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Convergence Criteria for Fixed Point Problems and Differential Equations

Mircea Sofonea [1,‡,*] and Domingo A. Tarzia [2,‡]

1 Laboratoire de Mathématiques et Physique, University of Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France
2 Departamento de Matemática, FCE, Universidad Austral, Paraguay 1950, S2000FZF Rosario, Argentina and CONICET, Argentina; DTarzia@austral.edu.ar
* Correspondence: sofonea@univ-perp.fr
‡ These authors contributed equally to this work.

Abstract: We consider a Cauchy problem for differential equations in a Hilbert space X. The problem is stated in a time interval I, which can be finite or infinite. We use a fixed point argument for history-dependent operators to prove the unique solvability of the problem. Then, we state and prove convergence criteria for both a general fixed point problem and the corresponding Cauchy problem. These criteria provide necessary and sufficient conditions on a sequence $\{u_n\}$ which guarantee its convergence to the solution of the corresponding problem, in the space of both continuous and continuously differentiable functions. We then specify our results in the study of a particular differential equation governed by two nonlinear operators. Finally, we provide an application in viscoelasticity and give a mechanical interpretation of the corresponding convergence result.

Keywords: differential equation; Cauchy problem; fixed point; history-dependent operator; convergence criterion; viscoelastic constitutive law

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1. Introduction

Convergence results represent an important topic in Functional Analysis, Numerical Analysis, Differential and Partial Differential Equations Theory. They are important in the study of mathematical models which arise in Mechanics, Physics and Engineering Sciences, as well. Some elementary examples are the following: the convergence of the solution of a penalty problem to the solution of the original problem as the penalty parameter converges, the convergence of the discrete solution to the solution of the continuous problem as the time step or the discretization parameter converges to zero, the convergence of the solution of a viscoelastic problem to the solution of an elastic problem as the viscosity goes to zero, the convergence of the solution of a frictional problem to the solution of a frictionless problem as the coefficient of friction converges to zero.

For all these reasons, a considerable effort was done to obtain convergence results in the study of various mathematical problems including nonlinear equations, inequality problems, inclusions, fixed point problems, optimization problems, among others. Note that, in most of the cases, such results provide sufficient conditions which guarantee the convergence of a given sequence $\{u_n\}$ to the solution of the corresponding problem, denoted in what follows by $\mathcal{P}$. They do not describe all the sequences which have this property. Therefore, we naturally arrive to consider the following problem, associated to $\mathcal{P}$.

Problem $\mathcal{Q}_\mathcal{P}$. *Given a Problem $\mathcal{P}$ which has a unique solution u in a metric space Y, describe the convergence of a sequence $\{u_n\} \subset Y$ to the solution u. In the words, provide necessary and sufficient conditions for the convergence $u_n \to u$ in Y, i.e., provide a* convergence criterion.

Note that Problem $\mathcal{Q}_\mathcal{P}$ represents a major issue in the study of convegence results. Its solution depends on the structure of the original problem $\mathcal{P}$ and cannot be provided in this general framework. Results in solving Problem $\mathcal{Q}_\mathcal{P}$ have been obtained in [15], in the particular case when $\mathcal{P}$ is a variational inequality, a fixed point problem and a minimization problem.

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In this current paper we continue our research in [15] with the case when $\mathcal{P}$ is a Cauchy problem of the form

$$\dot{u}(t) = F(t, u(t))) \quad \forall\, t \in I, \tag{1}$$
$$u(0) = u_0 \tag{2}$$

Here and everywhere below X represent either a Banach space endowed with the norm $\|\cdot\|_X$ or a Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, $I \subset \mathbb{R}$ is an interval of time, $F : I \times X \to X$ and $u_0 \in X$ is a given initial data. Moreover, the dot above represents the derivative with respect to the time variable. We consider both the case when I is a bounded interval of the form $I = [0, T]$ with $T > 0$ and the case when $I = \mathbb{R}_+$ and, when no specification is made, I will represent whichever of these intervals. Under appropriate assumptions which guarantee that problem (1)–(2) has a unique solution $u \in C^1(I; X)$, our aim is to indicate necessary and sufficient conditions which garantee the convergence of a given sequence $\{u_n\} \in C^1(I; X)$ to the solution u, both in the spaces $C(I; X)$ and $C^1(I; X)$.

Note that the study of problem (1)–(2) is related to the study of the fixed problem

$$u(t) = \Lambda u(t) \qquad \forall\, t \in I, \tag{3}$$

where $\Lambda : C(I; X) \to C(I; X)$ is an operator which will be specified later. For this reason, we start with convergence results concerning this auxiliary fixed point problem. To conclude, in this paper we shall provide an answer to Problem $\mathcal{Q}_{\mathcal{P}}$ in the case when $\mathcal{P}$ both the Cauchy problem (1)–(2) and the fixed point problem (3) while the space Y is the space $C^1(I; X)$ and the space $C(I; X)$, respectively. Our study is motivated by possible applications in Analysis and Solid Mechanics.

The rest of the manuscrit is structured as follows. In Section 2 we introduce some preliminary material. Then, in Section 3 we state and prove a convergence criterion in the study of the fixed problem (3). Next, in Section 4 we state and prove two different convergence criteria in the study of the Cauchy problem (1)–(2), in the space of continuous and continuously differentiable functions, respectively. We use these results in Section 5, in which we consider a particular form of the differential equation (1), governed by two nonlinear operators. Finly, in Section 6 we provide an applications of the abstract results in Section 5 in the study of a differential equations arising in viscoelasticity.

2. Preliminaries

In this section we introduce two spaces of functions and the class of history-dependent operators. Then, we state two elementary inequalities which will be used repeatedly in the next sections. We precise that everywhere in this manuscript m will denote a given positive integer and the limits are considered as $n \to \infty$, even if we do not mention it explicitely. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \le \varepsilon_n \to 0$. We extend this notation to a sequence $\{\varepsilon_n^m\} \subset \mathbb{R}_+$ (with m given) which converges to zero and, therefore, we write $0 \le \varepsilon_n^m \to 0$.

Space of continuously and continuously differentiable functions. We start with some properties of the spaces $C(I; X)$ and $C^1(I; X)$ defined by

$$C(I; X) = \{\, v \colon I \to X \mid v \text{ is continuous} \,\},$$
$$C^1(I; X) = \{\, v \colon I \to X \mid v \in C(I; X) \text{ and } \dot{v} \in C(I; X) \,\}.$$

On occasion, these spaces will be denoted by $C([0, T]; X)$ and $C^1([0, T]; X)$, respectively, if $I = [0, T]$. The space $C([0, T]; X)$ will be equipped with the norm of the uniform convergence, that is

$$\|v\|_{C([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X. \tag{4}$$

It is well known that, endowed with this norm, this space is a Banach space. Moreover, the space $C^1([0,T];X)$ is a Banach space with the norm

$$\|v\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X + \max_{t\in[0,T]} \|\dot{v}(t)\|_X. \tag{5}$$

We now consider the case $I = \mathbb{R}_+$. It is well known that, if X is a Banach space, then $C(\mathbb{R}_+;X)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The details can be found in [3,7]. Here, we restrict ourselves to recall that the convergence of a sequence $\{v_n\} \subset C(\mathbb{R}_+;X)$ to the element $v \in C(\mathbb{R}_+;X)$ is characterized by the following equivalence:

$$\begin{cases} v_n \to v \ \text{ in } C(\mathbb{R}_+;X) \iff \\ \max_{t\in[0,m]} \|v_n(t) - v(t)\| \to 0 \ \text{ for all } \ m \in \mathbb{N}. \end{cases} \tag{6}$$

In other words, the sequence $\{v_n\}$ converges to the element v in the space $C(\mathbb{R}_+;X)$ if and only if it converges to v in the space $C([0,m];X)$ for any $m \in \mathbb{N}$. Next, the convergence of a sequence $\{v_n\}$ to the element v, in the space $C^1(\mathbb{R}_+;X)$, can be defined as follows:

$$\begin{cases} v_n \to v \quad \text{in } C^1(\mathbb{R}_+;X) \iff \\ \max_{t\in[0,m]} \|v_n(t) - v(t)\|_X \to 0 \ \text{ and} \\ \max_{t\in[0,m]} \|\dot{v}_n(r) - \dot{v}(t)\|_X \to 0, \ \text{ for all } \ m \in \mathbb{N}. \end{cases} \tag{7}$$

The equivalences (6) and (7) will be used repeatedly in the next sections, in order to prove various convergence results when working on the framework of an unbounded interval of time.

