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Intrinsic Geometric Structure of Subcartesian Spaces

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Abstract: Every subset S of a Cartesian spaces $\mathbb{R}^d$, endowed with differential structure $C^\infty(S)$ generated by restrictions to S of functions in $C^\infty(\mathbb{R}^d)$, has a canonical partition $\mathfrak{M}(S)$ by manifolds, which are orbits of the family $\mathfrak{X}(S)$ of all derivations of $C^\infty(S)$ that generate local one-parameter groups of local diffeomorphisms of S. This partition satisfies the frontier condition, Whitney's conditions A and B. If $\mathfrak{M}(S)$ is locally finite, then it satisfies all definitions of stratification of S. This result extends to Hausdorff locally Euclidean differential spaces.

Keywords: subcartesian differential space; orbits of family of vector fields

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1. Introduction

In the second half of twentieth century the idea of using differential geometry to study spaces with singularities was floating in the air. In 1955, Satake introduced a notion of a V-manifold in terms of an atlas of charts with values in quotients of connected open subsets of $\mathbb{R}^n$ by a finite group of linear transformations, [14].

In 1961, Cerf, introduced the notion generalized manifold, now known as manifold with corners, defined in terms of an atlas of charts with values in open subsets of $[0,\infty)^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^n$, where $k = 0, 1, ...n$, [6]. Cerf had all elements of the definition of general class of differential spaces, but he did not develop the corresponding general theory. He preferred to investigate its example provided by manifolds with corners.

In 1966, Smith introduced his notion of *differentiable structure* on a topological space, which consists of a family of continuous functions on the space, deemed to be smooth, which carry all the information about the geometry of the space, [17]. Smith used the term *differentiable spaces,* and he he studied the de Rham Theorem on differentiable spaces.

In 1967, Sikorski generalized the approach of Smith and used it to discuss the notion of an abstract covariant derivative, [15]. Sikorski used the term *differential structure* for the collection of functions on a topological space deemed to be smooth, and the term *differential space* for a topological space endowed with a differential structure. In 1974, Sikorski published a book on differential geometry, in which he started with developement of the theory of differential spaces and later specified the spaces under consideration to be smooth manifolds, [16]. Sikorski used his book as the text in his master level course of differential geometry at the University of Warsaw. Even though Sikorski's book was written in Polish, it was appreciated by a sizeable group of of international scientists. Also in 1967, Aronszajn introduced, in the abstract to his presentation at a Meeting of the American Mathematical Society, [1], the notion of a *subcartesian space,* as a Hausdorff topological space that is locally diffeomorphic to a subset of a Cartesian (Euclidean) space. The local diffeomorphisms used by Aronszajn formed an atlas, similar to that introduced by Cerf. A more comprehensive presentations of this theory and its applications were given by Aronszajn and Szeptycki in 1975, [2], and in 1980, [3].

There are other theories allowing for study of differential geometry of singular spaces. For a more comprehensive review see [5].

Here, we concentrate on theories of Aronszajn and Sikorski. The strength of Aronszajn's approach is his choice of assumptions, which are satisfied by most finite dimensional examples. On the other hand, Sikorski made the weakest assumptions. It leads to simplicity of the basic presentation of the theory, and makes other theories to be special cases of Sikorski's theory of differential spaces.The relation between the theories of Aronszajn and of Sikorski was discussed first by Walczak in 1973, [20].

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In 2021, we exhibited a natural transformation from the category od subcartesian spaces to the category of Hausdorff locally Euclidean differential spaces, [7]. Since Hausdorff locally Euclidean differential spaces can be identified with corresponding subcartesian spaces, we treat the terms *Hausdorff locally Euclidean differential space* and *subcartesian space* as synonims and use tham interchangeably. Aronszajn's term is shorter and it is well known to experts, but it does not convey much information to uninitiated. That is why we use the longer term in the abstract and explanations. In the proofs we use the shorter term.

The theory of differential spaces attracted a fair amount of interest, see [19] and references cited there. In the next section, we give a brief review of the elements of this theory that are essential for subsequent developement.

In Section 3, we give a more comprehensive review of results on derivations of the differential structure of a differential space and their integration. We introduce the term *vector fields* on a subcartesian space S (Hausdorff locally Euclidean differential space) for derivations of $C^\infty(S)$ that generate one-parameter groups of local diffeomorphisms of S. In [18] it was proved that orbits of the family of all vector fields on a subcartesian space S form a partition $\mathfrak{M}(S)$ of S by smooth manifolds.

In Section 4, we study the partition $\mathfrak{M}(S)$ of a diffferential space S by orbits of the family of all vector fields on S, which is the main objective of this paper. In the case when the differential space under consideration is a connected manifold M, the Lie algebra of local one-parameter groups of local diffeomorphisms of M acts transitively of M, which means that the corresponding partition of M is trivial, it consists of a single orbit. We show that the partition $\mathfrak{M}(S)$ satisfies the frontier condition, Whitney's conditions A and B, and it leads to a filtration of S by closed subsets.

In Section 5, we compare the results of Section 4 with various definitions of stratifications. If the partition $\mathfrak{M}(S)$ is locally finite then it satisfies all definitions of a stratification of a closed subset of a smooth manifold.

In Section 6, we briefly relate derivations that are not vector fields to transient vector fields on manifolds with boundary discussed by Percel [12]. These derivations generate transitions between different manifolds of the partition $\mathfrak{M}(S)$.

In Section 7, we apply our approach to manifolds with corners. According to Cerf's definition, [6], a manifold with corners S is a locally closed subcartesian space. Following Joyce's formulation of the theory of manifolds with corners, [9], we show that the depth function stratification of S coincides with the partition $\mathfrak{M}(S)$, and it satisfies Whitney's conditions A and B.

The second author is greatly indebted to Dominic Joyce for helpful and stimulating e-mails.

2. Differential Spaces

Definition 2.1. *A differential structure on a topological space S is a family $C^\infty(S)$ of real valued functions on S that satisfy the following conditions.*

1. *The family*

$$\{f^{-1}(I) \mid f \in C^\infty(S) \text{ and } I \text{ is an open interval in } \mathbb{R}\}$$

is a sub-basis of the topology of S.

2. *If $f_1, ..., f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, ..., f_n) \in C^\infty(S)$.*

3. *If $f : S \to \mathbb{R}$ is a function such that, for each $x \in S$, there is an open neighbourhood V of x in S and a function $f_x \in C^\infty(S)$ satisfying*

$$f_{x|V} = f_{|V},$$

then $f \in C^\infty(S)$.

A topological space S endowed with a differential structure $C^\infty(S)$ is called differential space.

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A simple way of definig a differential structure on a set S is as follows. Choose a family of functions $\mathcal{F}$ on S. Endow S with the topology generated by a sub-basis

$$\{f^{-1}(I) \mid f \in \mathcal{F} \text{ and } I \text{ is an open interval in } \mathbb{R}\}. \tag{1}$$

The differential structure $C^\infty(S)$ *generated by* $\mathcal{F}$ consists of functions $h : S \to \mathbb{R}$ such that, for each $x \in S$, there exist an open neighbourhood V of x, an integer $n \in \mathbb{N}$, functions $f_1, ..., f_n \in \mathcal{F}$, and $F \in C^\infty(\mathbb{R}^n)$ such that

$$h_{|V} = F(f_1, ..., f_n)_{|V}. \tag{2}$$

It is easy to see that that the differential structure $C^\infty(S)$ generated by $\mathcal{F}$ satisfies all conditions of Definition 2.1.

Below, we are using the method, outlined above, to generate differential structures of products, subsets and quotients of differential spaces.

Definition 2.2. *Let $(S, C^\infty(S))$ and $(R, C^\infty(R))$ be differential spaces. Choose*

$$\mathcal{F} = \{S \times R \to R : (x, y) \mapsto f(x)g(y) \mid f \in C^\infty(S) \text{ and } g \in C^\infty(R)\},$$

where $f(x)g(y)$ is the product in $\mathbb{R}$ of the numbers $f(x)$ and $g(y)$. It is easy to see that, for this choice of $\mathcal{F}$, equation (1) gives a sub-basis of the product topology on $S \times R$.The diferential structure $C^\infty(S \times R)$ generated by $\mathcal{F}$ is called the product differential structure.

Definition 2.3. *Let $(S, C^\infty(S))$ be a differential space, and let R be a subset of S. Let*

$$\mathcal{R}(R) = \{f_{|R} \mid f \in C^\infty(S)\}$$

be the family of restrictions to R of smooth functions on S. Equation (1) with $\mathcal{F} = \mathcal{R}(R)$ gives a sub-basis of the topology in R induced by its inclusion in S. The differential structure of R generated by $\mathcal{F} = \mathcal{R}(R)$ is called the subspace differential structure, *and we refer to R as a differential subspace of S. We also refer to the differential structure of $R \subseteq S$ generated by $\mathcal{F} = \mathcal{R}(R)$ as the differential structure induced by the inclusion of R in S.*

Definition 2.4. *Let $(S, C^\infty(S))$ be a differential space. An equivalence relation $\sim$ on S defines a subset R of $S \times S$ such that, if $(x, y) \in S \times S$, then*

$$(x, y) \in R \text{ if and only if } x \sim y.$$

For each $x \in S$ we denote by $[x]$ the $\sim$ equivalence class x. Let $Q = S/\sim$ be the set ot equivalence classes of the relation $\sim$ in S, and let $\pi : S \to Q$ be the projection map given by $\pi(x) = [x]$ for every $x \in S$. The quotient differential structure of Q is

$$C^\infty(Q) = \{f : Q \to \mathbb{R} \mid \pi^* f = f \circ \pi \in C^\infty(S)\}$$

It should be noted that the topology of Q defined by the differential structure $C^\infty(Q)$ need not coincide with the quotient topology of Q.

Let $(S, C^\infty(S))$ and $(R, C^\infty(R))$ be differential spaces.

Definition 2.5. *A continuous map $\varphi : S \to R$ is* smooth *if for each $f \in C^\infty(R)$ the pull back $\varphi^* f = f \circ \varphi$ is in $C^\infty(S)$. A smooth map $\varphi : S \to R$ is a* diffeomorphism *if it is invertible and its inverse is smooth.*

Note that, if $(S, C^\infty(S))$ and $(R, C^\infty(R))$ are differential spaces, and a map $f : S \to R$ is smooth, then it is a homeomorphism of the underlying topological spaces. Differential spaces and smooth maps form a category.

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Sikorski's theory of differential spaces is the most general approach to C^∞-differential geometry of singular spaces. Of special interest here are differential spaces that are locally diffeomorphic to differential subspaces of Euclidean spaces.

Definition 2.6. *A differential space* $(S, C^\infty(S))$ *is* locally Euclidean *if, for every* $x \in S$*, there exists an open neighbour* V *of* x *in* S*,a subset* W *of some* $\mathbb{R}^n$ *and a diffeomorphism* $\alpha : V \to W$*,where* V *is endowed with the differential structure* $C^\infty(V)$ *induced by its inclusion in* S *and* W *is endowed with the differential structure* $C^\infty(W)$ *induced by its inclusion in* $\mathbb{R}^n$*.*

Definition 2.7. *A Hausdorff locally Euclidean differential space* $(S, C^\infty(S))$ *is a subcartesian space of Aronszajn.*

Proof. Since $(S, C^\infty(S))$ is a locally Euclidean differential space, local diffeomorphisms $\alpha : V_\alpha \to W_\alpha$, where V_α is an open differential subspace of S and W_α a differential subspace of some $\mathbb{R}^{d_\alpha}$ generate a complete atlas $\mathfrak{A}(S) = \{\alpha : V_\alpha \to W_\alpha\}$ of S, which satisfies the following conditions:
(1). The family $\{V_\alpha \mid \alpha \in \mathfrak{A}(S)\}$ of open sets in S forms a covering of S.

