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Article

Equilibrium Figures for a Rotating Compressible Capillary Two-Layer Liquid

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Abstract: The paper proves the existence of a family of axisymmetric equilibrium figures as solutions of a stationary problem with unknown boundaries for the Navier–Stokes equations corresponding to the slow rotation of a viscous compressible two-layer liquid mass about some axis. It is assumed that the liquids are barotropic, capillary and have different viscosities, the internal fluid being bounded by a closed surface. This interface does not intersect with the external boundary of the cloud. The proof is based on implicit function theorem and carried out in the Hölder spaces.

Keywords: equilibrium figures; viscous compressible two-layer fluid; capillary forces; interface problem for the Navier–Stokes system; the Hölder spaces

MSC: Primary 35Q30; 76T06; 76D05; 76D06; Secondary 35R35; 76D03

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1. Introduction

Existence of an equilibrium surface for an isolated compressible liquid mass rotating about a fixed axis was first proved in [9]. Our aim is to prove the existence of equilibrium figures for a rotating compressible two-layer fluid.

The problem of the rotation of an isolated incompressible liquid mass about a fixed axis as a rigid body was considered by many famous mathematicians, among them were Newton, Maclaurin, Jacobi, Kovalevskaya, Lyapunov, Poincare and others [1–3], who mainly studied the movement without capillarity. The capillary fluids were first investigated by Globa-Mikhailenko [4], Boussinesq and Charrueau in the begining of 20th century. The latter gave a detailed analysis of the problem, calculated the shape of equilibrium figures, including the toroidal case, and considered some aspects of the stability [5,6]. These results were included in a big review on this subject presented in the book of Appell [7]. Stability problem for various ellipsoidal equilibrium figures is analyzed in monograph [8].

Now we state, in a complete setting, the problem on unsteady motion of two compressible barotropic fluids of finite volume separated by a closed unknown interface.

At the initial instant $t = 0$, let a fluid with dynamic viscosities μ^+, μ_1^+ be in a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$, and in the domain Ω_0^-, surrounding it, there be a fluid with dynamic viscosities μ^-, μ_1^-;

$$\mu^\pm > 0, \qquad 2\mu^\pm + 3\mu_1^\pm \geq 0.$$

The domain $\Omega_0 \equiv \overline{\Omega_0^+} \cup \Omega_0^-$ is bounded by the free surface Γ_0^- and includes the closed interface $\Gamma_0^+ \equiv \partial\Omega^+$; $\Gamma_0^\pm$ are given. This two-component cloud rotates about the vertical axis x_3 with an angular velocity ω.

For $t > 0$, it is necessary to find the surfaces Γ_t^-, Γ_t^+, as well as velocity vector field $v(x,t) = (v_1, v_2, v_3)$ and the density $\rho(x,t) > 0$ of the fluids satisfying diffraction problem for the Navier – Stokes system

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$$
\begin{aligned}
&\rho\big(\mathcal{D}_t\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big)-\nabla\cdot\mathbb{T}=0,\quad \mathcal{D}_t\rho+\nabla\cdot(\rho\boldsymbol{v})=0\quad\text{in } \Omega_t^-\cup\Omega_t^+,\quad t>0,\\
&\qquad \boldsymbol{v}|_{t=0}=\boldsymbol{v}_0(x),\quad \rho|_{t=0}=\rho_0(x)\quad \text{in } \Omega_0^-\cup\Omega_0^+,\\
&\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\big|_{\Gamma_t^-}=\sigma^-H^-\boldsymbol{n}\quad\text{on } \Gamma_t^-,\\
&[\boldsymbol{v}]\big|_{\Gamma_t^+}\equiv\lim_{\substack{x\to x_0\in\Gamma_t^+,\\ x\in\Omega_t^+}}\boldsymbol{v}(x,t)-\lim_{\substack{x\to x_0\in\Gamma_t^+,\\ x\in\Omega_t^-}}\boldsymbol{v}(x,t)=0,\quad [\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}]\big|_{\Gamma_t^+}=\sigma^+H^+\boldsymbol{n}\quad\text{on } \Gamma_t^+,\\
&V_{\boldsymbol{n}}=\boldsymbol{v}\cdot\boldsymbol{n}\quad\text{on } \Gamma_t\equiv\Gamma_t^+\cup\Gamma_t^-,
\end{aligned}
\tag{1}
$$