Using the properties of the integral it is easy to see that if $f \in C(I;X)$ then the function $g\colon I \to X$ given by

$$g(t) = \int_0^t f(s)\,ds \ \text{ for all } \ t \in I$$

belongs to $C^1(I;X)$ and, moreover, $\dot{g} = f$. In addition, we recall that for a function $v \in C^1(I;X)$ the following equality holds

$$v(t) = \int_0^t \dot{v}(s)\,ds + v(0) \ \text{ for all } \ t \in I. \tag{8}$$

Finally, we mention that, when no confusion arises, we shall use the notation 0_X for the zero element in both spaces X, $C(I;X)$, $C^1(I;X)$, $C([0,m];X)$ and $C^1([0,m];X)$, for any $m \in \mathbb{N}$.

History-dependent operators. We now introduce a class of operators defined on the space of continuous functions $C(I;X)$.

Definition 1. *An operator* $\Lambda\colon C(I;X) \to C(I;X)$ *is called* history-dependent *if:*

a) $I = [0,T]$ *and there exists* $L > 0$ *such that*

$$\|\Lambda u_1(t) - \Lambda u_2(t)\|_X \le L \int_0^t \|u_1(s) - u_2(s)\|_X\,ds \tag{9}$$
$$\text{for all } \ u_1,\, u_2 \in C([0,T];X), \ \ t \in [0,T].$$

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b) $I = \mathbb{R}_+$ *and for any* $m \in \mathbb{N}$ *there exists* $L_m > 0$ *such that*

$$\|\Lambda u_1(t) - \Lambda u_2(t)\|_X \le L_m \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \qquad (10)$$
$$\text{for all } \ u_1,\, u_2 \in C(\mathbb{R}_+; X),\ t \in [0, m].$$

Note that here and below, when no confusion arises, we use the shorthand notation $\Lambda u(t)$ to represent the value of the function Λu at the point t, i.e., $\Lambda u(t) = (\Lambda u)(t)$, for all $t \in I$. Also, we recall that the term "history-dependent operator" was introduced in [12] and since it has been used in many papers, see [8,9,11,13], for instance. Exemples of history dependent operators will be provided in the next sections of this manuscript.

Finally, using Definition 1 and the convergences (4) and (6) it is easy to see that any history operator $\Lambda : C(I; X) \to C(I; X)$ is continuous, that is

$$u_n \to u \qquad \text{in } \ C(I; X) \qquad \Longrightarrow \quad \Lambda u_n \to \Lambda u \qquad \text{in } \ C(I; X). \qquad (11)$$

An important property of history-dependent operators is the following fixed point property, proved in [10,14].

Theorem 1. *Let* X *be a Banach space and* $\Lambda\colon C(I; X) \to C(I; X)$ *be a history-dependent operator. Then,* Λ *has a unique fixed point, i.e., there exists a unique element* $u \in C(I; X)$ *such that* $\Lambda u = u$.

Theorem 1 is useful to prove the unique the solvability of various classes of nonlinear equations and variational inequalities. An example is provided by the following result which will be used in Section 6 in this paper.

Theorem 2. *Let* X *be a Hilbert space,* $A : X \to X$ *a strongly monotone Lipschitz continuous operator and* $\Lambda\colon C(I; X) \to C(I; X)$ *a history-dependent operator. Then, for any* $f \in C(I; X)$ *there exists a unique function* $u \in C(I; X)$ *such that*

$$Au(t) + \Lambda u(t) = f(t) \qquad \forall\, t \in I. \qquad (12)$$

A proof of Theorem 2 can be found in [13], based on the fixed point result provided by Theorem 1.

Two elementary inequalities. We now recall two elementay inequalities which will be used in many places below. To this end, we use the notation $C(I)$ for the space of real-valued continuous functions defined on the interval I, that is, $C(I) = C(I; \mathbb{R})$. The first inequality we recall is the well-known Gronwall inequality and is stated as fallows.

Lemma 1. *Let* $f, g \in C(I)$ *and assume that there exists* $c > 0$ *such that*

$$f(t) \le g(t) + c \int_0^t f(s)\, ds \quad \text{for all } \ t \in I. \qquad (13)$$

Then,

$$f(t) \le g(t) + c \int_0^t g(s)\, e^{c\,(t-s)}\, ds \quad \text{for all } \ t \in I. \qquad (14)$$

Moreover, if g *is nondecreasing, then*

$$f(t) \le g(t)\, e^{c\,t} \quad \text{for all } \ t \in I.$$

A proof of Lemma 1 can be found in [13, p.60] and, therefore, we skip it.

The second inequality we need is the following.

Lemma 2. *Let X be a Hilbert space, $x \in X$ and $\varepsilon \geq 0$. Then the follwing equivalence holds:*

$$\|x\|_X \leq \varepsilon \quad \Longleftrightarrow \quad (x,v)_X + \varepsilon\|v\|_X \geq 0 \quad \forall\, v \in X. \tag{15}$$

Proof. Assume that $\|x\|_X \leq \varepsilon$ and $v \in X$. Then, it is easy to see that

$$(x,v)_X + \varepsilon\|v\|_X \geq -\|x\|_X\|v\|_X + \varepsilon\|v\|_X = (\varepsilon - \|x\|_X)\|v\|_X$$

and, therefore $(x,v)_X + \varepsilon\|v\|_X \geq 0$. Conversely, assume that $(x,v)_X + \varepsilon\|v\|_X \geq 0$ for any $v \in X$. We take $v = -x$ in this inequality to find that $-(x,x)_X + \varepsilon\|x\|_X \geq 0$ which implies that $\|x\|_X^2 \leq \varepsilon\|x\|_X$. We deduce from here that $\|x\|_X \leq \varepsilon$, which concludes the proof. □

3. The fixed point problem

In this section we state and prove a convergence criterion for the solution of the fixed point problem (3). To this end, everywhere in this section we assume that X is a Hibert space and, under the assumption of Theorem 1, we denote by $u \in C(I;X)$ the fixed point of operator Λ. Moreover, given an arbitrary sequence $\{u_n\} \subset C(I;X)$ we consider the following statements:

$$u_n \to u \qquad \text{in } \ C(I;X). \tag{16}$$

$$u_n - \Lambda u_n \to 0_X \qquad \text{in } \ C(I;X). \tag{17}$$

$$\begin{cases} I = [0;T] \text{ and there exists } 0 \leq \varepsilon_n \to 0 \text{ such that} \\ (u_n(t),v)_X + \varepsilon_n\|v\|_X \geq (\Lambda u_n(t),v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I. \end{cases} \tag{18}$$

$$\begin{cases} I = \mathbb{R}_+ \text{ and for any } m \in \mathbb{N} \text{ there exists } 0 \leq \varepsilon_n^m \to 0 \text{ such that} \\ (u_n(t),v)_X + \varepsilon_n^m\|v\|_X \geq (\Lambda u_n(t),v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in [0,m]. \end{cases} \tag{19}$$

Our main result in this section is the following.