(2). For every $\alpha, \beta \in \mathfrak{A}(S)$, and every $x \in V_\alpha \cap V_\beta$, there exists a C^∞-mapping Φ_α of an open neighbourhood U_α of $\alpha(x) \in \mathbb{R}^{d_\alpha}$ to $\mathbb{R}^{d_\beta}$, which extends the mapping

$$\beta \circ \alpha^{-1} : \alpha(V_\alpha \cap V_\beta) \to \beta(V_\alpha \cap V_\beta),$$

and a C^∞-mapping Φ_β of an open neighbourhood U_β of $\beta(x) \in \mathbb{R}^{d_\beta}$ to $\mathbb{R}^{d_\alpha}$, which extends the mapping

$$\alpha \circ \beta^{-1} : \beta(V_\alpha \cap V_\beta) \to \alpha(V_\alpha \cap V_\beta).$$

(3). $C^\infty(S)$ consists of continuous function $f : S \to \mathbb{R}$ on S such that, for every chart $\alpha : V_\alpha \to W_\alpha \subseteq \mathbb{R}^{d_\alpha}$, there exists an open set U_α in $\mathbb{R}^{d_\alpha}$ containing W_α, and a smooth function $F \in C^\infty(U_\alpha)$ such that $f \circ \alpha^{-1} : W_\alpha \to \mathbb{R}$ is the restriction of F to $W_\alpha \subseteq U_\alpha$.
These conditions, together with the assumptionthat S is Hausdorff, define a subcartesian space of Aronszajn, [1]. and [?].

Proof. In order to complete the proof, we show that that a continuous map $\varphi : S \to R$ of Hausdorff locally Euclidean differential spaces with complete atlases $\mathfrak{A}(S) = \{\alpha : V_\alpha \to W_\alpha\}$ and $\mathfrak{A}(R) = \{\beta : V_\beta \to W_\beta\}$, respectively, is smooth if and only if, for every $x \in S$, there exist $\alpha \in \mathfrak{A}(S)$ and $\beta \in \mathfrak{A}(R)$ such that $x \in V_\alpha$, $\varphi(V_\alpha) \subseteq V_\beta$ and the mapping

$$\Phi_{\alpha\beta} = \beta \circ \varphi \circ \alpha^{-1} : W_\alpha \to W_\beta : \boldsymbol{x} \mapsto \boldsymbol{y} = \Phi_{\alpha\beta}(\boldsymbol{x}) \tag{3}$$

extends to a C^∞ mapping

$$F_{\alpha\beta} : U_\alpha \to U_\beta : \boldsymbol{x} \mapsto \boldsymbol{y} = F_{\alpha\beta}(\boldsymbol{x}), \tag{4}$$

where U_α is an open subset of $\mathbb{R}^{d_\alpha}$ containing W_α and U_β is an open subset of $\mathbb{R}^{d_\beta}$ containing W_β.
(a). Let $\varphi : S \to R$ be a map between subcartesian spaces. Assume that every transition map $\Phi_{\alpha\beta}$, given by equation (3) has a smooth extension $F_{\alpha\beta}$ such that

$$\Phi_{\alpha\beta} : W_\alpha \to W_\beta : \boldsymbol{x} \mapsto \boldsymbol{y} = F_{\alpha\beta|W\alpha}(\boldsymbol{x}). \tag{5}$$

The components of $\Phi_{\alpha\beta}$ with respect to the Cartesian coordinates $(y^1, ..., y^{d_R})$ on $\mathbb{R}^{d_R}$ are

$$\Phi^i_{\alpha\beta} = \Phi^*_{\alpha\beta} y^i_{|W_\beta} \ \text{ for } \ i = 1, ..., d_R.$$

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Since $\Phi_{\alpha\beta} = F_{\alpha\beta|W_\beta}$, it follows that each component, $\Phi^i_{\alpha\beta} = \Phi^*_{\alpha\beta} y^i_{|W_\beta}$ of Φ, is the restriction of the corresponding component of

$$F_{\alpha\beta}\Phi^i_{\alpha\beta} = (F_{\alpha\beta|W_\alpha})^* y^i_{|W_\beta} \; for \; i = 1, ..., d_{R.} \tag{6}$$

□

Next, given $h \in C^\infty(R)$, we want to show that $f = \varphi^* h = h \circ \varphi \in C^\infty(S)$. In terms of the charts $\alpha : V_\alpha \to W_\alpha$ on S and $\beta : V_\beta \to W_\beta$ on R, given above, $h_{|V_\beta} : V_\beta \to \mathbb{R}$, and $h \circ \beta^{-1} : W_\beta \to \mathbb{R}$. Similarly, $f \circ \alpha^{-1} : W_\alpha \to \mathbb{R}$, and

$$f \circ \alpha^{-1} = (\varphi^* h) \circ \alpha^{-1} = h \circ \varphi \circ \alpha^{-1} = (h \circ \beta^{-1}) \circ \beta \circ \varphi \circ \alpha^{-1} = (h \circ \beta^{-1}) \circ \Phi_{\alpha\beta}. \tag{7}$$

Since R is a locally Euclidean differential space, and $h \in C^\infty(R)$, it follows that there exists a smooth function $h_\beta : U'_\beta \to R$, where U'_β is open in R and contains W_β, such that $(\beta^{-1})^* h = h \circ \beta^{-1} = h_{\beta|W_\beta}$. Without loss of generality, we may assume that $U'_\beta = U_\beta$. Hence, equations (7) and (5) imply that

$$f \circ \alpha^{-1} = (h \circ \beta^{-1}) \circ \Phi_{\alpha\beta} = h_{\beta|W_\beta} F_{\alpha\beta|W_\alpha}.$$

Hence, $f \circ \alpha^{-1} : W_\alpha \to \mathbb{R}$ is the restriction to W_α of a C^∞ function $h_\beta F_{\alpha\beta} : U_\alpha \to U_\beta$.

This result holds for all $x \in S$ and every pair of charts (α, β) such that $x \in V_\alpha$, $\varphi(x) \in V_\beta$, and $\varphi \circ \alpha^{-1}(W_\alpha) \subseteq V_\beta$. Hence , $f = \varphi^* h = h \circ \varphi \in C^\infty(S)$ for every $h \in C^\infty(R)$. Therefore, the map $\varphi : S \to R$ is smooth.

(b). In order to prove the implication in the opposite direction assume that a map $\varphi : S \to R$ is smooth in the sense of differential spaces. That is, $\varphi^* h \in C^\infty(S)$ for every $h \in C^\infty(R)$. Equations (3) and (7) yield

$$(\varphi^* h) \circ \alpha^{-1} = (h \circ \beta^{-1}) \circ \Phi_{\alpha\beta}. \tag{8}$$

For $x \in V_\alpha$ and $y = \varphi(x) \in V_\beta$, let $y^1, ..., y^{d_\beta}$ be Cartesian coordinates in $\mathbb{R}^{d_\beta}$. We are going to construct functions $h_1, ..., h_{d\beta}$ in $C^\infty(R)$ such that, for each $i = 1, ..., d_{R\beta}$, $h_i \circ \beta^{-1} = y^i$ in a neighbourhood of $\beta(y)$ in $W_\beta \subseteq \mathbb{R}^d$.

There exists an open set V_y in R such that $y \in V_y \subseteq \overline{V}_y \subseteq V_\beta$. Since $V_y \subseteq \overline{V}_y \subseteq V_\beta$, continuity of $\varphi : S \to R$ implies that $V_x = V_\alpha \cap \varphi^{-1}(V_y)$ is an open subset of S. Moreover, $\varphi(V_x) \subseteq V_y$ and

$$\varphi(\overline{V}_x) \subseteq \overline{\varphi(V_x)} \subseteq \overline{V}_y \subseteq V_\beta.$$

Hence, $\overline{V}_x \subseteq \varphi^{-1}(V_\beta) \subseteq V_\alpha$ so that

$$V_x \subseteq \overline{V}_x \subseteq V_\alpha. \tag{9}$$

For each $i = 1, ..., d_\beta$, $\beta^* y^i_{|W_\beta}$ is a smooth function on V_β. Using partition of unity in R, we can construct a function $h^i_y \in C^\infty(R)$ such that

$$h^i_{y|V_y} = (\beta^* y^i_{|W_\beta})_{|V_y} \quad \text{and} \quad h^i_{y|R\setminus V_\beta} = 0. \tag{10}$$

Since $\varphi : S \to R$ is a smooth map of differential spaces, with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively, and $h^1_y, ..., h^{d_R}_y \in C^\infty(R)$, it follows that $\varphi^* h^1_y, ..., \varphi^* h^{d_R}_y$ are in $C^\infty(S)$. Moreover, $V_x = V_\alpha \cap \varphi^{-1}(V_y)$ is an open neighbourhood of x in $V_\alpha \subseteq S$, $\alpha(V_x)$ is an open subset of W_α, and $\varphi(V_x) \subseteq V_y$. Hence, for every $i = 1, ..., d_\beta$, the restriction of $\varphi^* h^i_y$ to V_x is

$$\varphi^* h^i_{y|V_x} = h^i_y \circ \varphi_{|V_x} = h^i_{y|V_\beta} \circ \varphi_{|V_x} = (\beta^* y^i_{|W_\beta}) \circ \varphi_{|V_x} = y^i_{|W_\beta} \circ (\beta \circ \varphi)_{|V_x}, \tag{11}$$

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so that

$$(h^i_y \circ \varphi \circ \alpha^{-1})_{\alpha(V_x)} = y^i_{|W_\beta} \circ (\beta \circ \varphi \circ \alpha^{-1})_{|\alpha(V_x)}. \tag{12}$$

This implies that the restriction of $\Phi_{\alpha\beta}$ to $\alpha(V_x)$ is given in Cartesian coordinates on $\mathbb{R}^{d_\beta}$ by

$$\begin{aligned} \Phi_{\alpha\beta|\alpha(V_x)} &: \alpha(V_x) \to W_\beta : \boldsymbol{x} \mapsto \boldsymbol{y} = \Phi_{\alpha\beta}(\boldsymbol{x}) \\ &= (((\varphi^* h^1_y) \circ \alpha^{-1})(\boldsymbol{x}), ..., (\varphi^*(h^{d_R}_y) \circ \alpha^{-1})(\boldsymbol{x})) \\ &= (((h^1_y \circ \varphi) \circ \alpha^{-1})(\boldsymbol{x}), ..., ((h^{d_R}_y \circ \varphi) \circ \alpha^{-1})(\boldsymbol{x})). \end{aligned} \tag{13}$$

Since $\varphi^* h^1_y, ..., \varphi^* h^{d_\beta}_y \in C^\infty(S)$, Definition 2.6(3) ensures that, for every $i = 1, ..., d_R$, there exists an open set U^i in $\mathbb{R}^{d_S}$ containing W_α, and a smooth function $F^i \in C^\infty(U^i)$ such that $\varphi^* h^i_y \circ \alpha^{-1} : W_\alpha \to \mathbb{R}$ is the restriction of F^i to $W_\alpha \subseteq U^i$. The intersection $U = \cap_{i=1}^{d_R} U^i$ is open in $\mathbb{R}^{d_S}$ and $\alpha(V_x) \subseteq \alpha(V_\alpha) = W_\alpha$. Hence $\Phi_{\alpha\beta|\alpha(V_x)} : \alpha(V_x) \to W_\beta$ is the restriction to $\alpha(V_x)$ of $(F^1_{|U}, ..., F^{d_\beta}_{|U}) \in C^\infty(U, \mathbb{R}^{d_R})$ to domain $\alpha(V_x)$ and codomain W_β.

The above result can be established for every $x \in S$. Therefore, the map $\varphi : S \to R$ is a smooth map between subcartesian spaces. □

In view of Proposition 2.1.7 we identify the terms *Hausdorff locally Euclidean differential space* and *subcartesians space*. The first term is more transparent, while the second term is well known to specialists in the field.

3. Derivations and Vector Fields

Definition 3.1. *Let S be a differential space. A derivation of $C^\infty(S)$ is a linear map $X : C^\infty(S) \to C^\infty(S) : f \mapsto Xf$ satisfying Leibniz's rule*

$$X(f_1 f_2) = (Xf_1)f_2 + f_1(Xf_2) \tag{14}$$

for every $f_1, f_2 \in C^\infty(S)$.