where
$$\mathbb{T}=\big(-p(\rho)+\mu_1^{\pm}\nabla\cdot\boldsymbol{v}\big)\mathbb{I}+\mu^{\pm}\mathbb{S}(\boldsymbol{v})$$
is stress tensor, $\big(\mathbb{S}(\boldsymbol{v})\big)_{ij}=\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}$ is double strain rate tensor, $\mathbb{I}$ is identity matrix; $\mu^{\pm}$, $\mu_1^{\pm}$ are step functions of dynamic viscosities, equal to μ^+, μ_1^+ in Ω_t^+ and μ^-, μ_1^- in Ω_t^- respectively; $p(\rho)$ is fluid pressure given by a known smooth density function; $\boldsymbol{v}_0$ and $\rho_0>0$ are initial distributions of velocity and density of the liquids, $\boldsymbol{n}$ is the outward normal vector to the union Γ_t; $H^{\pm}(x,t)$ are twice the mean curvatures of the surfaces $\Gamma_t^{\pm}$ (moreover, $H^+<0$ at points of convexity Γ_t^+ towards Ω_t^-); $\sigma^-,\sigma^+>0$ are surface tension coefficients on Γ_t^- and Γ_t^+, respectively; $V_{\boldsymbol{n}}$ is the rate of evolution of Γ_t in the direction $\boldsymbol{n}$. We assume that the Cartesian coordinate system $\{x\}$ is introduced in the space $\mathbb{R}^3$. The central dot denotes the Cartesian scalar product.

We mean summation over repeated indices from 1 to 3 if they are denoted by Latin letters, and from 1 to 2 if they are Greek. We mark vectors and vector spaces in bold. The notation $\nabla\cdot\mathbb{T}$ denotes the vector with the components $(\nabla\cdot\mathbb{T})_j=\frac{\partial T_{ij}}{\partial x_i}$, $j=1,2,3$.

The kinematic boundary condition $\boldsymbol{v}\cdot\boldsymbol{n}=V_{\boldsymbol{n}}$ excludes mass transfer across fluid boundaries. It follows from our assumption that the fluid particles do not leave the boundaries $\Gamma_t^{\pm}$ during the time.

Local (in time) solvability was proved for problem (1) in the whole space $\mathbb{R}^3$ with a closed interface between the fluids. The result was obtain both in the Sobolev – Slobodetskiĭ classes of functions [10] and in the Hölder ones [11]. One can get similar results for a two-component domain bounded by a free boundary if one takes into account the estimates for a model problem in a half-space [12,13].

As we have mentioned, we suppose the liquids to be barotropic which implies that the pressure p is a known increasing function of the density: $p'(\rho)>0$. Let, in addition, $\rho=\rho(|x'|)$, $x'=(x_1,x_2,0)$.

We assume that equilibrium figures $\mathcal{F}^+$, $\mathcal{F}$ are nearly globular domains with the radiuses $R_0^{\pm}$ ($R_0^+<R_0^-$), and the motion of fluids is close to the state of rest, i.e., the velocity $\boldsymbol{v}$ is small, and the density ρ differs little from a step function $\rho^{\pm}>0$. This picture is schematically presented in Figure 1. We denote the balls $\{x\in\mathbb{R}^3:|x|\le R_0^{\pm}\}$ by $B_{R_0^{\pm}}$.

We are going to prove the existence of $\mathcal{G}^+$ and $\mathcal{G}^-$, the boundaries of the figures $\mathcal{F}^+$ and $\mathcal{F}$, respectively. We follow the plan of paper [9].