Theorem 3. *Let X be a Hilbert space, $T > 0$ and $\Lambda : C(I;X)$ a history-dependent operator.*

a) *If $I = [0;T]$ then the statements (16), (17) and (18) are equivalent.*

b) *If $I = \mathbb{R}_+$ then the statements (16), (17) and (19) are equivalent.*

Proof. a) We start with the case $I = [0,T]$. Assume that (16) holds. Then, using (11) it is easy to see that $u_n - \Lambda u_n \to u - \Lambda u$ in $C(I;X)$ and, since u is the solution of the fixed point problem (3), we deduce that (17) holds.

Next, assume that (17) holds which shows that

$$\max_{s\in[0,T]} \|u_n(s) - \Lambda u_n(s)\|_X \to 0 \quad \text{as } \ n \to \infty. \tag{20}$$

For which $n \in \mathbb{N}$ denote

$$\varepsilon_n = \max_{s\in[0,T]} \|u_n(s) - \Lambda u_n(s)\|_X. \tag{21}$$

Then, (20) shows that $0 \leq \varepsilon_n \to 0$ and, moreover, definition (21) implies that for any $t \in I$ we have

$$\|u_n(t) - \Lambda u_n(t)\|_X \leq \varepsilon_n. \tag{22}$$

We now use inequality (22) and Lemma 2 to see that condition (18) holds.

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Finally, assume that (18) holds. Let $n \in \mathbb{N}$ and $t \in [0,T]$. We take $v = u(t) - u_n(t)$ in this inequality to see that

$$(u_n(t), u(t) - u_n(t))_X + \varepsilon_n \|u(t) - u_n(t)\|_X \geq (\Lambda u_n(t), u(t) - u_n(t))_X$$

and, using equality $u(t) = \Lambda u(t)$, we find that

$$(u_n(t) - u(t), u(t) - u_n(t))_X + \varepsilon_n \|u(t) - u_n(t)\|_X \geq (\Lambda u_n(t) - \Lambda u(t), u(t) - u_n(t))_X.$$

Thus,

$$\|u_n(t) - u(t)\|_X^2 \leq \varepsilon_n \|u_n(t) - u(t)\|_X + \|\Lambda u_n(t) - \Lambda u(t)\|_X \|u_n(t) - u(t)\|_X$$

and, therefore,

$$\|u_n(t) - u(t)\|_X \leq \varepsilon_n + \|\Lambda u_n(t) - \Lambda u(t)\|_X.$$

We now use inequality (9) to see that

$$\|u_n(t) - u(t)\|_X \leq \varepsilon_n + L \int_0^t \|u_n(s) - u(s)\|_X \, ds$$

and, employing the Gronwall argument provided by Lemma 1, we find that

$$\|u_n(t) - u(t)\|_X \leq \varepsilon_n e^{Lt}.$$

We now use the convergence $\varepsilon_n \to 0$ and inequality $t \leq T$ to see that

$$\max_{t \in [0,T]} \|u_n(t) - u(t)\|_X \to 0 \quad \text{as } n \to \infty,$$

which implies that (16) holds.

To conclude, we proved the implications (16) $\Longrightarrow$ (17) $\Longrightarrow$ (18) $\Longrightarrow$ (16) which shows the equivalence of the statements (16), (17) and (18).

b) We continue with the case $I = \mathbb{R}_+$. To this end we fix $m \in \mathbb{N}$ and we use the first part of the theorem with $T = m$, combined with the remark that the quantity ε_n defined in (21) depends on T and, therefore, since $T = m$, we denote it by ε_n^m. We deduce from here the equivalences of the following statements:

$$u_n \to u \qquad \text{in } C([0,m]; X). \tag{23}$$

$$u_n - \Lambda u_n \to 0_X \qquad \text{in } C([0,m]; X). \tag{24}$$

$$\begin{cases} \text{there exists } 0 \leq \varepsilon_n^m \to 0 \text{ such that} \\ (u_n(t), v)_X + \varepsilon_n^m \|v\|_X \geq (\Lambda u_n(t), v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in [0,m]. \end{cases} \tag{25}$$

Recall that the equivalence of these statements is valid for any $m \in \mathbb{N}$. We now use (6) to see that the convergences (23) and (24) can be replaced by the convergences (16) and (17), respectively, which concludes the proof. □

We remark that Theorem 3 provides an answer to Problem $\mathcal{Q}_\mathcal{P}$ in the particular case when Problem $\mathcal{P}$ is the fixed point problem (3). Indeed, it provides a convergence criterion to the solution of this problem, both in the case $I = [0,T]$ and $I = \mathbb{R}_+$.

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We end this section with the remark that in the case $I = \mathbb{R}_+$, we cannot skip the dependence on m for the constants ε_n^m which appear in (25). More precisely, we claim that in the case $I = \mathbb{R}_+$, condition

$$\begin{cases} \text{there exists } 0 \leq \varepsilon_n \to 0 \text{ such that} \\ (u_n(t), v)_X + \varepsilon_n \|v\|_X \geq (\Lambda u_n(t), v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I \end{cases} \tag{26}$$

is not equivalent with the convergence (16). The proof of this claim follows from the following example.

Example 1. *Let $X = \mathbb{R}$, $I = \mathbb{R}_+$ and let $\Lambda : C(I) \to C(I)$ be the operator defined by*

$$\Lambda u(t) = \int_0^t u(s)\, ds \tag{27}$$

for all $u \in C(I)$, $t \in I$ and $n \in \mathbb{N}$. Then, it is easy to see that Λ is a history-dependent operator and its unique fixed point is the function $u(t) = 0$ for all $t \in I$. Consider now the function

$$u_n(t) = \frac{1}{n}\, e^{\frac{n+1}{n} t} \qquad \forall\, n \in \mathbb{N},\ t \in \mathbb{R}_+. \tag{28}$$

Then, it is easy to see that

$$\max_{t \in [0,m]} |u_n(t)| = \frac{1}{n}\, e^{\frac{n+1}{n} m} \leq \frac{1}{n} e^{2m} \to 0 \quad \text{as } n \to \infty,\ \forall\, m \in \mathbb{N}$$

and, therefore, (6) shows that $u_n \to 0$ in the space $C(I)$. Nevertheless, we shall prove that condition (26) does not hold. Indeed, arguing by contradiction, assume that the sequence $\{u_n\}$ satisfies this condition. Then, there exists a sequence $0 \leq \varepsilon_n \to 0$ such that the inequality in (26) holds and, using Lemma 2, we deduce that

$$|\Lambda u_n(t) - u_n(t)| \leq \varepsilon_n \qquad \forall\, n \in \mathbb{N},\ t \in \mathbb{R}_+. \tag{29}$$

Using now (27)–(29), we deduce that

$$\frac{1}{n(n+1)}\, e^{\frac{n+1}{n} t} + \frac{1}{n+1} \leq \varepsilon_n \qquad \forall\, n \in \mathbb{N},\ t \in \mathbb{R}_+.$$

We now take $t = n^2$ in the previous inequality, then we pass to the limit as $n \to \infty$ and arrive to a contradiction. We conclude from here that condition (26) does not hold.