Let Der $C^\infty(S)$ denote the space of deriviations of $C^\infty(S)$. It is a Lie algebra with Lie bracket

$$[X_1, X_2]f = X_1(X_2 f) - X_2(X_1 f) \tag{15}$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty(S)$ and every $f \in C^\infty(S)$. In addition, Der $C^\infty(S)$ is a module over the ring $C^\infty(S)$ with $[fX_1, X_2] = f[X_1, X_2]$ and

$$[X_1, fX_2] = (X_1 f)X_2 + f[X_1, X_2] \tag{16}$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty(S)$ and every $f \in C^\infty(S)$.

Definition 3.2. *Let $\varphi : R \to S$ be a smooth map of differential spaces with differential structures $C^\infty(R)$ and $C^\infty(S)$, respectively. Derivations X in* Der $C^\infty(S)$ *and Y in* Der $C^\infty(R)$ *are* φ-related *if*

$$\varphi^*(Y(f)) = X(\varphi^* f)$$

for every $f \in C^\infty(R)$.

Suppose that the map $\varphi : R \to S$ in Definition 3.1.2 is a diffeomorphism, that is $\varphi^{-1} : R \to S$ exists and is smooth. For every derivation $X \in \operatorname{Der} C^\infty(R)$ there exists a unique derivation $\varphi_* X \in \operatorname{Der} C^\infty(S)$,

$$\varphi_* X : C^\infty(S) \to C^\infty(S) : f \mapsto (\varphi_* X)f = (\varphi^{-1})^*(X(\varphi^* f)), \tag{17}$$

which is φ-related to X. It is called the *push-forward* of X by φ. Moreover,

$$\varphi_* : \operatorname{Der} C^\infty(R) \to \operatorname{Der} C^\infty(S) : X \mapsto \varphi_* X$$

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is a Lie algebra diffeomorphism.

Suppose now that $(S, C^\infty(S))$ is a differential space and V is an open subset of S. By Definition 2.3, the differential structure $C^\infty(V)$ of V is generated by

$$\mathcal{R}(V) = \{f_{|V} \mid f \in C^\infty(S)\}.$$

A continuous function $h : V \to \mathbb{R}$ is in $C^\infty(V)$ if and only if, for every $x \in V$, there exists an open subset U on V, and a function $f \in C^\infty(S)$, such that $h_{|U} = f_{|U}$. Since V is open in S, it follows that for every $f \in C^\infty(S)$ the restriction $f_{|V}$ of f to V is in $C^\infty(V)$. Hence, $\mathcal{R}(V) \subseteq C^\infty(V)$. Moreover, every $X \in \operatorname{Der} C^\infty(S)$ restricts to a derivation $X_{|V}$ of $\mathcal{R}(V)$, given by

$$X_{|V} : \mathcal{R}(V) \to \mathcal{R}(V) : f_{|V} \mapsto X_{|V} f_{|V} = (Xf)_{|V}.$$

We want to extend the derivation $X_{|V}$ to all functions in $C^\infty(V)$. Suppose that $h \in C^\infty(V) \backslash \mathcal{R}(V)$. For every $x \in V$, there exists an open subset U of V, and a function $f \in C^\infty(S)$, such that $h_{|U} = f_{|U}$. Since U is open in V and V is open in S, it follows that U is open in S. The argument above, applied to U, implies that every X in $\operatorname{Der} C^\infty(S)$ restricts to a derivation $X_{|U}$ of

$$\mathcal{R}(U) = \{f_{|U} \mid f \in C^\infty(S)\},$$

given by

$$X_{|U} : \mathcal{R}(U) \to \mathcal{R}(U) : f_{|U} \mapsto X_{|U} f_{|U} = (Xf)_{|U}.$$

Since, U is open in V, it follows that

$$\mathcal{R}(U) = \{f_{|U} \mid f \in C^\infty(S)\} = \{(f_{|V})_{|U} \mid f \in C^\infty(S)\} = \{(f_{|V})_{|U} \mid f_{|V} \in \mathcal{R}(V)\}$$

and

$$X_{|U} f_{|U} = (Xf)_{|U} = X_{|U} f_{|U} = ((Xf)_{|V})_{|U} = (X_{|V} f_{|V})_{|U} = (X_{|V})_{|U} (f_{|V})_{|U} = (X_{|V})_{|U} f_{|U},$$

so that $X_{|U} = (X_{|V})_{|U}$. Thus, $h_{|U} = f_{|U}$ implies that we may extend the definition of $X_{|V}$ to $h \in C^\infty(V) \backslash \mathcal{R}(V)$ by setting

$$(X_{|V} h)_{|U} = X_{|U} f_{|U}$$

whenever $h_{|U} = f_{|U}$ for $f \in C^\infty(S)$ and U is open in V.

We need to verify that this definition is consistent. Suppose that U' is another open subset of V such that $U \cap U' \neq \emptyset$. Then $U \cap U'$ is open in V and we may evaluate$(X_{|V} h)_{|U \cap U'}$ in two ways:

$$\begin{aligned} (X_{|V} h)_{|U \cap U'} &= ((X_{|V} h)_{|U})_{|U \cap U'} = (X_{|U} f_{|U})_{|U \cap U'} = X_{|U \cap U'} f_{|U \cap U'}, \\ (X_{|V} h)_{|U \cap U'} &= ((X_{|V} h)_{|U'})_{|U \cap U'} = (X_{|U'} f_{|U'})_{|U \cap U'} = X_{|U \cap U'} f_{|U \cap U'}, \end{aligned}$$

obtaining the same result.

Conclusion 3.3. *If V is an open differential subspace of S, then every $X \in \operatorname{Der} C^\infty(S)$ restricts to a derivation $X_{|V}$ of $C^\infty(V)$. If $\iota : V \to S$ is the inclusion map then*

$$\iota^*(X(f)) = (Xf)_{|V} = X_{|V} f_{|V} = X_{|V}(\iota^* f)$$

for every $f \in C^\infty(S)$. In other words, X and $X_{|V}$ are related by the inclusion map.

Next, we show that derivations of the differential structure of a subcartesian space admit unique maximal integral curves.

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We begin with a review of the notion of integral curves of vector fields on manifolds. Let M be a smooth manifold and X a vector field on M. A smooth map $c : I \to M$ of an interval $I \subseteq \mathbb{R}$ is an integral curve of X if

$$\frac{d}{dt} f(c(t)) = (Xf)(c(t)) \text{ for every } f \in C^\infty(M) \text{ and every } t \in I. \tag{18}$$

In other words, $c : I \to S$ is an integral curve of X if $Tc(t) = X(c(t))$ for every $t \in I$.

If $t_0 \in I$ and $x_0 = c(t_0)$, we may reparametrize the curve by a shift $s : I \to \tilde{I} : t \mapsto \tilde{t} = t - t_0$, obtaining an integral curve $\tilde{c} : \tilde{I} \to M : \tilde{t} \mapsto \tilde{c}(\tilde{t}) = c(t - t_0)$ of X such that $\tilde{c}(0) = x_0$. We say that $\tilde{c} : \tilde{I} \to M$ is an integral curve is an integral curve of X that originates at x_0.

We generalize this definition to subcartesian spaces. Let $c : I \to S$ be a smooth map of an interval I in $\mathbb{R}$, containing 0, to a subcartesian space S, a derivation X of $C^\infty(S)$ and a point $x_0 \in S$. Suppose that

$$c(0) = x_0 \text{ and } \frac{d}{dt} f(c(t)) = (Xf)(c(t)) \text{ for every } f \in C^\infty(S) \text{ and every } t \in I. \tag{19}$$

If the interval I has non-empty interior, then the conditions above are well defined and we may call $c : I \to S$ an itegral curve of X originating at x_0. However, there exist subcartesian spaces in which no two distinct points are arc connected.

Example 3.4. *Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$, and $C^\infty(\mathbb{Q})$ consists of restrictions to $\mathbb{Q}$ of smooth functions on $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows from equation (2) that every derivation of $C^\infty(\mathbb{R})$ induces a derivation of $C^\infty(\mathbb{Q})$. Let X be the derivation of $C^\infty(\mathbb{Q})$ induced by the derivative $\frac{d}{dx}$ on $C^\infty(\mathbb{R})$. In other words, for every $f \in C^\infty(\mathbb{Q})$ and every $x_0 \in \mathbb{Q}$,*

$$(Xf)(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

where the limit is taken over $x \in \mathbb{Q}$. On the other hand, no two distinct points in Q can be connected by a continuous curve.

In order to avoid saying that that in the Example 3.4 non-zero derivations have no integral curves, we redefine the notion of an integral curve by allowing its domain I shrink to a point. With this modification, every derivation X of $C^\infty(\mathbb{Q})$ has an integral curve $c : I \to \mathbb{Q}$ originating at $x_0 \in \mathbb{Q}$ with $I = (0)$ and $c(0) = x_0$. Thus, we adopt the following formal definition.

Definition 3.5. *Let S be a subcartesian space and X a derivation of $C^\infty(S)$. An integral curve of X originating at $x_0 \in S$ is a map $c : I \to S$, where I is a connected subset of $\mathbb{R}$ containing 0, such that $c(0) = x_0$ and*

$$\frac{d}{dt} f(c(t)) = (Xf)(c(t)) \textit{ for every } f \in C^\infty(S) \textit{ and every } t \in I, \tag{20}$$

whenever the interior of I is not empty.

Integral curves of a given derivation X of $C^\infty(S)$ starting at x_0 can be ordered by inclusion of their domains. In other words, if $c_1 : I_1 \to S$ and $c_2 : I_2 \to S$ are two integral curves of X, such that $c_1(0) = c_2(0) = x_0$, and $I_1 \subseteq I_2$, then $c_1 \preceq c_2$. An integral curve $c : I \to S$ of X is *maximal* if $c \preceq c_1$ implies that $c = c_1$.

Theorem 3.6. *Let S be a subcartesian space and let X be a derivation of $C^\infty(S)$. For every $x \in S$, there exists a unique maximal integral curve c of X such that $c(0) = x$.*

Proof. The modification of the definition of an integral curve, given in Definition 3.5, allows for closing the hole in the proof of Theorem 3.2.1 in [19]. For the sake of clarity, we include here the complete proof.

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(i) Local existence. Consider the defining equation for an integral curve $c : I \to S$ of X originating at x :

$$c(0) = x \text{ and } \frac{d}{dt} f(c(t)) = (Xf)(c(t)) \text{ for every } f \in \mathcal{C}^\infty(S) \text{ and every } t \in I.$$

If $I = \{0\}$, then the integral curve c consists of one point $c(0) = x$. If I has non-empty interior, then it is an interval in R, possibly unbounded, and we need to consider the differential equation (20).

Let φ be a diffeomorphism of a neighbourhood V of x in S onto a differential subspace R of $\mathbb{R}^n$. Let $Z = \varphi_* X_{|V}$ be a derivation of $\mathcal{C}^\infty(R)$ obtained by pushing forward the restriction of X to V by φ. In other words,

$$Z(f) \circ \varphi = X_{|V}(f \circ \varphi)$$

for all $f \in \mathcal{C}^\infty(R)$. Without loss of generality, we may assume that there is an extension of Z to a vector field Y on $\mathbb{R}^n$.

Let $z = \varphi(x)$, and c_0 be an integral curve in $\mathbb{R}^n$ of the vector field Y such that $c_0(0) = z$. Let I_x be the connected component of $c_0^{-1}(R)$ containing 0, and let $c : I_x \to R$ be the restriction of c_0 to I_x. Clearly, $c(0) = z$. We have to consider two cases: (1) $I_x = \{0\}$ and (2) I_x is an interval in $\mathbb{R}$. In the first case, $c : \{0\} \to S : 0 \mapsto x$ is an integral curve of X originating at x.[1] In the second case, for each $t_0 \in I_x$ and each $f \in \mathcal{C}^\infty(R)$ there exists a neighbourhood U of $c(t_0)$ in R and a function $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f_{|U} = F_{|U}$. Therefore,

$$\begin{aligned} \frac{d}{dt} f(c(t))_{|t=t_0} &= \frac{d}{dt} F(c(t))_{t=t_0} = (Y(F))(c(t_0)) \\ &= (Y(F))_{|U}(c(t_0)) = (Z(f))(c(t_0)), \end{aligned}$$

which implies that $c : I_x \to R$ is an integral curve of Z through z.