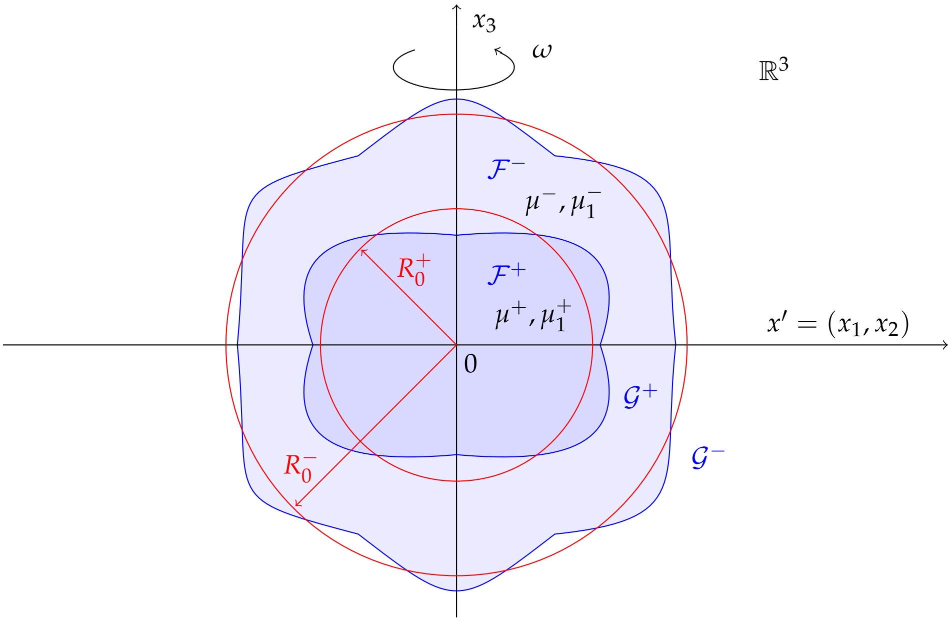

Figure 1. Equilibrium Figures for a Two-Layer Compressible Fluid.

At rest, the bubble consisted of nested spherical two layers $B_{R_0^+}$ and $B_{R_0^-} \setminus B_{R_0^+}$ with uniform distributions of densities $\rho^{\pm}$ has the piecewise constant pressure:

$$\begin{aligned} p(\rho^-) &= \frac{2\sigma^-}{R_0^-} \quad \text{in } B_{R_0^-} \setminus B_{R_0^+}, \\ p(\rho^+) &= \frac{2\sigma^+}{R_0^+} + \frac{2\sigma^-}{R_0^-} \quad \text{in } B_{R_0^+}. \end{aligned} \tag{2}$$

The masses of the layers are

$$\rho^+|B_{R_0^+}| = m^+, \qquad \rho^-\left(|B_{R_0^-}| - |B_{R_0^+}|\right) = m^-. \tag{3}$$

Steady motion of a two-layer gaseous body $\mathcal{F} \equiv \overline{\mathcal{F}^+} \cup \mathcal{F}^-$ uniformly rotating about the axis x_3 with a constant angular velocity ω is governed by the homogeneous stationary Navier–Stokes equations

$$\rho(\boldsymbol{\mathcal{V}} \cdot \nabla)\boldsymbol{\mathcal{V}} - \nabla \cdot \mathbb{T} = 0, \qquad \nabla \cdot (\rho \boldsymbol{\mathcal{V}}) = 0 \quad \text{in } \cup \mathcal{F}^{\pm} \tag{4}$$

(here the density ρ and velocity $\boldsymbol{\mathcal{V}}$ depend only on x) and the boundary conditions

$$\begin{aligned} &\mathbb{T}(\boldsymbol{\mathcal{V}}, \mathcal{P})\boldsymbol{n}\big|_{\mathcal{G}^-} = \sigma^-\mathcal{H}^-\boldsymbol{n} \quad \text{on } \mathcal{G}^-, \\ &[\boldsymbol{\mathcal{V}}]\big|_{\mathcal{G}^+} = 0, \qquad [\mathbb{T}(\boldsymbol{\mathcal{V}}, \mathcal{P})\boldsymbol{n}]\big|_{\mathcal{G}^+} = \sigma^+\mathcal{H}^+\boldsymbol{n} \quad \text{on } \mathcal{G}^+, \\ &\boldsymbol{\mathcal{V}} \cdot \boldsymbol{n} = 0 \quad \text{on } \mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-, \end{aligned} \tag{5}$$

where $\mathcal{H}^-, \mathcal{H}^+$ are twice the mean curvatures of $\mathcal{G}^-, \mathcal{G}^+$, respectively. The last relation follows from the boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = V_{\boldsymbol{n}}$. The pressure $\mathcal{P}$ depends on ρ.