4. The Cauchy Problem

We now proceed with the study of the Cauchy problem (1)–(2) and, to this end, we consider the following assumptions.

$$\begin{cases} \text{(a) } F : I \times X \to X. \\ \text{(b) The mapping } t \mapsto F(t, u) : I \to X \text{ is continuous} \\ \qquad \text{for all } \ u \in X. \\ \text{(c) If } I = [0, T] \text{ then there exists } L_F > 0 \text{ such that} \\ \qquad \|F(t, u_1) - F(t, u_2)\|_X \leq L_F \|u_1 - u_2\|_X \\ \qquad \text{for all } \ u_1, u_2 \in X,\ t \in [0, T]. \\ \text{(d) If } I = \mathbb{R}_+ \text{ then for any } m \in \mathbb{N} \text{ there exists } L_F^m > 0 \text{ such that} \\ \qquad \|F(t, u_1) - F(t, u_2)\|_X \leq L_F^m \|u_1 - u_2\|_X \\ \qquad \text{for all } \ u_1, u_2 \in X,\ t \in [0, m]. \end{cases} \tag{30}$$

$$u_0 \in X. \tag{31}$$

Our first result in this section is the following.

Theorem 4. *Let X be a Banach space, $T > 0$ and assume* (30)(a), (b) *and* (31). *Then, problem* (1)–(2) *has a unique solution $u \in C^1(I;X)$ in the following two cases:*

a) *$I = [0;T]$ and F satisfy condition* (30)(c);

b) *$I = \mathbb{R}_+$ and F satisfy condition* (30)(d).

Proof. Let $u_0 \in X$ and let $\Lambda\colon C(I;X) \to C(I;X)$ be the operator defined by

$$\Lambda u(t) = \int_0^t F(s,u(s))\,ds + u_0 \ \text{ for all } \ u \in C(I;X),\ t \in I. \tag{32}$$

Note that assumptions (30) (a),(b) imply that for any function $u \in C(I;X)$, the function $t \mapsto F(t,u(t))$ is continuous on I and, therefore, the operator Λ is well defined. In addition, using condition (30) (c) it is easy to see that in the case when $I = [0,T]$, this operator satisfies inequality (9) and, therefore, Definition 1 a) guarantees that it is a history-dependent operator. Moreover, if $I = \mathbb{R}_+$, using condition (30) (d) it follows that the operator Λ satisfies inequality (10) and, therefore, Definition 1 b) guarantees that it is a history-dependent operator, too. Therefore, using Theorem 1 we deduce that there exists a unique function $u \in C(I;X)$ such that

$$u(t) = \Lambda(t) \ \text{ for all } \ t \in I. \tag{33}$$

Hence, using (33) and (32) we deduce the existence of a unique function $u \in C(I;X)$ such that

$$u(t) = \int_0^t F(s,u(s))\,ds + u_0 \ \text{ for all } \ t \in I. \tag{34}$$

On the other hand, it is easy to see that a function $u \in C^1(I;X)$ is a solution to the Cauchy problem (1)–(2) if and only if $u \in C(I;X)$ and (34) holds. We combine this equivalence with the unique solvability of the integral equation (34) to end the proof. □

The proof of Theorem 4 establish a link between the Cauchy problem (1)–(2) and the fixed point problem (3) with Λ given by (32). Based on this link, in the case when X il a Hilbert space, we can easily deduce a convergence criterion to the solution of the Cauchy problem (1)–(2). More precisely, we write the statements (16)–(19) in the particular case of the operator (32):

$$u_n \to u \qquad \text{in } \ C(I;X). \tag{35}$$

$$u_n - \int_0^t F(s,u_n(s))\,ds - u_0 \to 0 \qquad \text{in } \ C(I;X). \tag{36}$$

$$\begin{cases} I = [0;T] \text{ and there exists } 0 \le \varepsilon_n \to 0 \text{ such that} \\ (u_n(t),v)_X + \varepsilon_n \|v\|_X \ge \Big(\int_0^t F(s,u_n(s))\,ds + u_0, v\Big)_X \\ \qquad\qquad \forall\, v \in V,\ n \in \mathbb{N},\ t \in I. \end{cases} \tag{37}$$

$$\begin{cases} I = \mathbb{R}_+ \text{ and for any } m \in \mathbb{N} \text{ there exists } 0 \le \varepsilon_n^m \to 0 \text{ such that} \\ (u_n(t),v)_X + \varepsilon_n^m \|v\|_X \ge \Big(\int_0^t F(s,u(s))\,ds + u_0, v\Big)_X \\ \qquad\qquad \forall\, v \in V,\ n \in \mathbb{N},\ t \in [0,m]. \end{cases} \tag{38}$$

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Then, using the convergence criterion provided by Theorem 3 we deduce the following result.

Corollary 1. *Let X be a Hilbert space, $T > 0$ and assume* (30)(a), (b) *and* (31).

a) *If $I = [0; T]$ and* (30)(c) *holds then the statements* (35), (36) *and* (37) *are equivalent.*

b) *If $I = \mathbb{R}_+$ and* (30)(d) *holds then the statements* (35), (36) *and* (38) *are equivalent.*

Note that Corollary 1 provides a convergence criterion for the solution of the Cauchy problem (1)-(2), in the space $C(I; X)$. Nevertheless, recall that the solution u of the problem belongs to the space $C^1(I; X)$. The example below shows that this criterion is not valid in the space $C^1(I; X)$.

Example 2. *Let X be a Hilbert space, $I = [0, T]$, $f \in X$, $f \neq 0_X$ and consider the Cauchy problem of finding a function $u : I \to X$ such that*

$$\dot{u}(t) + u(t) = f \qquad \forall\, t \in [0, T], \quad u(0) = f. \tag{39}$$

Then, it is easy to see that this problem is of the form (1)–(2) *with $F(t, u) = f - u$ for each $t \in I$, $u \in X$ and $u_0 = f$.It is easy to see that the assumptions of Corollary* 1 a) *are satisfied and, moreover, the solution of this problem is given by*

$$u(t) = f \qquad \forall\, t \in I.$$

Consider now the sequence $\{u_n\} \subset C^1(I; X)$ defined by

$$u_n(t) = \Big(1 + \frac{1}{n}\sin nt\Big) f \qquad \forall\, t \in I.$$

Then, it is easy to see that conditions (35) *and* (36) *are satisfied. Nevertheless, the convrgence $u_n \to u$ in $C^1([0, T]; X)$ does not hold since, for instance, the sequence of derivatives $\{\dot{u}_n\}$ does not converge to zero in the space $C([0, T]; X)$.*

In order to provide a convergence criterion to the solution the Cauchy problem (1)–(2) in the space $C^1(I; X)$ we consider the following statements.

$$u_n \to u \qquad \text{in } \; C^1(I; X). \tag{40}$$

$$\dot{u}_n - F(\cdot, u_n) \to 0_X \quad \text{in } \; C(I; X) \quad \text{ and } \quad u_n(0) \to u_0 \quad \text{in } \; X. \tag{41}$$

$$\begin{cases} I = [0; T] \text{ and there exists } 0 \le \varepsilon_n \to 0 \text{ such that} \\ (\dot{u}_n(t), v)_X + \varepsilon_n \|v\|_X \ge (F(t, u_n(t)), v)_X \quad \forall\, v \in X,\, n \in \mathbb{N},\, t \in I, \\ \|u_n(0) - u_0\|_X \le \varepsilon_n \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{42}$$

$$\begin{cases} I = \mathbb{R}_+ \text{ and for any } m \in \mathbb{N} \text{ there exists } 0 \le \varepsilon_n^m \to 0 \text{ such that} \\ (\dot{u}_n(t), v)_X + \varepsilon_n^m \|v\|_X \ge (F(t, u_n(t)), v)_X \quad \forall\, v \in X,\, n \in \mathbb{N},\, t \in [0, m], \\ \|u_n(0) - u_0\|_X \le \varepsilon_n^m \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{43}$$

Our next result in this section is the following.