Since I_x is an interval, $c_x = \varphi^{-1} \circ c : I_x \to V \subseteq S$ satisfies $c_x(0) = \varphi^{-1}(c(0)) = \varphi^{-1}(z) = x$. Moreover, for every $t \in I_x$ and $h \in \mathcal{C}^\infty(S)$, $f = h \circ \varphi^{-1} \in \mathcal{C}^\infty(R)$ and

$$\begin{aligned} \frac{d}{dt} h(c_x(t)) &= \frac{d}{dt} h(\varphi^{-1}(c(t))) = \frac{d}{dt}(h \circ \varphi^{-1})(c(t)) \\ &= \frac{d}{dt}(f(c(t)) = Z(f)(c(t)) \\ &= Z(h \circ \varphi^{-1})(\varphi \circ c_x(t)) = X(h)(c_x(t)). \end{aligned}$$

Thus, $c_x : I_x \to S$ is an integral curve of X through x.

(ii) Smoothness. It follows from the theory of differential equations that the integral curve c_0 in $\mathbb{R}^n$ of a smooth vector field Y is smooth. Hence, $c = c_{0|I_x}$ is smooth. Since φ is a diffeomorphism of a neighbourhood of x in S to R, its inverse φ^{-1} is smooth, and the composition $c_x = \varphi^{-1} \circ c$ is smooth.

(iii) Local uniqueness. This follows from the local uniqueness of solutions of first order differential equations in $\mathbb{R}^n$.

(iv) Maximality. If there are no integral curves $c : I \to S$ of X originating at x such that the interior of I is not empty, then $c : \{0\} \mapsto \{x\}$ is maximal. Otherwise, suppose that there is an integral curve $c : I \to S$ of X originating at x has domain I with endpoints $p < q$, where $p \leq 0$ and $q \geq 0$. If $q \in I$, $q = \infty$, or $\lim_{t \to q^-} c(t)$ does not exist, then the curve c does not extend beyond q. If $x_1 = \lim_{t \to q^-} c(t)$ exists, then it is unique because S is Hausdorff and we can repeat the construction of section (i) beginning with the point x_1. In this way, we obtain an integral curve $c_1 : I_1 \to S$ of X with the initial condition $c_1(0) = x_1$. Let $\tilde{I}_1 = I \cup \{t = q + s \mid s \in I_1 \cap [0, \infty)\}$, and $\tilde{c}_1 : \tilde{I}_1 \to S$ be given by $\tilde{c}_1(t) = c(t)$ if $t \in I$ and $\tilde{c}_1(t) = c_1(t - q)$ if $t \in \{q + s \mid s \in I_1 \cap [0, \infty)\}$. Clearly, $\tilde{c}_1$ is continuous.

[1] This argument was missing in the proof of Theorem 3.2.1 in [19].

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Moreover, since $x_1 = \lim_{t\to q^-} c(t)$, it follows that the lower end point p_1 of I_1 is strictly less than zero. Hence, the restriction of c to $(\max(p, p_1) + q, q)$ differs from the restriction of c_1 to $(\max(p, p_1), 0)$ by reparametrization $t \mapsto t - q$. Since c and c_1 are smooth, it follows that $\tilde{c}_1$ is smooth. Let q_1 be the upper limit q_1 of I_1. If $q_1 \in I_1$, $q_1 = \infty$, or $\lim_{t\to q_1^-} c_1(t)$ does not exist, then the curve c_1 does not extend beyond q_1. Otherwise, we can extend $\tilde{c}_1$ by an integral curve c_2 of X through $x_2 = \lim_{t\to q_1^-} c_1(t)$. Continuing the process we obtain a maximal extension for $t \geq 0$. In a similar way we can construct a maximal extension for $t \leq 0$.

(v) Global uniqueness. Let $c : I \to S$ and $c' : I' \to S$ be two maximal integral curves of X through x and

$$T^+ = \{t \in I \cap I' \mid t > 0 \text{ and } c(t) \neq c'(t)\}.$$

Suppose that $T^+ \neq \emptyset$. Since T^+ is bounded from below by 0, there exists a greatest lower bound l of T^+. This implies that $c(t) = c'(t)$ for $0 \leq t \leq l$ and, for every $\varepsilon > 0$, there exists $t_\varepsilon \in T^+$ such that $l < t_\varepsilon < l + \varepsilon$ and $c(t_\varepsilon) \neq c'(t_\varepsilon)$. Let $x_l = c(l) = c'(l)$ and $c_l : I_l \to S$ be an integral curve of X through x_l constructed as in section (i). We denote by q_l the upper end point of the interval I_l. If $q_l > 0$, the local uniqueness implies that $c(t) = c'(t) = c_l(t - l)$ for all $l \leq t \leq l + q_l$. Hence, we get a contradiction with the assumption that l is the greatest lower bound of T^+. If $q_l = 0$, then there is no extension of c_l to $t > 0$. Let q and q' be the upper endpoints of I and I', respectively. Since c and c' are maximal integral curves of X, it follows that $q = q' = l$. Hence, the set T^+ is empty. A similar argument shows that

$$T^- = \{t \in I \cap I' \mid t < 0 \text{ and } c(t) \neq c'(t)\} = \emptyset.$$

Therefore, $c(t) = c'(t)$ for all $t \in I \cap I'$. If $I \neq I'$, then we get a contradiction with the assumption that c and c' are maximal. Hence, $I = I'$ and $c = c'$. □

Let X be a derivation of $C^\infty(S)$. We denote by $e^{tX}(x)$ the point on the maximal integral curve of X, originating at x, corresponding to the value t of the parameter. Given $x \in S$, $e^{tX}(x)$ is defined for t in an interval I_x containing zero, and $e^{0X}(x)(x) = x$. If t, s, and $t + s$ are in I_x, $s \in I_{e^{tX}(x)}$, and $t \in I_{e^{sX}(x)}$, then

$$e^{(s+t)X}(x) = e^{sX}(e^{tX}(x)) = e^{tX}(x)(e^{sX}(x)).$$

Proposition 3.7. *For every derivation X of the differential structure $C^\infty(S)$ of a subcartesian space and a diffeomorphism $\varphi : S \to R$,*

$$e^{t\varphi_* X} = \varphi \circ e^{tX} \circ \varphi^{-1}.$$

Proof. For each $f \in C^\infty(R)$ and $y = \varphi(x) \in R$,

$$\begin{aligned}
& \frac{d}{dt} f((\varphi \circ e^{tX} \circ \varphi^{-1})(y)) = \frac{d}{dt} f(\varphi \circ e^{tX})(x) \\
= & \left(T\varphi\left(\frac{d}{dt} e^{tX}(x)\right)\right)(f) = \left(\frac{d}{dt} e^{tX}(x)\right)(\varphi^* f) \\
= & X(\varphi^* f)(e^{tX}(x)) && \text{by equation (19)} \\
= & \varphi^*(\varphi_* X(f))(e^{tX}(x)) && \text{by equation (17)} \\
= & (\varphi_* X(f))(\varphi(e^{tX}(x))) = (\varphi_* X(f))(\varphi(e^{tX}(\varphi^{-1}(y))) \\
= & (\varphi_* X(f))(\varphi \circ e^{tX} \circ \varphi^{-1})(y).
\end{aligned}$$

Hence, $t \mapsto (\varphi \circ e^{tX} \circ \varphi^{-1})(y)$ is an integral curve of $\varphi_* X$ through y. □

In the case when S is a manifold, the map e^{tX} is a local one-parameter group of local diffeomorphisms of S. For a subcartesian space S, $e^{tX} : x \mapsto e^{tX}(x)$ might fail to be a local diffeomorphism.

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Definition 3.8. *A* vector field *on a subcartesian space S is a derivation X of $\mathcal{C}^\infty(S)$ such that for every $x \in S$, there exists an open neighbourhood U of x in S and $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$, the map $e^{tX}(x)$ is defined on U, and its restriction to U is a diffeomorphism from U onto an open subset of S. In other words, X is a vector field on S if e^{tX} is a local 1-parameter group of local diffeomorphisms of S.*

Notation 3.9. *We denote by $\mathfrak{X}(S)$ the famılly of all vector fields on a subcartesian space S.*

Example 3.10. *Consider $S = [0, \infty) \subseteq \mathbb{R}$ with the structure of a differential subspace of $\mathbb{R}$. Let $(Xf) = \frac{df}{dx}$ for every $f \in \mathcal{C}^\infty([0, \infty))$ and $x \in [0, \infty)$. Note that the derivative at $x = 0$ is the right derivative; it is uniquely defined by $f(x)$ for $x \geq 0$. For this X, the map e^{tX} is given by $\mathrm{e}^{tX}(x) = x + t$ whenever x and $x + t$ are in $[0, \infty)$. In particular, for every neighbourhood U of 0 in $[0, \infty)$ there exists $\delta > 0$ such that $[0, \delta) \subseteq U$. Moreover, e^{tX} maps $[0, \delta)$ onto $[t, \delta + t)$, which is not an open neighbourhood of $t = \mathrm{e}^{tX}(0)$ in $[0, \infty)$. Hence, the derivation X is not a vector field on $[0, \infty)$. On the other hand, for every $f \in \mathcal{C}^\infty[0, \infty)$ such that $f(0) = 0$, the derivation fX is a vector field, because 0 is a fixed point of e^{tfX}.* □

Theorem 3.11. *Let S be a subcartesian space. A derivation X of $\mathcal{C}^\infty(S)$ is a vector field on S if the domain of every maximal integral curve of X is open in $\mathbb{R}$.*

Proof. [2] Theorem 3.6 ensures that maximal integral curves of vector fields have non-empty open domains. This implies that, if a derivation X of $\mathcal{C}^\infty(S)$ has a maximal integral curve of the type $c : \{0\} \to \{x\} : 0 \mapsto x$, then it cannot be a vector field. Hence, in the remaining of the proof we need not consider integral curves of this type.Consider the case when S is a differential subspace of $\mathbb{R}^n$. Let X be a derivation on S such that domains of all its integral curves are open in $\mathbb{R}$. In other words, for each $x \in S$, the domain I_x of the map $t \mapsto \mathrm{e}^{tX}(x)$ is an open interval in $\mathbb{R}$.

This implies that no maximal integral curve of X is defined only for $t = 0$. We need to show that the map $x \mapsto \mathrm{e}^{tX}(x)$ is a local diffeomorphism of S.

Given $x_0 \in S \subseteq \mathbb{R}^n$, there exists an open neighbourhood W_0 of x_0 such that the restriction of X to W_0 extends to a vector field Y on an open subset $\bar{U}_0 \subseteq \mathbb{R}^n$, containig W_0. We show first that the restriction of X to W_0 generates a local one-parameter group of local diffeomorphisms of W_0.

Since open sets in S are the intersections with S of open sets in $\mathbb{R}^n$, without loss of generality we can write $W_0 = U_0 \cap S$. Let e^{tY} denote the local one-parameter group of local diffeomorphisms of U_0 generated by Y. There exists an open neighbourhood U_1 of x_0, contained in U_0, and $\varepsilon > 0$ such that, for every $t \in (-\varepsilon, \varepsilon)$, the map $\mathrm{e}^{tY} : U_1 \to \mathrm{e}^{tY}(U_1) \subseteq U_0$ is a diffeomorphism of U_1 onto its image.