It is easily seen that velocity vector field

$$\boldsymbol{\mathcal{V}}(x) = \omega \boldsymbol{e}_3 \times \boldsymbol{x} \equiv \omega(-x_2, x_1, 0) \tag{6}$$

satisfies (4) together with pressure function gradient

$$\nabla \mathcal{P}(\rho) = \rho\omega^2 x' \equiv \rho\frac{\omega^2}{2}\nabla|x'|^2, \tag{7}$$

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where $\boldsymbol{e}_i$ is the ith basis vector, $|x'|^2 = x_1^2 + x_2^2$.

First, we consider the simple case when equality (7) coincides with the following one

$$\nabla \mathcal{P}(\rho) = \mathcal{P}'(\rho) \nabla \Big(\frac{\omega^2}{2} |x'|^2 \Big),$$

whence $\mathcal{P}'(\rho) = \rho(x)$ and $\mathcal{P}^{\pm}(\rho) \equiv \frac{\rho^2(x)}{2} + p^{\pm}$ in $\mathcal{F}^{\pm}$ with constants $p^{\pm}$, because pressure functions can differ each other in different domains by a constant. These constants can be found from relations (2):

$$\begin{aligned} p^- &= \frac{2\sigma^-}{R_0^-} - \frac{\rho^{-2}}{2}, \\ p^+ &= \frac{2\sigma^+}{R_0^+} + \frac{2\sigma^-}{R_0^-} - \frac{\rho^{+2}}{2}. \end{aligned} \tag{8}$$

Let S_1 be the unit sphere in $\mathbb{R}^3$ with the center in zero, $\xi = \frac{x}{|x|} \in S_1$. We suppose $\mathcal{G}^{\pm}$ to be given by functions $R^{\pm}(\xi)$ on S_1. In addition, let $R^{\pm}(\xi)$ be rotationally symmetric, i.e., they depend only on $|\xi'| = \sqrt{\xi_1^2 + \xi_2^2}$ and ξ_3, and be even in ξ_3.

By substituting $\boldsymbol{\mathcal{V}}$ given by (6) and $\mathcal{P} = \mathcal{P}^{\pm}$ into boundary conditions (5), we obtain the equations for the surface $\mathcal{G}^-$ of the domain $\mathcal{F}$ and for the interface $\mathcal{G}^+$ between the fluids:

$$\begin{aligned} &\sigma^- \mathcal{H}^-(x) + \mathcal{P}(\rho) = 0, \quad x \in \mathcal{G}^-, \\ &\sigma^+ \mathcal{H}^+(x) + [\mathcal{P}(\rho)]|_{\mathcal{G}^+} = 0, \quad x \in \mathcal{G}^+. \end{aligned} \tag{9}$$

Rotationally symmetry implies that $R^{\pm}(\xi)$ do not depend on $\arctan(\frac{\xi_2}{\xi_1})$. It is clear that $\boldsymbol{n} = (\frac{\xi_1}{c_0}, \frac{\xi_2}{c_0}, n_3)$, since the first two components of $\boldsymbol{n}$ are proportional to ones of the radial vector of the circles to be horizontal sections of $\mathcal{G}^{\pm}$. Therefore $\boldsymbol{\mathcal{V}} \cdot \boldsymbol{n} = 0$ on $\mathcal{G}^{\pm}$.

Obviously, the density is given by the formulas

$$\begin{aligned} \rho(x) &= \frac{\omega^2}{2} |x'|^2 + c^+ \quad \text{in } \mathcal{F}^+, \\ \rho(x) &= \frac{\omega^2}{2} |x'|^2 + c^- \quad \text{in } \mathcal{F}^- \end{aligned} \tag{10}$$

with arbitrary positive constants c^+ and c^-, equations (9) taking the form

$$\begin{aligned} &\sigma^- \mathcal{H}^-(x) + \mathcal{P}^-\big(\frac{\omega^2}{2}|x'|^2 + c^-\big) = 0, \quad x \in \mathcal{G}^-, \\ &\sigma^+ \mathcal{H}^+(x) + \mathcal{P}^+\big(\frac{\omega^2}{2}|x'|^2 + c^+\big) - \mathcal{P}^-\big(\frac{\omega^2}{2}|x'|^2 + c^-\big) = 0, \quad x \in \mathcal{G}^+. \end{aligned} \tag{11}$$

