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Article

An Explicit Form of Ramp Function

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Abstract: In this paper, an analytical exact form of Rump Function is presented. This seminal function constitutes a fundamental concept of digital signal processing theory and is also involved in many other areas of applied sciences and engineering. In particular, Rump Function is performed in a simple manner as the limit of a sequence of real functions letting n tend to infinity. This limit is zero for strictly negative values of the real variable x whereas it coincides with the independent variable x for strictly positive values of the variable x. The novelty of this work when compared to other research studies concerning analytical expressions of the Ramp Function, is that the proposed formula is not exhibited in terms of miscellaneous special functions, e.g. Gamma Function, Biexponential Function or any other special functions such as Error Function, Hyperbolic Function, Orthogonal polynomials etc. Hence, this formula may be much more practical, flexible and useful in the computational procedures which are inserted into digital signal processing techniques and other engineering practices.

Keywords: Ramp Function; analytical expression; absolute value

Mathematical subject classification: Special Functions

1. Introduction

The Ramp Function, the notation of which is $\mathrm{R}(x)$, is a discontinuous single - valued function of a real variable with a point discontinuity located at zero. For negative arguments, $\mathrm{R}(x)$ vanishes whilst for positive arguments, $\mathrm{R}(x)$ is simply x [1]. In addition, its first derivative is the Heaviside Step function, also known as the Unit Step Function, whereas its second derivative is the Dirac delta distribution (or δ distribution), also known as the unit impulse [2]. Step, ramp and parabolic functions are called singularity functions [2,3]. In fact, the ramp function has many applications in applied sciences/engineering and is mainly involved in digital signal processing and electrical engineering. Actually it constitutes a signal the amplitude of which varies linearly with respect to time and can be expressed by several definitions [4,5]. In digital signal processing, the unit ramp function is a discrete time signal that starts from zero and increases linearly. Here one may emphasize that the basic Continuous - Time (CT) and Discrete - Time (DT) signals include impulse, step, ramp, parabolic, rectangular pulse, triangular pulse, signum function, sinc function, sinusoid and finally real along with complex exponentials [4–6]. Ramp Function states that the signal will start from time zero and instantly will take a slant shape and depending upon given time characteristics (i.e. either positive or negative, here positive) the signal will follow the straight slant path either towards right or left, here towards right [6,7]. In this context, the ramp function constitutes a type of elementary function which exists only for the positive side and is zero for negative [8–10]. Moreover, the impulse function is obtained by differentiating the ramp function twice [9,10]. On the other hand, it is well - known that an electrical network consists of passive elements like resistors, capacitors and inductors. They are connected in series, parallel and series parallel combinations [2,11,12]. The currents through and voltages across these elements are obtained by solving integro-differential equations. Alternatively, the elements in the network are transformed from the time domain and an algebraic equation is obtained which is expressed in terms of input and output [11,12]. The commonly used inputs are impulse, step, ramp, sinusoids, exponentials etc. In addition to the aforementioned above that

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demonstrate the central role of this function in digital signal processing and electrical engineering, we have to mention the fact that the Ramp Function has many other applications in finance as well as in applied statistics (e.g regression models) etc [2,4,12]. There are many explicit forms of this significant function that can be found in the literature.

In particular in Ref. [4] an elegant explicit representation of this function was proposed by means of the following representation:

$$R(x)=\frac{x}{2}+\frac{x}{\pi}\cdot(arctan(x)+arctan(\frac{1}{x})) \tag{1}$$

In Ref. [13] the following exact form of this function was performed

$$R(x)=\frac{x}{2}+i\ \frac{ln\,(x)\ -ln\,(-x)}{2\pi} \tag{2}$$

Moreover, on the basis of Ref. [14] where Heaviside Step Function was analytically treated, another closed form of Ramp Function can be obtained as the next formula states

$$R(x)=\frac{3x}{4}+\frac{x}{\pi}\cdot(arctan(x-1)+arctan(\frac{x-2}{x})) \tag{3}$$

Further, on the basis of Ref. [15] the Ramp Function can be calculated as

$$R(x)=\frac{x}{\pi}\cdot\left(arctan(x^n)+2\ arctan(\frac{x^n}{x^{2n}+1})+arctan(\frac{1}{x^n})+2\ arctan(\frac{x^{2n}-x^n\ +1}{x^{2n}+x^n+1})\right) \tag{4}$$

