A + A^T(P+X) + \alpha(P+X) + Y + Y^T + S + hR$.

To find the 'smallest' bound for the reachable set, one may propose a simple optimisation problem. That is, maximise δ subject to $\delta I \leq P$, which can be transformed to the following optimisation problem for a scalar $\delta > 0$:

$$\begin{aligned} & min\ \ \bar{\delta}\ ,\ (\bar{\delta} = \tfrac{1}{\delta}) \\ & s.t \begin{cases} (a) & \begin{bmatrix} \bar{\delta} & I \\ I & P \end{bmatrix} \geq 0 \\ (b) & (21), (22) \end{cases} \end{aligned} \tag{23}$$

Remark 3.3 If we let $B = 0, K = 0$ and $G = 0$ in (6),(7) of Theorem 3.1, the condition becomes the condition in [17], also (20) becomes (23) in Remark 3.2, in this respect, the conclusion can be seen as an extension of [17].

4. Numerical example

In this section, A example is given to illustrate our proposed approach. The simulation is carried out using Matlab and the LMI, a package for specifying and solving linear matrix inequalities.

Consider the linear state-delayed control system (1) with the following parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, w_m = 1$$

By solving optimization problem (20), we get the sizes of the ellipsoidal bound of a reachable set for various u when $h = 0.70$ and $h = 0.75$. These results are summarized in the following Tables 1 and 2, and are compared to the previous results in [17].

Table 1. Computed $\bar{\delta}$'s in Example 1 for $0 \le d(t) \le 0.7, |\dot{d}(t)| \le u \le 1$.

	u						
Method	0	0.1	0.2	0.3	0.4	0.5	0.6
[17]	2.2586	2.4970	2.8497	3.4355	4.5384	7.0915	16.8263
Theorem 3.1	1.4571	1.6372	1.8905	2.2702	2.9171	4.2496	8.1427

As we can see in the above table 1, when $h = 0.7, u = 0.6$, our results greatly reduce the size of the ellipsoid. At this point, the state feedback gain is $K = [-0.5209 \quad -1.3698], G = [0.9414 \quad -0.4495]$.

Table 2. Computed $\bar{\delta}$'s in Example 1 for $0 \le d(t) \le 0.75, |\dot{d}(t)| \le u \le 1$

	u						
Method	0	0.1	0.2	0.3	0.4	0.5	0.6
[17]	2.5077	2.8071	3.2462	3.9935	5.4419	8.9945	25.1048
Theorem 3.1	1.6222	1.8417	2.1363	2.5992	3.4134	5.1046	10.4056

It is obvious that the results of this paper are better than the autonomous systems in [17], which verifies the effectiveness of the proposed method.

5. Conclusion

In this paper, we deal with the problem of reachable set estimation and state-feedback controller design for linear time-delay control system with bounded disturbances. Firstly, by constructing an appropriate L-K functional, we obtained a delay-dependent condition, which determed the admissible bounding ellipsoid for the reachable set of the system we considered. Secondly, a sufficient condition in form of liner matrix inequalities is given to solve the problem of controller design with reachable set estimation.finally, by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. A comparative numerical example shows that the results of this paper are better than the autonomous systems in [17], which verifies the effectiveness of the proposed method. Due to the limitation of controller design, the conclusion of this paper is conservative.

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