Theorem 5. *Let X be a Hilbert space, $T > 0$ and assume* (30)(a), (b) *and* (31).

a) *If $I = [0; T]$ and* (30)(c) *holds then the statements* (40), (41) *and* (42) *are equivalent.*

b) *If $I = \mathbb{R}_+$ and* (30)(d) *holds then the statements* (40), (41) *and* (43) *are equivalent.*

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Proof. a) We start with the case $I = [0, T]$. Assume that (40) holds. Then, using (30) (a), (b), (c) it is easy to see that $\dot{u}_n - F(\cdot, u_n) \to \dot{u} - F(\cdot, u)$ in $C(I; X)$ and, since u is the solution of the Cauchy problem problem (3), we deduce that

$$\dot{u}_n - F(\cdot, u_n) \to 0 \qquad \text{in } \ C(I; X).$$

In addition, $u_n(0) \to u(0)$ in X and, since $u(0) = u_0$ we find that $u_n(0) \to u_0$ in X. It follows from here that (41) holds.

Next, assume that (41) holds which shows that

$$\max_{s\in[0,T]} \|\dot{u}_n(s) - F(s, u_n(s))\|_X \to 0 \quad \text{as} \ \ n \to \infty. \tag{44}$$

For which $n \in \mathbb{N}$ denote

$$\theta_n = \max_{s\in[0,T]} \|\dot{u}_n(s) - F(s, u_n(s))\|_X. \tag{45}$$

Then, (44) shows that $0 \le \theta_n \to 0$ and, moreover, definition (45) implies that for any $t \in I$ we have

$$\|\dot{u}_n(t) - F(t, u_n(t))\|_X \le \theta_n. \tag{46}$$

We now use inequality (46) and Lemma 2 to see that

$$(\dot{u}_n(t), v)_X + \theta_n \|v\|_X \ge (F(t, u_n(t)), v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I \tag{47}$$

Then, it is easy to see that condition (42) holds with

$$\varepsilon_n = \max\,\{\theta_n, \|u_n(0) - u(0)\|_X\}. \tag{48}$$

Finally, assume that (42) holds. Let $n \in \mathbb{N}$ and $t \in [0, T]$. We take $v = \dot{u}(t) - \dot{u}_n(t)$ in this inequality to see that

$$(\dot{u}_n(t), \dot{u}(t) - \dot{u}_n(t))_X + \varepsilon_n \|\dot{u}(t) - \dot{u}_n(t)\|_X \ge (F(t, u_n(t)), \dot{u}(t) - \dot{u}_n(t))_X$$

and, using equality $\dot{u}(t) = F(t, u(t))$, we find that

$$\begin{aligned} &(\dot{u}_n(t) - \dot{u}(t), \dot{u}(t) - \dot{u}_n(t))_X + \varepsilon_n \|\dot{u}(t) - \dot{u}_n(t)\|_X \\ &\quad \ge (F(t, u_n(t)) - F(t, u(t)), \dot{u}(t) - \dot{u}_n(t))_X. \end{aligned}$$

Thus,

$$\|\dot{u}_n(t) - \dot{u}(t)\|_X^2 \le \varepsilon_n \|\dot{u}_n(t) - \dot{u}(t)\|_X + \|F(t, u_n(t)) - F(t, u(t))\|_X \|\dot{u}_n(t) - \dot{u}(t)\|_X$$

and, therefore,

$$\|\dot{u}_n(t) - \dot{u}(t)\|_X \le \varepsilon_n + \|F(t, u_n(t) - F(t, u(t))\|_X.$$

We now use assumption (30)(c) to see that

$$\|\dot{u}_n(t) - \dot{u}(t)\|_X \le \varepsilon_n + L_F \|u_n(t) - u(t)\|_X$$

and, keeping in mind (8), after some algebra we find that

$$\|\dot{u}_n(t) - \dot{u}(t)\|_X \le \varepsilon_n + L_F \int_0^t \|\dot{u}_n(s) - \dot{u}(s)\|_X \, ds + L_F \|u_n(0) - u_0\|_X.$$

Next, we use the Gronwall lemma and inequality $\|u_n(0) - u_0\|_X \le \varepsilon_n$ in (42) to find that

$$\|\dot{u}_n(t) - \dot{u}(t)\|_X \le (1 + L_F)e^{L_F t}\varepsilon_n.$$

We now use the convergences $\varepsilon_n \to 0$ and inequality $t \le T$ to see that

$$\max_{t\in[0,T]} \|\dot{u}_n(t) - \dot{u}(t)\|_X \to 0 \quad \text{as} \quad n \to \infty. \tag{49}$$

On the other hand, using the identity

$$u_n(t) - u(t) = \int_0^t (\dot{u}_n(s) - \dot{u}(s))\,ds + u_n(0) - u_0$$

we find that

$$\|u_n(t) - u(t)\|_X \le \int_0^t \|\dot{u}_n(s) - \dot{u}(s))\|_X\,ds + \|u_n(0) - u_0\|_X.$$

Therefore, (49) and (42) imply that

$$\max_{t\in[0,T]} \|u_n(t) - u(t)\|_X \to 0 \quad \text{as} \quad n \to \infty. \tag{50}$$

The convergences (49) and (50) show that $u_n \to u$ in $C^1(I; X)$ and, therefore, (40) holds.

To conclude, we proved the implications (40) $\Longrightarrow$ (41) $\Longrightarrow$ (42) $\Longrightarrow$ (40) which shows the equivalence of the statements (40), (41) and (42).

b) We proceed with the case $I = \mathbb{R}_+$. To this end we fix $m \in \mathbb{N}$ and we use the first part of the theorem with $T = m$, combined with the remark that the quantity ε_n defined by (48), (45) depends on T and, since $T = m$, it we denote it what follows by ε_n^m. We deduce from here the equivalences of the following statements:

$$u_n \to u \qquad \text{in} \;\; C^1([0, m]; X). \tag{51}$$

$$\dot{u}_n - F(\cdot, u_n) \to 0_X \;\; \text{in} \;\; C([0, m]; X), \;\; u_n(0) \to u(0) \;\; \text{in} \;\; X. \tag{52}$$

$$\begin{cases} \text{there exists } 0 \le \varepsilon_n^m \to 0 \text{ such that} \\ (\dot{u}_n(t), v)_X + \varepsilon_n^m \|v\|_X \ge (F(t, u_n(t)), v)_X \quad \forall\, n \in \mathbb{N},\; t \in [0, m], \\ \|u_n(0) - u_0\|_X \le \varepsilon_n^m \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{53}$$

Recall that the equivalence of these statements is valid for any $m \in \mathbb{N}$. We now use (5) and the equivalence (6) to see that the convergences (51) and (52) can be replaced by the convergences (40) and (41), respectively, which concludes the proof. □

Note that Theorem 5 provides a convergence criterion for the solution of the Cauchy problem (1)–(2), in the space $C^1(I; X)$. Therefore, it provides an answer to Problem $\mathcal{Q}_Q$ in the case when $\mathcal{P}$ represents the above mentioned Cauchy problem.