In the meanwhile, there are many smooth analytical approximations to the Ramp Function as it can be seen in the literature [16–19]. One of the simplest approximations to this function is the following [16]

$$R(x)=\frac{x}{2}+\left(1+\frac{x}{\sqrt{x^2+\varepsilon^2}}\right) \tag{5}$$

where $\varepsilon\ \in(0,1)$ such that $\varepsilon\ \ll 1$

On the other hand, in Ref. [20] an analytical form of the Unit Step Function was proposed and besides a qualitative study on Ramp and Signum Function was carried out.

Concurrently, as it was signified beforehand, there are numerous applications of Ramp Function in applied sciences and engineering as it can be observed in the literature.

In Ref. [21] a detailed study on neural networks operators by the aid of ramp functions was carried out, whilst an analogous valuable investigation took place in Ref. [22] where an interpolation by neural network operators was activated by means of ramp functions.

In Ref. [23] a remarkable study on the application of fixed-point neuron models with threshold and the role of ramp and sigmoid activation functions was presented, whereas for a thorough study on approximate solutions of Volterra integral equations via an interpolation method based on ramp functions one may refer to Ref. [24].

In Ref. [25] the role of smooth ramp functions on the activation of network interpolation operators was examined.

In Ref. [26] quadratic programming with ramp functions and fast online QP-MPC solutions was performed.

In Ref. [27] a new implementation of the Simplex method for solving linear programming problems is developed, and its application for solving Model Predictive Control (MPC) problems on the basis of ramp functions, is described.

In Ref. [28] an implementation of Ramp Function to a fracture mechanics problem concerning multi-cracked simply supported beams is carried out where the determination of the response of beams is addressed under static loads and in presence of multiple cracks, whilst in Ref. [29] a substructure elimination method for evaluating bending vibration of beams is performed. In this valuable work, a vibration analysis method is presented on the basis of the substructure elimination method, for a general class of Bernoulli-Euler beams. Here, discontinuities were treated by the use of

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the Heaviside Step Function, whereas the non-smooth points were approached by means of Ramp Function. In fact, referring to Euler-Bernoulli beams and Timoshenko beams there are many remarkable investigations in the literature, where singularity functions such as Unit Step Function and Ramp function have been taken into consideration to carry out analytical treatments to discontinuity problems.

In Refs. [30–32] the jump discontinuities on Euler-Bernoulli beams and Timoshenko beams were analytically treated by the aid of singularity functions whereas in Refs. [33,34] some basic concepts of the well - known Timoshenko beam theory were revisited and discussed in depth.

In Ref. [35] a considerable study on the dynamics of viscoelastic discontinuous beams is accomplished. This investigation deals with the dynamics of beams with an arbitrary number of Kelvin–Voigt viscoelastic rotational joints, translational supports, and attached lumped masses.

In Ref. [36] Heaviside Step Function was implemented to approximate the discontinuities in Euler-Bernoulli discontinuous beams where the analytical solution is finally carried out by means of uniform-beam Green's functions.

In Ref. [37] an analytical treatment for Euler - Bernoulli vibrating discontinuous beams was carried out. Heaviside Step Function and Dirac's delta distribution (also known as the unit impulse), were taken into account towards the analytical approach of the beam discontinuities.

In Ref. [38] a remarkable study concerning the achievement of closed-form solutions for stochastic Euler-Bernoulli discontinuous beams was carried out, whilst in Ref. [39] a valuable theoretical investigation on a general category of Euler - Bernoulli simply supported discontinuous beams was performed.

In Ref. [40] an Euler-Bernoulli-like Finite Element Method (FEM) for Timoshenko beams was presented and discussed whereas in Ref. [41] an exact stochastic solution for a general class of linear elastic beams subjected to delta-correlated loads was accomplished. In addition, in Ref. [42] a considerable analytical study on the effect of axial load and thermal heating on the dynamic characteristics of axially moving Timoshenko Beams was presented. On the other hand, there are many other engineering problems where Ramp Function (along with other singularity functions) are involved and play key roles. For instance, the wavemaker problem is a fundamental and important issue in the study of marine and coastal engineering. In this context, in Ref. [43] the transient waves were generated by a vertical flexible wavemaker plate by means of a general ramp function. In addition, in Ref. [44] a numerical modeling framework based on complex analysis meshless methods, which can accurately and efficiently track arbitrary crack paths in two-dimensional linear elastic solids, is presented and discussed. In this investigation, the Ramp Function is applied in compatibility conditions in order to warrant that the deformations will leave the elastic continuum body in a compatible state.