Let $W_1 = U_1 \cap S \subseteq W_0$. Since $Y_{|W_0} = X_{|W_0}$, the assumption that maximal integral curves of vector fields have non-empty open domains ensures that, for every $x \in W_1 \subseteq W_0$, there is $\delta_x > 0$ such that $\mathrm{e}^{tY}(x) = \mathrm{e}^{tX}(x) \in W_0 = U_0 \cap S$ for all $t \in (-\delta_x, \delta_x)$. Let $\iota_{W_1} = \inf \{\delta_x \mid x \in W_1\}$ be the , the infimum of the set $\{\delta_x \mid x \in W_1 \subseteq \mathbb{R}^n\}$. Since each $\delta_x > 0$ it follows that $\iota_{W_1} \geq 0$.

(1) If $\iota_{w_1} > 0$, then there is a neigbourhood W_2 of x_0 contained in W_1 and $\varepsilon_1 \in (0, \varepsilon)$ such that, for every $t \in (-\varepsilon_1, \varepsilon_1)$, the map $\mathrm{e}^{tX} : W_2 \to \mathrm{e}^{tX}(W_2) \subseteq W_1$ is a diffeomorphism of W_2 onto its image. In this case, the restriction of X to $W_1 \ni x_0$ is a vector field on W_1.

(2) Suppose that $\iota_{W_1} = 0$. Since the domain of every maximal integral curve of X is open in $\mathbb{R}$, it follows that the closure $\overline{W}_1$ of W_1 has non-empty intersection with the part of the boundary $\bar{S} \cap (\overline{R^n \backslash S})$ of S that is not contained in S. In this case there exists an open set $V \subset S$ such that $x_0 \in V \subset \overline{V} \subset W_1$ so that $\overline{V}$ has empty intersection with the part of the boundary $\bar{S} \cap (\overline{R^n \backslash S})$ of S that is not contained in S. Then $\iota_V = \inf\{\delta_x \mid x \in V\} > 0$, and there exists a neighbourhood W_2 of x_0 contained in V and $\varepsilon_1 \in (0, \varepsilon)$ such that, for every $t \in (-\varepsilon_1, \varepsilon_1)$, the map $\mathrm{e}^{tX} : W_2 \to \mathrm{e}^{tX}(W_2) \subseteq V$ is a diffeomorphism of W_2 onto its image. In this case, the restriction of X to $V \ni x_0$ is a vector field on V.

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[2] This proof is an improvement of the proof of Proposition 3.2.6 in [19]. Not only it does not require the assumption that S is locally closed, but it is complete and more transparent.

These arguments can be repeated for every $x_0 \in W_0 \subseteq S$. Hence, the restriction of X to W_0 is a vector field on W. Similarly, we can repeat these arguments for every $x_0 \in S$, concluding that X is a vector field on S.

Consider now the case of a general subcartesian space S. Let X be a derivation of $\mathcal{C}^\infty(S)$ such that the domains of all its maximal integral curves are open. For every $x \in S$ there exists a neighbourhood W of x in S and a diffeomorphism $\chi$of W onto a differential subspace S_W of $\mathbb{R}^n$. Since W is open in S, maximal integral curves of the restriction $X_{|W}$ of X to W are open domais. The diffeomorphism $\chi : W \to S_W$ pushes-forward $X_{|W}$ of X to a derivation $\chi_* X_{|W}$ of $\mathcal{C}^\infty(S_W)$ with the same properties. That is all integral curves of $\chi_* X_{|W}$ have open domains. By the argument above, $\chi_* X_{|W}$ is a vector field on S_W.

Since $\chi : W \to S_W$ is a diffeomorphism, it follows that $X_{|W}$ is a vector field on W. This argument can be repeated at every point $x \in S$. Therefore, for every $x \in S$, the derivation X restricts to a vector field in a an open neighbourhood of of x.

Therefore, X is a vector field on S. $\square$

For $X_1, \ldots, X_n \in \mathfrak{X}(S)$ consider a piece-wise smooth integral curve c in S,originating at $x_0 \in S$, given by a sequence of steps. First, we follow the integral curve of X_1 through x_0 for time τ_1; next we follow the integral curve of X_2 though $x_1 = \varphi^X_{\tau_1}(x_0)$ for time τ_2; and so on. For each $i = 1, \ldots, n$ let J_i be $[0, \tau_i] \subseteq \mathbb{R}$ if $\tau_i > 0$ or $[\tau_i, 0]$ if $\tau_i < 0$. Note that $\tau_i < 0$ means that the integral curve of X_i is followed in the negative time direction. For every i, J_i is contained in the domain $I_{x_{i-1}}$ of the maximal integral curve of X_i starting at x_{i-1}. In other words, for $t = \tau_1 + ... + \tau_{n-1} + \tau_n$,

$$c(t) = c(\tau_1 + \tau_2 + ... + \tau_{n-1} + \tau_n) = \varphi^{X_n}_{\tau_n} \circ \varphi^{X_{n-1}}_{\tau_{n-1}} \circ .... \circ\ \varphi^{X_1}_{\tau_1}(x_0).$$

Definition 3.12. *The* orbit through x_0 *of the family* $\mathfrak{X}(S)$ *of vector fields on* S *is the set* M *of points* x *in* S *that can be joined to* x_0 *by a piecewise smooth integral curve of vector fields in* $\mathfrak{X}(S)$*;*

$$M = \{\varphi^{X_n}_{t_n} \circ \varphi^{X_{n-1}}_{t_{n-1}} \circ .... \circ \varphi^{X_1}_{t_1}(x_0) \mid X_1, ..., X_n \in \mathfrak{X}(S)\ t_1, ..., t_n \in \mathbb{R},\ n \in \mathbb{N}\}.$$

Theorem 3.13. *Orbits* M *of the family* $\mathfrak{X}(S)$ *of vector fields on a subcartesian space* S *are submanifolds of* S*. In the manifold topology of* M*, the differential structure on* M *induced by its inclusion in* S *coincides with its manifold differential structure.*

Proof. See reference [18]. $\square$

4. Partition of S by Orbits of $\mathfrak{X}(S)$

In this section, we study consequences of Theorem 3.13 to our understanding of the geometry of subcartesian spaces.

Notation 4.1. *We denote by* $\mathfrak{M}(S)$ *the family of orbits of* $\mathfrak{X}(S)$*.*

By Theorem 3.13 each orbit M of $\mathfrak{X}(S)$ is a manifold. Moreover, the manifold structure of M is its differential structure induced by the inclusion of M in S. Hence, M is a submanifold of the differential space S. The orbits of $\mathfrak{X}(S)$, give a partition $\mathfrak{M}(S)$ of S by connected smooth manifolds. Since the notion of a vector field on a subcartesian space S is intrinsically defined in terms of its differential structure, it follows that every subcartesian space has a natural partition by connected smooth manifolds. In particular, every subset S of $\mathbb{R}^n$ has natural partition by connected smooth manifolds.

Proposition 4.2. *Let* X *be a derivation of* $C^\infty(S)$*. If, for each* $M \in \mathfrak{M}(S)$ *and each* $x \in M$*, the maximal integral curve of* X *originating at* $x \in M$ *is contained in* M*, then* $X \in \mathfrak{X}(S)$*, that is,* X *is a derivation of* $C^\infty(S)$ *that generates local one parameter groups of local diffeomorphisms of* S*.*

Proof. Suppose that X is a derivation of $C^\infty(S)$ satisfying the assumptions of Proposition 4.2. By Theorem 3.13, every $M \in \mathfrak{M}(S)$ is a submanifold of the differential space S. This means that the manifold structure $C^\infty(M)$ of M is induced by the restrictions to M of functions in $C^\infty(S)$. Since all integral curves of X originating at points of M are contained in M, it follows that the restriction $X_{|M}$ of X to M is a derivation of $C^\infty(M)$. But, for a manifold M, all derivations of $C^\infty(M)$ are vector fields on M in the sense that their integral curves generate local one parameter groups of local diffeomorphisms of M. Moreover, domains of maximal integral curves of vector fields on a manifold are open. By assumption, this holds to every $M \in \mathfrak{M}(S)$. Since S is the union of all manifolds $M \in \mathfrak{M}(S)$, it follows that every integral curve of X has open domain. Theorem 3.11 ensures that X is a vector field on S in the sense that it generates local one parameter groups of local diffeomorphisms of S. □

Theorem 4.3. *The family $\mathfrak{X}(S)$ of all vector fields on a subcartesian space S is a Lie subalgebra of the Lie algebra* $\mathrm{Der}C^\infty(S)$ *of derivations of* $C^\infty(S)$.[3]

Proof. For $X \in \mathfrak{X}(S)$ and $f \in C^\infty(S)$, the product $f\,X \in \mathrm{Der}C^\infty(S)$. By construction, for every $M \in \mathfrak{M}(S)$, $X_{|M}$ is a vector field on the submanifold M of S, and $f_{|M} \in C^\infty(M)$. Hence, $(fX)_{|M} = f_{|M}X_{|M}$ is a derivation of $C^\infty(M)$. Therefore, for every $x \in M$, the maximal integral curve of fX originating at x, is the maximal integral curve of $(fX)_{|M}$ originating at x. But M is a manifold, which implies that the derivation $(fX)_{|M}$ of $C^\infty(M)$ is a vector field on M so that every maximal integral curve of $(fX)_{|M}$ has open domain.

The argument above is valid for every manifold . Since $S = \cup_{M\in\mathfrak{M}(S)} M$, it follows that every integral curve of fX has open domain. Theorem 3.3.8 ensures that fX is a vector field on S, that is $fX \in \mathfrak{X}(S)$.

Suppose that $X, Y \in \mathfrak{X}(S)$. Then $X + Y \in \mathrm{Der}C^\infty(S)$. As before, for every $M \in \mathfrak{M}(S)$, the restrictions $X_{|M}$ and $Y_{|M}$ are vector fields on the manifold M, so that $(X+Y)_{|M} = X_{|M} + Y_{|M}$ is a vector field on M. Hence integral curves of $X + Y$ originating at points in M have open domains. This is valid for every $M \in \mathfrak{M}(S)$, which implies that all integral curves of $X + Y$ have open domains. Therefore, $X + Y \in \mathfrak{X}(S)$.

Replacing $+$ in the arguments of the preceding paragraph by the Lie bracket $[\cdot,\cdot]$, we can show that, for every $X, Y \in \mathfrak{X}(S)$, their Lie bracket $[X, Y] \in \mathfrak{X}(S)$. Therefore, the family $\mathfrak{X}(S)$ of all vector fields on S is a Lie subalgebra of $\mathrm{Der}C^\infty(S)$. □

Proposition 4.4 (Frontier Condition)**.** *For $M, M' \in \mathfrak{M}(S)$, if $M' \cap \overline{M} \neq \emptyset$, then either $M' = M$ or $M' \subset \overline{M} \backslash M$.*

Proof. Let M and M' be orbits of $\mathfrak{X}(S)$ such that $M' \cap \overline{M} \neq \emptyset$, where $\overline{M}$ denotes the closure of M in S. Suppose that $x_0 \in M' \cap \overline{M}$ with $M' \neq M$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points in M converging to x_0. For every $X \in \mathfrak{X}(S)$, there is an open neighbourhood U_0 of x_0 in S and $t_0 > 0$ such that $\exp(tX)(x)$ is defined for every $0 \leq t \leq t_0$ and every $x \in U_0$. Moreover, if $0 \leq t \leq t_0$, the map $U_0 \to S : x \mapsto \exp(tX)(x)$ is continuous. Therefore, for $0 \leq t \leq t_0$,

$$\lim_{k\to\infty} \exp(tX)(x_k) = \exp(tX)(x_0).$$

Since M is the orbit of $\mathfrak{X}(S)$, it is invariant under the family of one-parameter local groups of local diffeomorphisms of S generated by vector fields, and $\{x_k\}_{k\in\mathbb{N}} \subseteq M$, it follows that $\lim_{k\to\infty} \exp(tX)(x_k) \in \overline{M}$. Therefore, $\exp(tX)(x_0) \in \overline{M}$. On the other hand, $M'$is the orbit of $\mathfrak{X}(S)$ through x_0, so that $\exp(tX)(x_0) \in M'$. Hence, $\exp(tX)(x_0) \in M' \cap \overline{M}$. By assumption, $M' \neq M$, which

[3] This result was first obtained by Watts in Ph.D. Thesis, Corollary 4.71, [21]. Here, we give an alternative proof.