5. A particular case

Everywhere in this section we assume that X is a Hilbert space. We use the results in Section 4 in the study of the Cauchy problem

$$A\dot{u}(t) + Bu(t) = f(t) \quad \forall\, t \in I, \tag{54}$$

$$u(0) = u_0, \tag{55}$$

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in which $A : X \to X$ and $B : X \to X$ are given nonlinear operators, $f : I \to X$ and u_0 is an initial data. In the study of this problem we assume that A is a strongly monotone Lipschitz continuous operator, that is, there exists two constants $m_A > 0$ and $L_A > 0$ such that

$$(Au - Av, u - v)_X \geq m_A \, \|u - v\|_X^2 \qquad \forall\, u,\, v \in X, \tag{56}$$

$$\|Au - Av\|_X \leq L_A \|u - v\|_X \qquad \forall\, u,\, v \in X. \tag{57}$$

We also assume that B is a Lipschitz continuous operator with constant $L_B > 0$, i.e.,

$$\|Bu - Bv\|_X \leq L_B \|u - v\|_X \qquad \forall\, u,\, v \in X \tag{58}$$

and, finally, we assume that the function f and the initial data have the following regularity:

$$f \in C(I; X), \tag{59}$$

$$u_0 \in X. \tag{60}$$

It is well known that conditions (56) and (57) imply that the operator is invertible and, moreover, its inverse $A^{-1} \colon X \to X$ is a strongly monotone Lipschitz continuous operator, with constants $\frac{m_A^2}{L_A}$ and $\frac{1}{m_A}$, respectively. A proof of this result can be found in [13, p. 23], for instance. Therefore,

$$(A^{-1}u - A^{-1}v, u - v)_X \geq \frac{m_A}{L_A^2} \, \|u - v\|_X^2 \qquad \forall\, u,\, v \in X, \tag{61}$$

$$\|A^{-1}u - A^{-1}v\|_X \leq \frac{1}{m_A} \|u - v\|_X \qquad \forall\, u,\, v \in X. \tag{62}$$

The unique solvability of the problem (54)–(55) is provided by the following result.

Theorem 6. *Let X be a Hilbert space and assume (56)–(60). Then, problem (54)–(55) has a unique solution $u \in C^1(I; X)$.*

Proof. We use the inverse of the operator A to see that problem (54)–(55) is equivalent to the problem of finding a function $u \in C^1(I; X)$ such that

$$\dot{u}(t) = A^{-1}(f(t) - Bu(t)) \;\; \text{for all} \;\; t \in I, \tag{63}$$

$$u(0) = u_0. \tag{64}$$

Denote by $F \colon I \times X \to X$ the function given by

$$F(t, u) = A^{-1}(f(t) - Bu) \qquad \forall\, t \in I,\; u \in X. \tag{65}$$

Then, using the properties (62), (58) of the operators A^{-1} and B, respectively, as well as the regularity (59) of the function f, it is easy to see that the function F defined before satisfies conditions (30). Therefore, Theorem 6 is a direct consquence of Theorem 4, which guarantees the unique solvability of the Cauchy problem (63)–(64). □

We provide a convergence criterion to the solution the Cauchy problem (63)–(64) and, to this end, we consider the following statements.

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$$u_n \to u \quad \text{in } C^1(I;X). \tag{66}$$

$$A\dot{u}_n + Bu_n \to f \quad \text{in } C(I;X) \quad \text{and} \quad u_n(0) \to u_0 \quad \text{in } X. \tag{67}$$

$$\begin{cases} I = [0;T] \text{ and there exists } 0 \le \varepsilon_n \to 0 \text{ such that} \\ (A\dot{u}_n(t), v)_X + (Bu_n(t), v)_X + \varepsilon_n \|v\|_X \ge (f(t), v)_X \\ \qquad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I, \\ \|u_n(0) - u_0\|_X \le \varepsilon_n \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{68}$$

$$\begin{cases} I = \mathbb{R}_+ \text{ and for any } m \in \mathbb{N} \text{ there exists } 0 \le \varepsilon_n^m \to 0 \text{ such that} \\ (A\dot{u}_n(t), v)_X + (Bu_n(t), v)_X + \varepsilon_n^m \|v\|_X \ge (f(t), v)_X \\ \qquad \forall\, v \in X,\ n \in \mathbb{N},\ t \in [0, m], \\ \|u_n(0) - u_0\|_X \le \varepsilon_n^m \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{69}$$

Our main result in this section is the following.

Theorem 7. *Let X be a Hilbert space, $T > 0$ and assume (56)–(60).*

a) *If $I = [0;T]$ then the statements (66), (67) and (68) are equivalent.*

b) *If $I = \mathbb{R}_+$ then the statements (66), (67) and (69) are equivalent.*

Proof. a) We assume that $I = [0, T]$. We use Theorem 5 with F given by (65) to see that the there statements below are equivalent.

$$u_n \to u \quad \text{in } C^1(I;X). \tag{70}$$

$$\dot{u}_n - A^{-1}(f - Bu_n) \to 0_X \quad \text{in } C(I;X) \quad \text{and} \quad u_n \to u_0 \ \text{in } X. \tag{71}$$

$$\begin{cases} I = [0;T] \text{ and there exists } 0 \le \theta_n \to 0 \text{ such that} \\ (\dot{u}_n(t), v)_X + \theta_n \|v\|_X \ge (A^{-1}(f(t) - Bu_n(t)), v)_X \\ \qquad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I, \\ \|u_n(0) - u_0\|_X \le \theta_n \quad \forall\, n \in \mathbb{N}. \end{cases} \tag{72}$$

Let $n \in \mathbb{N}$ and $t \in I$. We write

$$\dot{u}_n(t) - A^{-1}(f(t) - Bu_n(t)) = A^{-1}(A\dot{u}_n(t)) - A^{-1}(f(t) - Bu_n(t))$$

then we use the property (62) of the operator A^{-1} to deduce that

$$\|\dot{u}_n(t) - A^{-1}(f(t) - Bu_n(t))\|_X \le \frac{1}{m_A}\|A\dot{u}_n(t) + Bu_n(t) - f(t)\|_X. \tag{73}$$

A similar argument, based on the identity

$$A\dot{u}_n(t) + Bu_n(t) - f(t) = A\dot{u}_n(t) - A(A^{-1}(f(t) - Bu_n(t)))$$

and the property (57) of the operator A, yields

$$\|A\dot{u}_n(t) + Bu_n(t) - f(t)\|_X \le L_A\|\dot{u}_n(t) - A^{-1}(f(t) - Bu_n(t))\|_X. \tag{74}$$

Therefore, inequalities (73) and (74) show that

$$\text{the convergence (67) holds if and only the convergence (71) holds.} \tag{75}$$

Assume now that (72) holds. Then Lemma 2 guarantees that

$$\|\dot{u}_n(t) - A^{-1}(f(t) - Bu_n(t))\|_X \leq \theta_n \quad \forall\, n \in \mathbb{N},\ t \in I$$

and, using (74) we deduce that

$$\|A\dot{u}_n(t) + Bu_n(t) - f(t)\|_X \leq L_A\theta_n \quad \forall\, n \in \mathbb{N},\ t \in I.$$