Now, in the present study, which constitutes a theoretical investigation on this special function, Rump Function is exhibited as the limit of a sequence of real functions letting n tend to infinity. This limit is proved to be zero for strictly negative values of the real variable x whereas it is proved to be simply x for strictly positive values of x. Here, one may also emphasize that the proposed exact formula is not expressed in terms of miscellaneous special functions, (elliptic integrals etc) a fact that may render this formula much more practical and helpful in the computational procedures which are inserted into digital signal processing techniques along with other engineering practices.

2. Towards an explicit form of Ramp Function

Let us introduce the following single – valued function $f: R \to R^{+}$ with

$$f(x) = \lim_{n\to+\infty}\left(\frac{x \cdot n^{|x|+1} + \ln\left(n^2 \cdot (|x| + 2n)\right)}{2n^{|x|+1}} + \frac{x \cdot (\exp(n \cdot x) - 1)}{2\exp(n \cdot x) + 2}\right) \tag{6}$$

where $n \in N$.

3. Claim

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The function f coincides with Ramp Function over the set $(-\infty, 0) \cup [0, +\infty)$.

4. Proof

We shall prove that the values of the single - valued function f vanish for strictly negative arguments and coincide with the values of the real variable x for strictly positive arguments as well as at $x = 0$. To this end, let us distinguish the following three cases concerning the independent variable x.

i) $x \in (0, +\infty)$

In this context, one may deduce that

$$\lim_{n\to+\infty} (n \cdot x) = +\infty \Rightarrow \lim_{n\to+\infty} exp\,(n \cdot x) = +\infty \tag{7}$$

and therefore

$$\lim_{n\to+\infty} \frac{1}{exp\,(n \cdot x)} = 0 \tag{8}$$

Next, to calculate the infinitesimal quantity $\lim_{n\to+\infty} \frac{x \cdot n^{|x|+1} + ln\,(n^2 \cdot (|x|+2n))}{2n^{|x|+1}}$, one may observe that the above fraction which the limiting operation is applied to, can be equivalently expanded as follows:

$$\frac{x \cdot n^{|x|+1} + ln\,(n^2 \cdot (|x|+2n))}{2n^{|x|+1}} = \frac{x}{2} + \frac{ln\,(n)}{n^{|x|+1}} + \frac{ln\,(\,|x|+2n)}{2n^{|x|+1}} \Leftrightarrow$$

$$\frac{x \cdot n^{|x|+1} + ln\,(n^2 \cdot (|x|+2n\,))}{2n^{|x|+1}} = \frac{x}{2} + \frac{ln\,(n)}{n^{|x|+1}} + \frac{ln\,(n)}{2n^{|x|+1}} + \frac{ln\,(\frac{|x|}{n}+2)}{2n^{|x|+1}} \Leftrightarrow \tag{9}$$

$$\frac{x \cdot n^{|x|+1} + ln\,(n^2 \cdot (|x|+2n))}{2n^{|x|+1}} = \frac{x}{2} + \frac{3}{2} \cdot \frac{ln\,(n)}{n^{|x|+1}} + \frac{ln\,(\frac{|x|}{n}+2)}{2n^{|x|+1}}$$

Now, since the exponent $(\,|x| + 1)$ which appears on the denominator of the fractions above is always a strictly positive quantity, i.e. $|x| + 1 > 0 \ \forall\, x \in (-\infty, 0) \cup [\,0, +\infty)$ one also may deduce that

$$\lim_{n\to+\infty} \frac{ln\,(n)}{n^{|x|+1}} = 0 \tag{10}$$

In addition, since the positive term $|x|$ does not vary with respect to the integer variable n, which is a natural number as it was signified beforehand, the quotient $\frac{|x|}{n}$ vanishes, letting n tend to infinity, as it is known from Calculus [45] that every sequence in the general form: $a(n) = \frac{c}{n}$, $c \in R$ is a convergent sequence. In this framework, one may also infer:

$$\lim_{n\to+\infty} \frac{ln\,(\frac{|x|}{n}+2)}{2n^{|x|+1}} = 0 \tag{11}$$

Hence, one obtains

$$\lim_{n\to+\infty} \frac{x \cdot n^{|x|+1} + ln\,(n^2 \cdot (|x|+2n))}{2n^{|x|+1}} = \frac{x}{2} \tag{12}$$