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implies that $\exp(tX)(x_0) \in \overline{M}\backslash M$. This holds for every $X \in \mathfrak{X}(S)$ and $x_0 \in M' \cap \overline{M}\backslash M$. Therefore, $M' \subset \overline{M}\backslash M$. □

Proposition 4.5 (Whitney's Conditions A and B)**.** *Consider a differential subspace S of $\mathbb{R}^n$. Let $y \in M' \subseteq \overline{M}\backslash M$,where $M, M' \in \mathfrak{M}(S)$, and let $m = \dim M$.*
(A). If x_i is a sequence of points in M such that $x_i \to y \in M'$, and $T_{x_i}M$ converges to some m-plane $E \subseteq T_yS \subseteq T_y\mathbb{R}^n$ then $T_yM' \subseteq E$.
(B). If y_i be a sequence of points in M' also converging to y. Suppose that $T_{x_i}M$ converges to an m-plane $E \subseteq T_yS \subseteq T_y\mathbb{R}^n$ and the secant $\overleftrightarrow{x_iy_i}$ converges to some line in $L \subseteq \mathbb{R}^n$. Then $L \subseteq E$.

Proof. (A). Since M is a submanifold of the differential subspace S of $\mathbb{R}^n$, and $\overline{M}$ is the closure of M in S, then $\overline{M}$ is a differential subspace of S. Moreover, M and M' are submanifolds of $\overline{M}$. Hence, for a sequence x_i in M, such that $y = \lim_{i\to\infty} x_i \in M'$, we have

$$T_y\overline{M} = \lim_{i\to\infty} T_{x_i}M.$$

Since $M'$is a submanifold of $\overline{M}$, it follows that $T_yM' \subseteq T_y\overline{M}$.

In order to write the result in the form used in the statement of the proposition, we use the identification $\mathbb{R}^n \times \mathbb{R}^n \equiv T\mathbb{R}^n$ such that the following diagram commutes

$$\begin{array}{ccccc} & \mathbb{R}^n \times \mathbb{R}^n & \equiv & T\mathbb{R}^n & \\ \mathrm{pr}_1 & \downarrow & & \downarrow & \tau \ , \\ & \mathbb{R}^n & = & \mathbb{R}^n & \end{array}$$

where $\mathrm{pr}_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the first factor, and $\tau : T\mathbb{R}^n \to \mathbb{R}^n$ is the tangent bundle projection. Moreover, for every $f \in C^\infty(\mathbb{R}^n)$ and $v = (\boldsymbol{x}, \boldsymbol{v}) \in T\mathbb{R}^n$, the derivation of f by v is $vf = \langle df \mid \boldsymbol{v}\rangle(\boldsymbol{x})$. With this identification, the m-plane $T_y\overline{M} \subseteq T_yS \subseteq T_y\mathbb{R}^n$, can be expressed as $T_y\overline{M} = (y,E)$, where $E \subseteq \mathbb{R}^n$. Hence, $T_yM' \subseteq (y, E)$.

(B) The sequence $\overleftrightarrow{x_iy_i}$ of secants, if it converges as $i \to \infty$, defines a derivation $v \in T_y\overline{M}$ such that, for every $f \in C^\infty(\overline{M})$,

$$vf = \lim_{i\to\infty} \frac{f(x_i) - f(y_i)}{\|x_i - y_i\|},$$

where $\|x_i - y_i\| = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$. The limiting line of the sequence $\overleftrightarrow{x_iy_i}$ of secants is the line L through y in direction v. Since, $v \in T_y\overline{M}$, in the identification used above, $L \subseteq E$. □

For each $n = 0, 1, 2, ...$, let

$$\mathfrak{M}_n(S) = \{M \in \mathfrak{M}(S) \mid \dim M = n\}, \tag{21}$$

and

$$S_n = \coprod_{M \in \mathfrak{M}_n(S)} M. \tag{22}$$

Since elements of $\mathfrak{M}_n(S)$ are mutually disjoint n-dimensional manifolds , it follows that S_n is a manifold of dimension n, and the connected manifolds $M \in \mathfrak{M}_n(S)$ are connected components of S_n. Since S is a subcartesian space, the dimension n of S_n is locally bounded. For every chart $\alpha : V_\alpha \to W_\alpha \subseteq \mathbb{R}^{d_\alpha}$, $\dim S_n \cap V_\alpha \leq d_\alpha$. Hence,

$$S = \coprod_{n=0}^{\infty} S_n. \tag{23}$$

In general, the partition $\mathfrak{M}(S)$ of S by orbits of $\mathfrak{X}(S)$ need not be locally finite, as is shown in the following example.

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Example 4.6. *Let $S = \mathbb{Q} \times \mathbb{R} \subseteq \mathbb{R}^2$, where $\mathbb{Q}$ is the set of rational numbers. The discussion following Example 3.3.1 shows that that a derivation $X \in \mathrm{Der}\, S$ is a vector field only if it is tangent to the second factor $\mathbb{R}$. In other words, if $f \in C^\infty(S)$ is written in terms of the coordinates $(x_1, x_2) \in \mathbb{Q} \times \mathbb{R}$, then $X \in \mathfrak{X}(S)$ if and only if, there exists $a \in C^\infty(S)$ such that*

$$(Xf)(x_1, x_2) = a(x_1, x_2)\frac{\partial f(x_1, x_2)}{\partial x_2}$$

for every $f \in C^\infty(S)$ and every $(x_1, x_2) \in \mathbb{Q} \times \mathbb{R}$.
Since the space $\mathfrak{X}(\mathbb{R})$ of vector fields on $\mathbb{R}$ acts transitively on $\mathbb{R}$, it follows that in our example, for every $x = (x_1, x_2) \in S$, the orbit M of $\mathfrak{X}(S)$ through $x = (x_1, x_2)$ is $\{x_1\} \times \mathbb{R}$. Thus, the space $\mathfrak{M}(S)$ of orbits of $\mathfrak{X}(S)$ for $S = \mathbb{Q} \times \mathbb{R}$ is parametrized by $\mathbb{Q}$, and it is not locally finite.

Example 4.7. *Let $S = \{x \in \mathbb{R} \mid x = 0$ or $x = \frac{1}{n}$ for $n \in \mathbb{N}\}$. In this case, the only vector field on S is $X = 0$, and every $M \in \mathfrak{M}(S)$ is a single point. There is no neighbourhood of $0 \in S$ that contains only finite number of points of S. Hence, $\mathfrak{M}(S)$ is not locally finite.*

5. Comparison with Stratification

There are several definitions of stratification of a closed subset S of a smooth[4] manifold. The definition used by Goresky and MacPherson, [8], adapted to the set up considered here, can be reformulated as follows.

Definition 5.1. *A partition of a subcartesian space S by submanifolds of S is a* decomposition *of S if it is locally finite and satisfies Frontier Condition, that is the statement of Proposition 4.3. A* Whitney stratification *of S is a decomoposition of S that satisfies Whitney's conditions A and B, that is the statement of Proposition 4.4.*

If S is a closed subset of a smooth manifold M, then composing the inclusion of S into M with the charts for M we get an atlas $\mathfrak{A}(S) = \{\alpha : V_\alpha \to W_\alpha\}$, where V_α an open subset of S and W_α is a locally closed subset of $\mathbb{R}^{d_\alpha}$. In other words, S is a l*ocally closed subcartesian space.* Propositions 4.3 and 4.4 ensure that, if S is a locally closed subcartesian space and the partition $\mathfrak{M}(S)$ is locally finite, then $\mathfrak{M}(S)$ is a Whitney stratification of S.

Mather, uses the term *prestratification* for a decomposition of S by submanifolds and the term *stratification* for the sheaf $\mathcal{S}$ of germs of manifolds of prestratification, [11]. If S is locally closed and $\mathfrak{M}(S)$ is locally closed, then $\mathfrak{M}(S)$ is a prestratification of S and the sheaf $\mathcal{S}$ of germs of manifolds in $\mathfrak{M}(S)$ is the induced stratification.

Prestratifications of S that induce the same sheaf of germs $\mathcal{S}$ can be partially ordered by inclusion. Pflaum, [13], identifies the sheaf $\mathcal{S}$ of germs of the manifolds of prestratification with the coarsest prestratification in this class. If S is locally closed and $\mathfrak{M}(S)$ is locally closed, then the coarsest prestratification in the sense of Pflaum is $\{S_n\}_{n=0}^{\infty}$, where $S_n = \coprod_{M \in \mathfrak{M}_n(S)} M$, see equation (22).

We have seen that, for every definition of stratification discussed above, if S is a locally closed subcartesian space and $\mathfrak{M}(S)$ is locally finite, then the decomposition $\mathfrak{M}(S)$ of S corresponds to a stratification of S. It should be noted that, in this case, our approach corresponds to an algorithm leading to discovery of the of the stratification of S. Once S is chosen and its differential structure is established, there is no room for choice. The main step is to determine the family $\mathfrak{X}(S)$, consisting of all derivations of $C^\infty(S)$ that generate local one-parameter groups of local diffeomorphisms of S. Theorem 3.8 helps us to make this determination.

[4] We consider here only the C^∞ category.

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6. Transient Derivations

Up to now, we have concentrated on orbits of the Lie algebra $\mathfrak{X}(S)$ of vector fields on S, that is derivations of $C^\infty(S)$ that generate local one-parameter local groups of diffeomorphisms. In this section, we consider the role played by derivation of $C^\infty(S)$ that do not generate local one-paremeter groups of local diffeomorphisms of S.

Definition 6.1. Transient derivation[5] *on a subcartesian space S is a derivation of $C^\infty(S)$ that does not generate local one-paremeter groups of local diffeomorphisms of S.*

Let X be a transient derivation on a subcartesian space S. By Theorem 3.6, for every $x_0 \in S$, there exists a unique maximal integral curve c_0 of X such that $c_0(0) = x_0$. If, for every $x_0 \in S$, the maximal integral curve c_0 of X through $x_0 \in M \in \mathfrak{M}(S)$ is contained in M_0, then Proposition 4.2 ensures that X is generates local one paremeter local groups of diffeomorphisms of S, which contradicts the assumption that X is a transient derivation. Therefore, there must exist a maximal integral curve $c : I \to S$ of X such that, for some $t_1 \in I$, the curve c crosses from a manifold $M \in \mathfrak{M}(S)$ to a manifold $M' \subseteq \overline{M} \backslash M$. It follows that transient derivations provide integral curves joinining manifolds of $\mathfrak{M}(S)$

7. Manifolds with Corners

Manifolds with corners are a basic example of stratified subcartesian spaces. Here, we rely on the presentation of the theory of manifolds with corners given in [9]. We begin with a defininition of manifold with corners, as a locally Euclidean Hausdorff manifold, see Definition 2.6. This definition is equivalent to the original definition by Cerf, [6], used in [9].

Definition 7.1. *A d-dimensional manifold with corners is a paracompact Hausdorff topological space S equipped with a maximal d-dimensional atlas $\mathfrak{A} = \{\alpha : V_\alpha \to W_\alpha\}$, where α is a homeomorphism of an open subset V_α of S onto an open subset W_α of $\mathbb{R}^d_{k_\alpha} = [0,\infty)^{k_\varphi} \times \mathbb{R}^{d-k_\alpha} \subseteq \mathbb{R}^d$, in the topology induced by its inclusion in $\mathbb{R}^d$, which satisfies the conditions listed below.*
(1). The sets $\{V_\varphi \mid \alpha \in \mathfrak{A}\}$ form a covering of S.