Then, using again Lemme 2, we deduce that

$$(A\dot{u}_n(t), v)_X + (Bu_n(t), v)_X + L_A\theta_n\|v\|_X \geq (f(t), v)_X \quad \forall\, v \in X,\ n \in \mathbb{N},\ t \in I.$$

It follows from here that the statement (68) holds with $\varepsilon_n = \max\{L_A\theta_n, \theta_n\}$, for all $n \in \mathbb{N}$. This shows that the statement (72) implies the statement (68). A similar argument, based on inequality (73), shows that the converse of this implication holds, too. We conclude from here that

$$\text{the statement (68) holds if and only the statement (72) holds.} \tag{76}$$

The equivalence of the statements (66), (67) and (68) is now a direct consequence of the equivalences the statements (70), (71) and (72), guaranteed by Theorem 5, combined with the equivalences (75) and (76).

b) Assume now that $I = \mathbb{R}_+$. Then, the equivalences of the statements (66), (67) and (69) follows arguments similar to those used in the first part of the theorem. Since the modifications are straight, we skip the details. □

Consider now two sequences $\{f_n\}$ and $\{u_{0n}\}$ such that, for each $n \in \mathbb{N}$ the following condition hold.

$$f_n \in C(I; X), \tag{77}$$

$$u_{0n} \in X. \tag{78}$$

Then, it followws from Theorem 6 that for each $n \in \mathbb{N}$ there exists a unique function $u_n \in C^1(I; X)$ such that

$$A\dot{u}_n(t) + Bu_n(t) = f_n(t) \quad \forall\, t \in I, \tag{79}$$

$$u_n(0) = u_{0n}, \tag{80}$$

We have the following result.

Corollary 2. *Let X be a Hilbert space, and assume (56)–(60), (77) and (78). Then, the solution u_n of Problem (79)–(80) converges in $C^1(I; X)$ to the solution u of Problem (54)–(55) if and only if*

$$f_n \to f \quad \text{in } C^1(I; X) \quad \text{and} \quad u_{0n} \to u \quad \text{in } X.$$

Proof. Corollary 2 is a direct consequence of the equivalence of statements (66) and (67) in Theorem 7. □

Note that Corollary 2 provides, in particular, a continuous dependence result of the solution of the Cauchy problem (54)–(55) with respect to the date f and u_0. Similar results can be obtained by considerating the perturbation of the operators A or B as well as various perturbations of the left hand side of the differential equation (54). Such an example will be presented in the next section, in the study to a viscoelastic problem.

6. An application in Solid Mechanics

Our results in the previous sections are usefull in the study of various boundary value problems in Solid Mechanics. References in the field are the books [13,14], for instance. Here, to keep the paper in a reasonable length, we provide only one simplified exemple and, to this end, we need to introduce some additional notations.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ and denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. We recall that the canonical inner product and the corresponding norm on $\mathbb{S}^d$ are given by

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Here and below in this section the indices i, j, k, l run between 1 and 3, and, unless stated otherwise, the summation convention over repeated indices is used. We consider the space

$$Q = L^2(\Omega)_s^{d\times d} = \{\, \boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ 1 \le i, j \le d \,\}$$

which, recall, is a Hilbert spaces with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx$$

and the associated norm, denoted by $\|\cdot\|_Q$. Moreover, we need the space of symmetric fourth order tensors $\mathbf{Q}_\infty$ given by

$$\mathbf{Q}_\infty = \{\, \mathcal{C} = (c_{ijkl}) \mid c_{ijkl} = c_{jikl} = e_{klij} \in L^\infty(\Omega),\ 1 \le i, j, k, l \le d \,\}.$$

It is easy to see that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{C}\|_{\mathbf{Q}_\infty} = \max_{0 \le i,j,k,l \le d} \|c_{ijkl}\|_{L^\infty(\Omega)}$$

and, in addition,

$$\|\mathcal{C}\boldsymbol{\tau}\|_Q \le d\, \|\mathcal{C}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \forall\, \mathcal{C} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{81}$$

Below we denote by $\mathbf{0}_\infty$ the zero element of the spaces $C(I; \mathbf{Q}_\infty)$ and $C([0, m]; \mathbf{Q}_\infty)$ with $m \in \mathbb{N}$. Moreover, $\mathbf{0}_Q$ will represent the zero element of the space Q. Finally, let I be a time interval of interest which can be either bounded (i.e., of the form $I = [0, T]$ with $T > 0$), or unbounded (i.e., $I = \mathbb{R}_+$) and recall that, as usual, we use a dot above to denote the derivative with respect to the time variable.

We now turn to the viscoelastic problem we consider, which is governed by two given operators $A : Q \to Q$ and $B : Q \to Q$. It can be formulated as follows.

Problem $\mathcal{P}$. *Given a function* $\boldsymbol{\sigma} \in C(I; Q)$ *and an element* $\boldsymbol{\varepsilon}_0 \in Q$, *find a function* $\boldsymbol{\varepsilon} \in C(I; Q)$ *such that*

$$\boldsymbol{\sigma}(t) = A\dot{\boldsymbol{\varepsilon}}(t) + B\boldsymbol{\varepsilon}(t) \qquad \forall\, t \in I, \tag{82}$$

$$\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0. \tag{83}$$

This problem describes the behaviour of a viscoelastic body in the time interval I. Here, Ω represents the reference configuration of the body, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ represents the linearized

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strain tensor and equation (82) is related to the consitutive law of the material, asumed to be viscoelastic with short memory. The operator A represents the viscosity operator and B is the elasticity operator. Finally, the function ε_0 is the initial deformation. More details on the constitutive laws which describe the bahaviour of viscoelastic materials can be found in [4–6,13,14], for instance.

We now consider a sequence $\{\mathcal{C}_n\}$ of functions defined on I with values in the space $\mathbf{Q}_\infty$ and, for each $n \in \mathbb{N}$, we consider the following problem.

Problem $\mathcal{P}_n$. *Given a function $\boldsymbol{\sigma} \in C(I;Q)$ and an element $\boldsymbol{\varepsilon}_0 \in Q$, find a function $\boldsymbol{\varepsilon}_n \in C(I;Q)$ such that*

$$\boldsymbol{\sigma}(t) = A\dot{\boldsymbol{\varepsilon}}_n(t) + B\boldsymbol{\varepsilon}_n(t) + \int_0^t \mathcal{C}_n(t-s)\dot{\boldsymbol{\varepsilon}}_n(s)\,ds \qquad \forall\, t \in I, \tag{84}$$

$$\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0. \tag{85}$$

Note that the mechanical significance of Problem $\mathcal{P}_n$ is similar to that of Problem $\mathcal{P}$. The difference arise in the fact that the viscoelastic constitutive law with short memory (82) was replaced by the viscoelastic constitutive law with long memory (84), in which $\mathcal{C}_n$ represents a relaxation tensor. Such constitutive laws have been used in the literature in order to model the behavior of real materials like rubbers, rocks, metals, pastes and polymers. References in the field are [1,2], for instance.

In the study of Problems $\mathcal{P}$ and $\mathcal{P}_n$ we consider the following assumtions:

$$A : Q \to Q \text{ is a strongly monotone Lipschitz continuous operator.} \tag{86}$$

$$B : Q \to Q \text{ is a Lipschitz continuous operator.} \tag{87}$$

$$\boldsymbol{\sigma} \in C(I;Q). \tag{88}$$

$$\boldsymbol{\varepsilon}_0 \in Q. \tag{89}$$

$$\mathcal{C}_n \in C(I;\mathbf{Q}_\infty) \quad \forall\, n \in \mathbb{N}. \tag{90}$$

$$\mathcal{C}_n \to \mathbf{0}_\infty \quad \text{in } C(I;\mathbf{Q}_\infty). \tag{91}$$

Our main result in this section is the following.