Eqn. (6) can be combined with eqns. (7), (8) and (12) respectively, to yield

$$f(x) = \frac{x}{2} + \frac{1}{2} \lim_{n\to+\infty} \frac{x - \frac{x}{exp\,(n \cdot x)}}{1 + \frac{1}{exp\,(n \cdot x)}} \tag{13}$$

Moreover, since the real variable x does not vary with respect to the integer variable n one may also deduce that

$$\lim_{n\to+\infty} \frac{x - \dfrac{x}{exp\,(n\cdot x)}}{1 + \dfrac{1}{exp\,(n\cdot x)}} = x \tag{14}$$

Eqn. (13) can be combined with eqn. (14) to yield

$$f(x) = \frac{x}{2} + \frac{1}{2}\cdot\frac{x - 0}{1 + 0} \Rightarrow$$
$$f(x) = x \tag{15}$$

Thus, it was proved that the value of the function $f(x)$ is simply x for strictly positive arguments, i.e. when $x \in (0, +\infty)$.

ii) $x \in (-\infty, 0)$

In this context, since the real variable x as well as the natural number n (which evidently is an integer variable) do not agree in sign, one may deduce that

$$\lim_{n\to+\infty} (n\cdot x) = -\infty \Rightarrow \lim_{n\to+\infty} exp\,(n\cdot x) = 0 \tag{16}$$

Here, one may also pinpoint that eqn. (12) which was previously derived when we considered the variable x to be strictly positive, still holds. This significant observation is attributed to a fact that we shall discuss just below. By focusing on the fraction $\frac{x\cdot n^{|x|+1} + ln\,(n^2\cdot(|x|+2n))}{2n^{|x|+1}}$ which appears on eqn. (12) one may pinpoint that the real variable x occurs in the denominator of this fraction only by its absolute value i.e. $|x|$.

Thus the sign of this real variable, cannot influence the sign of the limit of the denominator in this aforementioned fraction, i.e. the term $2n^{|x|+1}$, letting the integer variable n tend to infinity.

Indeed, the limit of the real quantity $2n^{|x|+1}$ letting n tend to infinity, always is equal to $+\infty$, regardless of the sign and the values of the real variable x even if it equals zero.

In continuing, eqn. (6) can be combined with eqns. (12) and (16) respectively to yield

$$f(x) = \frac{x}{2} + \frac{1}{2}\cdot\frac{\lim_{n\to+\infty}(x\cdot \exp(n\cdot x) \quad - x)}{\lim_{n\to+\infty}(\exp(n\cdot x) + 1)} \Rightarrow$$
$$f(x) = \frac{x}{2} + \frac{1}{2}\cdot\frac{x\cdot \lim_{n\to+\infty}(exp(n\cdot x)) \quad - x}{\lim_{n\to+\infty}(exp(n\cdot x)) \; + 1} \tag{17}$$

At this point, we have to elucidate that we have taken into account the fact that the real variable x does not vary with respect to the integer variable n. In this framework, we were able to pull it out of the limiting operation, letting n tend to infinity.

Thus, on the basis of eqn. (17), it implies that

$$f(x) = \frac{x}{2} + \frac{1}{2}\cdot\frac{0\cdot x - x}{0 + 1} \Rightarrow$$
$$f(x) = \frac{x}{2} - \frac{x}{2} = 0 \tag{18}$$

Hence, it was proved that the values of $f(x)$ vanish for strictly negative arguments i.e. when the real variable $x \in (-\infty, 0)$. In addition, we have to emphasize that eqn. (12) which was derived in the first case of the problem we studied, i.e. when $x \in (0, +\infty)$, is always valid over the set of real