(2). For every $\alpha, \beta \in \mathfrak{A}$, and every $x \in V_\alpha \cap V_\beta$, there exist:
(a) a C^∞-mapping Φ_α of an open neighbourhood U_α of $\alpha(x) \in \mathbb{R}^{n_\alpha}$ to $\mathbb{R}^{n_\beta}$, which extends the mapping

$$\beta \circ \alpha^{-1} : \alpha(V_\alpha \cap V_\beta) \to \beta(V_\alpha \cap V_\beta),$$

(b) a C^∞-mapping Φ_β of an open neighbourhood U_β of $\beta(x) \in \mathbb{R}^{n_\beta}$ to $\mathbb{R}^{n_\alpha}$, which extends the mapping

$$\alpha \circ \beta^{-1} : \beta(V_\alpha \cap V_\beta) \to \alpha(V_\alpha \cap V_\beta).$$

(3). A continuous function $f : S \to \mathbb{R}$ on S is smooth *if and only if, for every chart $\alpha : V_\alpha \to W_\alpha \subseteq \mathbb{R}^d$, there exists an open set U_α in $\mathbb{R}^d$ containing W_α, and a smooth function $F \in C^\infty(U_\alpha)$ such that $f \circ \alpha^{-1} : W_\alpha \to \mathbb{R}$ is the restriction of F to $W_\alpha \subseteq U_\alpha$. We denote by $C^\infty(S)$ the space of smooth functions on S.*
(4). A map $\varphi : S \to R$ between manifolds with corners S and R is is smooth if it is continuous and, for every pair of charts $\alpha : V_\alpha \to W_\alpha \subseteq \mathbb{R}^{d_S}$ in $\mathfrak{A}(S)$ and $\beta : V_\beta \to W_\beta \subseteq \mathbb{R}^{d_R}$ in $\mathfrak{A}(R)$, such that $\varphi \circ \alpha^{-1}(W_\alpha) \subseteq V_\beta$, there exist open subsets $U_\alpha \subseteq \mathbb{R}^{d_S}$, $U_\beta \subseteq \mathbb{R}^{d_R}$ and $F_{\alpha\beta} \in C^\infty(U_\alpha, U_\beta)$ such that: (i) $W_\alpha \subseteq U_\alpha$, (ii) $W_\alpha \subseteq U_\alpha$ and, for every $\boldsymbol{x} \in W_\alpha$,

$$F_{\alpha\beta|W_\alpha}(\boldsymbol{x}) = \beta \circ \varphi \circ \alpha^{-1}(\boldsymbol{x}).$$

The fundamental notion on a manifold with corners S, leading to the stratification structure of S, is the depth functions

[5] The term *transient derivation* is an extension of the notion of *transient vector field* used in the theory of manifolds with boundary, [12].

$$\text{depth}_S : S \to \mathbb{Z}_{\geq 0} : x \mapsto \text{depth}_S x = \min_{\alpha \in \mathfrak{A}} \{k_\alpha \mid x \in V_\alpha\}.$$

It is easy to show that the function $\text{depth}_S x$ is well defined by the differential structure $C^\infty(S)$ of the manifold with corners S under consideration.

Definition 7.2. *For each $k \geq 0$, the depth k stratum of S is*

$$S^k = \{x \in S \mid \text{depth}_S x = k\}.$$

Proposition 7.3. *Let S be a d-dimensional manifold with corners.*

(a) S is a disjoint union of S^k, for $k = 0, ..., d$,

$$S = \coprod_{k=0}^{d} S^k.$$

(b) Each S^k has the structure of an $(d-k)$-dimensional manifold (without boundary or corners).

(c) If $\overline{S}^k \cap S^l \neq \emptyset$, then either $S^l = S^k$, or $S^l \subseteq \overline{S}^k \backslash S^k$, where $\overline{S}^k$ denotes the closure of S^k in S.

(d) For every $k = 0, ..., d$,

$$\overline{S}^k = \coprod_{l=k}^{d} S^l \tag{24}$$

is a manifold with corners.

Proof. (a) The depth of $x \in S$ is uniquely defined by the maximal n-dimensional atlas $\mathfrak{A}$. Hence $S^k \cap S^l = \emptyset$ if $k \neq l$. Moreover, $k = 0, ..., d$. Hence, S is a disjoint union of S^k, for $k = 0, ..., d$.

(b) Definition 3.1 enures that S has an atlas $\mathfrak{A} = \{\alpha : V_\alpha \to W_\alpha\}$, where α is a homeomorphism of an open subset V_α of S onto an open subset W_α of $\mathbb{R}^d_{k_\alpha} = [0,\infty)^{k_\varphi} \times \mathbb{R}^{d-k_\alpha} \subseteq \mathbb{R}^d$, in the topology induced by its inclusion in $\mathbb{R}^d$. For each $x \in S^k \subseteq S$, there exists a chart $\alpha : V_\alpha \to W_\alpha$ for S such that $x \in V_\alpha$, and $W_\alpha = ([0,\infty)^k \times \mathbb{R}^{d-k}) \cap U_\alpha$, where U_α is an open subset of $\mathbb{R}^d$. Moerover $\alpha(V_\alpha \cap S^k) = ([0]^k \times \mathbb{R}^{d-k}) \cap U_\varphi$. Note that $[0]^k \times \mathbb{R}^{d-k} \cong \mathbb{R}^{d-k}$ and $\mathbb{R}^{d-k} \cap U_\varphi$ is an open subset of $\mathbb{R}^{d-k}$. The collection of charts

$$\begin{aligned} \mathfrak{A}_{S^k} &= \{\varphi_{|V_\varphi \cap S^k} : V_\varphi \cap S^k \to \varphi(V_\varphi \cap S^k) \\ &= \mathbb{R}^{d-k} \cap U_\varphi \mid \text{for all } \varphi \in \mathfrak{A} \text{ such that } \varphi(V_\varphi \cap S^k) \\ &= ([0]^k \times \mathbb{R}^{d-k}) \cap U_\varphi\} \end{aligned}$$

is a $(d-k)$-manifold atlas for S^k. It satisifes the condition (2) of Definition 3.1because the atlas $\mathfrak{A}$ satisfies this condition.

(c) Recall that a manifold with corners S is defined as a topological space satisfying certain conditions. Therefore, by the closure $\overline{S^k}$ of S^k we mean the closure of S^k in S. If S were a subset of some other topological space T, then the closure of S^k in S is the intersection with S of the closure of S^k in the topology induced by its embedding of S into T.

If If $\overline{S^k} \cap S^l \neq \emptyset$, there exists $x_0 \in \overline{S^k} \cap S^l \subseteq S$. Since $x_0 \in \overline{S^k}$, every open neighbourhood V of x has non-empty intersection with S^k. Since $x_0 \in S^l$, it follows that $\text{depth}\, x_0 = l$, and there exists a chart $\alpha : V_\alpha \to W_\alpha$ be such that such that $x_0 \in V_\alpha$, and $W_\alpha = ([0,\infty)^l \times \mathbb{R}^{d-l}) \cap U_\alpha$, where U_α is an open subset of $\mathbb{R}^{d-l}$.Without loss of generality, we may assume that, for each $x \in S^k \cap V_\alpha$, $\alpha(x) = (x^1, ..., x^l, x^{l+1}, ..., x^d)$, has first k of the l components $(x^1, ..., x^l)$ equal to zero. Hence, $l \geq k$. If $l = k$, then $S^l = S^k$. If $l > k$, then $x_0 \in \overline{S^k} \backslash S^k$. This argument holds for every $x \in \overline{S^k} \cap S^l$ with $l > k$. Hence, $S^l \subseteq \overline{S^k} \backslash S^k$.

(d) It follows from (a) and (c) that

$$\overline{S^k} = \overline{S^k} \cap S = \overline{S^k} \cap \coprod_{l=0}^{d} S^l = \coprod_{l=0}^{d} \overline{S^k} \cap S^l = \coprod_{l=k}^{d} \overline{S^k} \cap S^l = \coprod_{l=k}^{d} S^l.$$

It is easy to check that $\overline{S^k} = \coprod_{l=k}^{d} S^l$ satisfies the conditions for a manifold with corners. □

Definition 4.2 quotes the corresponding definition in [9], in which the term "depth k stratum" is used without explanation. It shows that the stratification structure of manifolds with corners is a common knowledge in this field. By Definiton 4.1, manifolds with corners are locally closed subcartesian spaces.

All definitions of stratifications, discussed in the preceding section, deal with closed subsets of a manifold. Every closed subset of a manifold is a locally closed subcartesian space. However, not every locally closed subcartesian space can be presented as a closed subset of a manifold. Hence, the use of the term "stratification" in the theory of manifolds with corners is a generalization of the classical notion of stratification which is convenient to adopt in the theory of differential spaces.

In order to relate the general theory of the preceding sections to the example of manifolds with corners, we have to establish what are vector fields on manifolds with corners. In other words, we have to establish the class of derivations of $C^\infty(S)$ which generate local one-parameter groups of local diffeomorphisms of S.

The depth function stratification $\{S^0, S^1, ..., S^k, ...S^d\}$ encodes the intrinsic geometric structure of the manifold with corners S. Therefore, we may expect that connected components of the strata of the stratification $\{S^0, S^1, ..., S^k, ...S^d\}$ are integral manifolds of the Lie algebra $\mathfrak{X}(S)$ of S. We establish this result in a series of propositions.

Proposition 7.4. *Let S be a manifold with corners. A derivation X of $C^\infty(S)$ is a vector field on S if and only if every maximal integral curve $c : I \to S$ of X is contained in a single stratum of the depth function stratification of S.*

Proof. Let X be a derivation of $C^\infty(S)$ of an d-manifold with corners. Suppose that every maximal integral curve $c : I \to S$ of X is contained in a single stratum in $\mathfrak{M}(S)$. Let M be a connected component of a stratum S^k of the depth function stratification of S. Since all integral curves of X are connected, it follows that all integral curves of X originating at points in M are contained in M. Therefore, the restriction $X_{|M}$ of X to M is a derivation of $C^\infty(M)$. But M is a manifold and all derivations of $C^\infty(M)$ are vector fields on M. Therefore, $X_{|M}$ generates local one-parameter group of local diffeomorphisms of M.

The argument above is valid for every connected component of each stratum of the depth function stratification of S. Therefore, the derivation X generates a local one-paremeter group of local diffeomorphisms of manifolds with corners that preserve the depth function stratification of S. Hence the derivation X is a vector field on S.

Let X be a vector field on S. That is, X generates a local one-parameter group of local diffeomorphisms of X. We need to show that every integral curve of X is contained in a connected component of a single stratum of the depth function stratification of S. We suppose opposite and derive a contradiction.

Suppose that there is an integral curve $c : I \to S$ of X such that, for $-\epsilon < t < 0$, $c(t)$ is in a connected component M of a stratum S^m and $c(0)$ is in a connected component N of a different stratum S^n of S. Since $c(0) = \lim_{t \to 0^-} c(t)$, Proposition 4.3(c) implies that $N \subseteq \overline{M} \backslash M$ so that $m \leq n-1$. Let $\alpha : V \to W \subseteq \mathbb{R}^d$ be a chart in $\mathfrak{A}$, where V is a neighbourhood of $c(0)$ in S and $W \subseteq \mathbb{R}^d_m \cap U = ([0,\infty)^m \times \mathbb{R}^{d-m}) \cap U \subseteq \mathbb{R}^d$ for some open neighbourhood U of $\mathbf{0}_d \in \mathbb{R}^d$, such that $\alpha(c(0)) = \mathbf{0}_d \in \mathbb{R}^d$. Moreover, $\alpha(M \cap V) = (\{\mathbf{0}_m\} \times \mathbb{R}^{d-m}) \cap U$ and, for every

$$\boldsymbol{x} = (x^1, ..., x^m, x^{m+1}, ...x^n, x^{n+1}, ..., x^d) \in \alpha(M \cap V),$$

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the first m coordinates $(x^1, ..., x^m)$ are equal to zero, Similarly, $\alpha(N \cap V) = (\{\mathbf{0}_n\} \times \mathbb{R}^{d-n}) \cap U$ and, for every $\boldsymbol{y} = (y^1, ..., y^m, y^{m+1}, ...y^n, x^{n+1}, ..., x^d) \in \alpha(N \cap V)$, the first n coordinates $(y^1, ..., y^m, y^{m+1}, ...y^n)$ are equal to zero.