Theorem 8. *Assume* (86)–(90). *Then:*

a) *Problem $\mathcal{P}$ has a unique solution $\boldsymbol{\varepsilon} \in C^1(I;Q)$ and, for each $n \in \mathbb{N}$, Problem $\mathcal{P}_n$ has a unique solution $\boldsymbol{\varepsilon}_n \in C^1(I;Q)$.*

b) *If, moreover,* (91) *holds, then*

$$\boldsymbol{\varepsilon}_n \to \boldsymbol{\varepsilon} \quad \text{in } C^1(I;Q). \tag{92}$$

Proof. a) The unique solvability of Problem $\mathcal{P}$ is a direct consequence of Theorem 6. Let $n \in \mathbb{N}$. To prove the unique solvability of Problem $\mathcal{P}_n$ we consider the operator $\Lambda_n : C(I;Q) \to C(I;Q)$ defined by

$$\Lambda_n\boldsymbol{\eta}(t) = B\Big(\int_0^t \boldsymbol{\eta}(s)\,ds + \boldsymbol{\varepsilon}_0\Big) + \int_0^t \mathcal{C}_n(t-s)\boldsymbol{\eta}(s)\,ds \quad \forall\, t \in I,\ \boldsymbol{\eta} \in C(I:Q). \tag{93}$$

Then, using assumptions (87), (90) and inequality (81) it is easy to see that Λ_n is a history-dependent operator. We now use Theorem 2 to deduce that there exists a unique function $\boldsymbol{\eta}_n \in C(I;Q)$ such that

$$A\boldsymbol{\eta}_n(t) + \Lambda_n\boldsymbol{\eta}_n(t) = \boldsymbol{\sigma}(t) \qquad \forall\, t \in I$$

or, equivalently,

$$A\boldsymbol{\eta}_n(t) + B\Big(\int_0^t \boldsymbol{\eta}_n(s)\,ds + \boldsymbol{\varepsilon}_0\Big) + \int_0^t \mathcal{C}_n(t-s)\boldsymbol{\eta}_n(s)\,ds = \boldsymbol{\sigma}(t) \qquad \forall\, t \in I. \tag{94}$$

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Denote by ε_n the function given by

$$\varepsilon_n(t) = \int_0^t \eta_n(s)\,ds + \varepsilon_0 \qquad \forall\, t \in I. \tag{95}$$

It follows from (94) and (95) that ε_n is a solution to Problem $\mathcal{P}_n$ with regularity $\varepsilon_n \in C^1(I;Q)$. This proves the existence of the solution of Problem $\mathcal{P}_n$. The uniqueness follows from the uniquenss of the solution of equation (94), guaranteed by Theorem 6.

b) Assume now that (91) holds. We start with the case when $I = [0,T]$ with $T > 0$. First, we prove that the sequence $\{\dot{\varepsilon}_n\}$ is bounded in the space $C(I;Q)$, see inequality (98) below. To this end, we fix $n \in \mathbb{N}$ and $t \in [0,T]$. Then, using (84) we obtain that

$$(A\dot{\varepsilon}_n(t), \dot{\varepsilon}_n(t))_Q + (B\varepsilon_n(t), \dot{\varepsilon}_n(t))_Q + (\int_0^t \mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\,ds, \dot{\varepsilon}_n(t))_Q = (\sigma(t), \dot{\varepsilon}_n(s))_Q$$

and, therefore,

$$(A\dot{\varepsilon}_n(t) - A\mathbf{0}_Q, \dot{\varepsilon}_n(t))_Q = (\sigma(t), \dot{\varepsilon}_n(s)) - (A\mathbf{0}_Q, \dot{\varepsilon}_n(t))_Q - (B\varepsilon_n(t), \dot{\varepsilon}_n(t))_Q$$
$$-(\int_0^t \mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\,ds, \dot{\varepsilon}_n(t))_Q.$$

Next, using the strong monotonicity of the operator A with constant m_A we deduce that

$$m_A\|\dot{\varepsilon}_n(t)\|_Q^2 \le \Big(\|\sigma(t)\|_Q + \|A\mathbf{0}_Q\|_Q + \|B\varepsilon_n(t)\|_Q + \int_0^t \|\mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\,ds\|_Q\Big)\|\dot{\varepsilon}_n(t)\|_Q,$$

which implies that

$$m_A\|\dot{\varepsilon}_n(t)\|_Q \le \|\sigma(t)\|_Q + \|A\mathbf{0}_Q\|_Q + \|B\varepsilon_n(t)\|_Q + \int_0^t \|\mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\,ds\|_Q. \tag{96}$$

We now use assumption (87) and inequality (81) to find that

$$\|\dot{\varepsilon}_n(t)\|_Q \le D + \|\varepsilon_n(t)\|_Q + d\int_0^t \|\mathcal{C}_n(t-s)\|_{\mathbf{Q}_\infty}\|\dot{\varepsilon}_n(s)\|_Q\,ds. \tag{97}$$

Here and below D represents a positive constant which does not depend on n and whose value will change from place to place. On the other hand, inequality (97) combined with assumption (91) and identity (8) yields

$$\|\dot{\varepsilon}_n(t)\|_X \le D + D\int_0^t \|\dot{\varepsilon}_n(s)\|_Q\,ds.$$

We now use the Gronwall argument to see that

$$\|\dot{\varepsilon}_n(t)\|_Q \le D. \tag{98}$$

Next, we use equation (84), again, inequality (81) and inequality (98), valid for any $t \in [0,T]$, to see that

$$\|A\dot{\varepsilon}_n(t) + B\varepsilon_n(t) - \sigma(t)\|_Q = \|\int_0^t \mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\,ds\|_Q$$
$$\le \int_0^t \|\mathcal{C}_n(t-s)\dot{\varepsilon}_n(s)\|_Q\,ds \le d\int_0^t \|\mathcal{C}_n(t-s)\|_{\mathbf{Q}_\infty}\|\varepsilon_n(s)\|_Q\,ds$$
$$\le d\max_{r\in[0,T]}\|\mathcal{C}_n(r)\|_{\mathbf{Q}_\infty}\int_0^t \|\dot{\varepsilon}_n(s)\|_Q\,ds \le D\max_{r\in[0,T]}\|\mathcal{C}_n(r)\|_{\mathbf{Q}_\infty}.$$

It follows now from assumption (91) that

$$\max_{t\in[0,T]} \|A\dot{\varepsilon}_n(t) + B\varepsilon_n(t) - \sigma(t)\|_Q \to 0$$

and, therefore, $A\dot{\varepsilon}_n + B\varepsilon_n \to \sigma$ in $C(I;X)$. We now use Theorem 7 a) to deduce that the convergence (92) holds.

Assume now that $I = \mathbb{R}_+$. Then, assumption (91) guarantee that $\mathcal{C}_n \to \mathbf{0}_\infty$ in $C([0,m];\mathbf{Q}_\infty)$, for any $m \in \mathbb{N}$. Therefore, using the part a) of the theorem we deduce that $\varepsilon_n \to \varepsilon$ in $C^1([0,m];Q)$ for any $m \in \mathbb{N}$. This implies that (92) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence result (91) it is important from the mechanical point of view since it shows that the viscoelastic constitutive law with short memory (82) can be approached by the viscoelastic constitutive law with long memory (84) for a small relaxation tensor.

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