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numbers: $(-\infty,0)\cup[0,+\infty)$, as we have previously shown. Moreover, we have taken into consideration the fact that the real variable x and the integer variable n are always independent of each other. In this framework, the real variable x can be pulled out of the limiting operations letting n tend to infinity, since it can be roughly said that it practically behaves as a real constant during the limiting operation.

ii) $x = 0$

Then, one obtains

$$limn\to+\infty exp\,(n\cdot x)= limn\to+\infty exp\,(n\cdot 0) \tag{19}$$

and therefore

$$limn\to+\infty exp\,(n\cdot x)=limn\to+\infty exp\,(0)=1 \tag{20}$$

Moreover, one may note that eqn. (12) still holds.
Now, eqn. (6) can be combined with eqns. (20) and (12) respectively to yield

$$f(x) = \lim_{n\to+\infty}\left(\frac{ln\,(2n^3)}{2n}+\frac{0}{2+2}\right) \Rightarrow$$
$$f(x) = \lim_{n\to+\infty}\left(\frac{ln\,(2n^3)}{2n}\right)+\frac{0}{4} \tag{21}$$

and therefore

$$f(x) = 0+\frac{0}{4}=0 \tag{22}$$

Thus, it was proved that the value of $f(x)$ vanish at $x = 0$ as the Rump Function also does. Afterall, one may come to the conclusion that the single - valued real function introduced by eqn. (6) is identical to the Ramp Function over the set of real numbers.

5. Discussion

In Section 2, an explicit form of Rump Function was proposed, as the limit of a sequence of real functions letting n tend to infinity. This limit was proved to be zero for strictly negative values of the real variable x whereas it was proved to be simply x for strictly positive values of x. In fact, the proposed single - valued function coincides with Rump Function over the set $(-\infty,0)\cup(0,+\infty)$. In addition, one may also observe that the single – valued function f introduced by eqn. (6) vanishes at $x = 0$. Indeed, the Ramp Function (by its definition) also vanishes at $x = 0$ since it is just x for positive arguments. In this framework, one may also conclude that the single - valued function introduced by eqn. (6) coincides with Ramp Function over the set of real numbers $(-\infty,0)\cup[0,+\infty)$.

In addition, by focusing on the infinitesimal quantity $\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)\ \ -1)}{2exp(n\cdot x)+2}\right)$ which can be equivalently written as: $\frac{1}{2}\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)\ \ -1)}{exp(n\cdot x)+1}\right)$ one may observe that the following relationship holds:

$$\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)\ \ -1)}{exp(n\cdot x)+1}\right)=|x| \tag{23}$$

As a matter of fact, the infinitesimal quantity $\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)\ \ -1)}{exp(n\cdot x)+1}\right)$ coincides with the absolute value of the real variable x over the set $(-\infty,0)\cup(0,+\infty)$ and also at x = 0, since it is obvious that the numerator of the above fraction vanishes at x = 0 whereas its denominator equals to 2.

Actually, the validity of eqn. (23) is attributed to the fact that the value of the infinitesimal quantity $\lim_{n\to+\infty} exp\,(n \cdot x)$ depends on the sign of the variable x which is definitely independent of the integer variable n.

In this context, the infinitesimal quantity $\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)\ \ -1)}{exp(n\cdot x)+1}\right)$ equals x for strictly positive arguments, whilst it equals $-\ x$ for strictly negative arguments and finally vanishes at x=0.

6. Conclusions

The objective of this theoretical investigation was to introduce an analytical representation of the Rump Function, which evidently is a very useful mathematical tool and it participates in many areas of applied and engineering mathematics and physics. The novelty of this work, when compared to other analytical treatments to this significant function, is that the proposed exact mathematical formula is not exhibited in terms of any miscellaneous special functions, or any other special functions such as Error Function, Hyperbolic Function, Orthogonal polynomials etc.

Nevertheless, one may also observe that an advantage of the proposed closed - form expression of this special function, is that it coincides with Ramp Function over the set $(-\infty, 0) \cup [\,0, +\infty)$ since it was proved to vanish at $x\ = 0$, as Ramp Function also does.

In closing, as a future work, one may also propose similar analytical representations to other singularity functions [46] e.g. Heaviside Step Function, Signum Function etc by taking into consideration this theoretical approach.

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