For every $t < 0$, there exists a neighbourhood V_t of $c(t)$ in V such that $V_t \cap N = \emptyset$. Therefore, there exists an open neighbourhood U_t of $\mathbf{0}_d \in \mathbb{R}^d$ such that

$$\alpha(V_t) = ([0, \infty)^m \times \mathbb{R}^{d-m}) \cap U_t\,.$$

On the other hand, if $V_0 \subseteq V$ is a neigbourhood of $c(0)$ in S, then

$$\alpha(V_0) = ([0, \infty)^n \times \mathbb{R}^{d-n}) \cap U_0$$

for a neighbourhood U_0 of $\mathbf{0}_d \in \mathbb{R}^d$. But, $m \neq n$, so that, for $t < 0$, $\alpha(V_t)$ is not diffeomorphic to $\alpha(V_0)$. Since $\alpha : V \to W$ is a diffeomorphism, it follows that V_t is not diffeomorphic to V_0 for every $t < 0$. This contradicts the assumption that X generates a local one-parameter group of local diffeomorphisms of S. □

Proposition 7.5. *Let S be a manifold with corners and X a derivation on S such that, for every connected component M of the depth function stratification of S, the restriction $X_{|M}$ of X to M is a vector field on the manifold M. Then X is a vector field on S.*

Proof. In view of Proposition 7.4, it suffices to show that every integral curve of X originating at a connected component M of the depth function stratification of S, is contained in M. Suppose that there is an integral curve $c : I \to S$ of X, originating at x_0 in a connected component M of a stratum of the depth function stratification of S, such that $x_1 = c(t_1) \in N \in \overline{M} \backslash M$, where $t_1 = \min\{t \in I \mid t > 0$ and $c(t) \in M\}$ and N is a connected component of another another stratum in S. Since X is of class C^∞, it follows that

$$\lim_{t \to t_1-} X(c(t)) = \lim_{t \to t_{1-}} X_{|M}(c(t)) = X(c(t_1)) = X_{|N}(c(t_1)).$$

Suppose that $X(x_1) = 0$. The equation

$$\frac{d}{dt} f(c(t)) = (Xf)(c(t))$$

for every $f \in C^\infty(S)$ implies that,

$$\frac{dt}{df}(c(t)) = \frac{1}{(Xf)(c(t))}.$$

Hence, $X(x_1) = 0$ implies that $t \to \infty$ as $c(t) \to x_1$. Therefore, $x_1 = \lim_{t\to\infty} c(t)$ and it is not in the range of the curve c contrary to the previous assumption.

Suppose now that $X(x_1) = X_{|N}(x_1) \neq 0$. Note that $X_{|N}$ is a vector field on the manifold N. Hence, there exists an integral curve $c_N : I_N \to N$ of $X_{|N}$ originating at x_1. Consider a chart $\alpha : V \to W$ in $\mathfrak{B}$ such that V is a neighbourhood of $x_1 = c(t_1)$, and $W \subseteq \mathbb{R}^d_n = [0, \infty)^n \times \mathbb{R}^{d-n} \subseteq \mathbb{R}^d$ contains $\alpha(x_1)$. By Proposition 3.1.6 in [19], there exist a neighbourhood V_1 of $x_1 \in V$, $U \subseteq \mathbb{R}^d$ such that $\alpha_{|V_1} : V_1 \to W_1 = \alpha(V_1) \subseteq W$ is a diffeomorphism, and a vector field Y defined on an open set $U \subseteq \mathbb{R}^d$ containing W_1 such that

$$(\alpha_* X)_{|W_1} = Y_{|W_1}.$$

Since $c_N : I_N \to N$ of $X_{|N}$ originates at $x_1 \in V_1$, it follows that there is a connected subset $\tilde{I}_N$ of I_N containing 0 such that the restriction $\tilde{c}_N$ of $(\alpha \circ c_N)$ to $\tilde{I}_N$ has its range in W_1. The equation above implies that $\tilde{c}_N : \tilde{I}_N \to W_1$ is an integral curve of Y originating at $\alpha(x_1)$. On the other hand, $x_1 = c(t_1)$. Hence $c' : I' \to S : t \mapsto c(t - t_1)$ is an integral curve of X originating at $x_1 = c(t_1)$, where I' is I shitfed by t_1. Let $\tilde{I}'$ be a connected neighbourhood of $0 \in I'$ such that the restriction $\tilde{c}'$ of $\alpha \circ c'$ to $\tilde{I}'$ has its range in W_1. As before, $\tilde{c}' : \tilde{I}' \to W_1$ is an integral curve of Y originating at $\alpha(x_1)$. But Y is a vector

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field on an open subset of $\mathbb{R}^d$, and the germ of its integral curve passing through $\alpha(x_1)$ is unique up to parametrization. However, $\tilde{c}_N$ and $\tilde{c}'$ are distinct integral curves of Y such that $\tilde{c}_N(0) = \tilde{c}'(0) = \alpha(x_1)$. Therefore, we have a contradiction with hypothesis that $X(x_1) \neq 0$. □

Proposition 7.6. *Let S be a d-manifold with corners. For every vector $\mathrm{v}_0 \in T_{x_0}S$ tangent to the stratum of the depth function stratification of S that contains $x_0 = \tau(\mathrm{v}_0)$, there exists a vector field X on S such extending v_0, that is $X(x_0) = \mathrm{v}_0$.*

Proof. If $\mathrm{v}_0 = 0$, then it extends to the vector field $X = 0$ on S. That is $Xf = 0$ for every $f \in C^\infty(S)$.

If $\mathrm{v}_0 \neq 0$, consider a chart $\alpha : V_\alpha \to W_\alpha \subseteq \mathbb{R}^d_n = [0, \infty)^n \times \mathbb{R}^{d-n} \subseteq \mathbb{R}^d$ on the manifold with corners S such that V_α is a neighbourhood of x in S. and $\mathbb{R}^d$. If depth${}_S x = n$ then, without loss of generality, we may assume that

$$\begin{aligned}\alpha(x_0) &= \boldsymbol{x}_0 = (x_0^1, ..., x_0^d), \text{ where } x_0^1 = ... = x_0^n = 0 \text{ and } x_0^{n+1} = ... = x_0^d = 1,\\ \alpha(V_\alpha \cap S^n) &= \{(x^1, ..., x^d) \in \mathbb{R}^d \mid x^1 = 0, ..., x^n = 0, (x^{n+1}, ..., x^d) \in U \subseteq \mathbb{R}^{d-n}\}\\ &= \{\mathbf{0}_n\} \times U \subset \mathbb{R}^d,\end{aligned}$$

where U is open in $\mathbb{R}^{d-n}$. For every $m \in 0, 1, ..., d$, the point $(x^1, ..., x^d) \in \alpha(V_\alpha \cap S^m)$ if and only if exactly m of the coordinates $x^1, ..., x^d$ are zero. A vector $\boldsymbol{v} = (v^1, ..., v^d)$ is tangent to $\alpha(V_\alpha \cap S^m)$ at $(x^1, ..., x^d) \in \alpha(V_\alpha \cap S^m)$ if and only if, for every $i = 1, ..., d$, $x^i = 0$ implies $v^i = 0$. Since $\alpha : V_\alpha \to W_\alpha \subseteq R^d$ is a diffeomorphism, and the definition of the depth function is independent of the chart, it follows that $v \in T_x S$ is tangent at x to S^m if and only if $\boldsymbol{v} = T\alpha(v)$ is tangent to $\alpha(V_\alpha \cap S^m)$ at the point $\alpha(x) = (x^1, ..., x^d)$.

Thus, for $x \in S^m \cap V_\alpha$ a vector $\mathrm{v} \in T_x S$ is in $T_x S^m$ if and only if $x^i v^i = 0$ for every $i = 1, ..., d$, where $(x^1, ..., x^d)$ are coordinates of $\alpha(x)$ in $\mathbb{R}^d$ and $(v^1, ..., v^d)$ are components of $T\alpha(\mathrm{v}) \in T_{\alpha(x)}\mathbb{R}^d \cong T_{\alpha(x_0)}\mathbb{R}^d$.

Since U is open in $\mathbb{R}^{d-n}$, there exists $\epsilon \in (0, \frac{1}{2})$ such that the set

$$\begin{aligned}W_\epsilon = \{\boldsymbol{x} = (x^1, ..., x^d) \in \mathbb{R}^d \quad &| \quad -\epsilon < x^i < \epsilon \text{ for } i = 1, ..., n\\ \text{and } 1 - \epsilon \quad &< \quad x^j < 1 + \epsilon \text{ for } j = n+1, ..., d\}\end{aligned}$$

is an open neighbourhood of $\alpha(x_0) = \boldsymbol{x}_0$ in $W_\alpha \subseteq \mathbb{R}^d$ and $\overline{W}_\epsilon \subseteq W_\alpha$. It follows from the discussion above that $W_\epsilon \subseteq \cup_{m=0}^n \alpha(V_\alpha \cap S^m)$.

Let $\boldsymbol{v}_0 = (v_0^1, ..., v_0^d) = T\alpha(\mathrm{v}_0) \in \mathbb{R}^d \cong T_{\alpha(x_0)}\mathbb{R}^d$. The assumptions about the chart $\alpha : V_\alpha \to W_\alpha$, made above, imply that $v_0^{n+1} = ... = v_0^d = 0$. By construction, for every $\boldsymbol{x} = (x^1, ..., x^d) \in W_\epsilon$, the coordinates $x^{n+1}, ..., x^d$ do not vanish, and some of the coordinates $x^1, ..., x^n$ may also be non-zero. Therefore, for every $\boldsymbol{x} = (x^1, ..., x^d) \in W_\epsilon$, a vector $\boldsymbol{v} = (v^1, ..., v^d) \in T_{\boldsymbol{x}}\mathbb{R}^d \cong \mathbb{R}^d$ such that $v^1 = ... = v^n = 0$ is tangent to $\alpha(V_\alpha \cap S^m)$ for every $m \leq n$. On the other hand, for every $m \geq n$, $W_\epsilon \cap \alpha(V_\alpha \cap S^m) = \emptyset$.

Choose a function $f \in C^\infty(\mathbb{R}^d)$ such that $f(\boldsymbol{x}_0) = 1$ and $f(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \notin W_\epsilon$, and consider a vector field Y on $\mathbb{R}^d$ given by

$$Y(\boldsymbol{x}) = f(\boldsymbol{x})\frac{\partial}{\partial x^{n+1}} + f(\boldsymbol{x})\frac{\partial}{\partial x^{n+2}} + ... + f(\boldsymbol{x})\frac{\partial}{\partial x^d}$$

for every $\boldsymbol{x} \in \mathbb{R}^d$. Since $f \in C^\infty(\mathbb{R}^d)$, it follows that integral curves of Y have open domains. The assumption that $f(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \notin W_\epsilon$ imply that that the integral curves of Y originating in $W_\alpha \supseteq \overline{W}_\epsilon$ are contained in W_α. Therefore, the restriction $Y_{|W_\alpha}$ of Y to W_α is a vector field on W_α. The the push-forward $\alpha_*^{-1} Y_{|W_\alpha}$ be the diffeomorphism $\alpha^{-1} : W_\alpha \to V_\alpha$ is a vector field on V_α, which can be extended to a vector field $X \in \mathfrak{X}(S)$ vanishing outside $\alpha^{-1}(W_\epsilon) \subseteq V_\alpha$. Since $f(\alpha(x_0)) = 1$, it follows that $X(x_0) = \mathrm{v}_0$, which completes the proof. □

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Corollary 7.7. *It follows from the the above results that connected components of strata of the depth function stratification of the manifold with corners S are orbits of the Lie algebra $\mathfrak{X}(S)$ of all vector fields on S. Hence the depth function stratification of S is given by the partition $\mathfrak{M}(S)$ of S by orbits of $\mathfrak{X}(S)$.*

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