One can determine the constants $c^{\pm}$ by prescribing the masses of fluids to be the same as that of the nested spherical liquid layers (3):

$$\int_{\mathcal{F}^{\pm}} \rho(x)\, dx \equiv \int_{\mathcal{F}^{\pm}} \Big(\frac{\omega^2}{2} |x'|^2 + c^{\pm} \Big)\, dx = m^{\pm}. \tag{12}$$

We consider the angular momentum to be one more parameter of the problem. It is given:

$$\beta \equiv \int_{\mathcal{F}^+ \cup \mathcal{F}^-} \rho(x) \boldsymbol{\mathcal{V}} \cdot \boldsymbol{\eta}_3\, dx = \omega \int_{\cup \mathcal{F}^{\pm}} \Big(\frac{\omega^2}{2} |x'|^2 + c^{\pm} \Big) |x'|^2\, dx, \tag{13}$$

where $\boldsymbol{\eta}_i = \boldsymbol{e}_i \times \boldsymbol{x}$ is the ith solid rotation vector, $i = 1, 2, 3$. Then the angular velocity ω is a function of β.

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We denote by C^s, $s > 0$, $s \notin \mathbb{N}$, the Hölder space of functions f on the sphere S_1 with the norm

$$|f|_{C^s(S_1)} \equiv \max_{k=\{1,\ldots,N\}} \Big(\sum_{|j|<s} \sup_{\xi\in\zeta_k} |\mathcal{D}^j f(\xi)| + \sum_{|j|=[s]} \sup_{\xi,\bar{\xi}\in\zeta_k} |\xi-\bar{\xi}|^{-(s-[s])} |\mathcal{D}^j f(\xi) - \mathcal{D}^j f(\bar{\xi})| \Big),$$

where $\mathcal{D}^j f$ is $|\boldsymbol{j}|$th derivative of f calculated in local coordinates on the subdomain $\zeta_k \subset S_1$, $\bigcup_{k=1}^{N} \zeta_k = S_1$. Under $\tilde{C}^s(S_1)$, we mean the subspace of $C^s(S_1)$ consisting of rotationally symmetric functions that are even with respect to ξ_3.

Theorem 1. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and let the data of problem (4), (5) be such that condition (26) holds. Then for an arbitrary β satisfying the estimate*

$$|\beta| < \varepsilon \tag{14}$$

with small enough ε, there exists a unique solution $(R^\pm, \omega, c^\pm) \in \tilde{C}^{2+\alpha}(S_1) \times \tilde{C}^{2+\alpha}(S_1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to system (11)–(13). It obeys the inequality

$$\sum_{\pm} \{ |R^\pm - R_0^\pm|_{C^{2+\alpha}(S_1)} + |c^\pm - \rho^\pm| \} + |\omega| < c|\beta|. \tag{15}$$

2. Proof of Theorem 1

Proof. In order to linearize system (11), we apply the formula for the first variation of a functional $\delta_0 R[r] = \frac{\mathrm{d}}{\mathrm{d}s} R[sr]\big|_{s=0}$. According to [14,15], the first variation of twice the mean curvature of $\mathcal{G}^\pm$ with respect to the double curvature of the sphere $S_{R_0^\pm}$ is

$$\delta_0 \big(\mathcal{H}^\pm[r(\xi)] + \frac{2}{R_0^\pm}\big) = \frac{1}{R_0^{\pm 2}} (\Delta_{S_1} r^\pm + 2r^\pm),$$

where Δ_{S_1} is the Laplace–Beltrami operator on S_1 and $r^\pm \equiv R^\pm - R_0^\pm$; $\mathcal{G}^\pm = \{x = R_0^\pm \xi + r^\pm(\xi)\boldsymbol{N},\ \xi \in S_1\}$, $\boldsymbol{N}$ is the outward normal to S_1. This yields

$$\begin{aligned} &\frac{\sigma^-}{R_0^{-2}} (\Delta_{S_1} r^- + 2r^-) = k^- + f^-, \quad \xi \in S_1, \\ &\frac{\sigma^+}{R_0^{+2}} (\Delta_{S_1} r^+ + 2r^+) = k^+ + f^+, \quad \xi \in S_1, \end{aligned} \tag{16}$$

where $k^- = \frac{2\sigma^-}{R_0^-} - \mathcal{P}^-(c^-) \equiv \mathcal{P}^-(\rho^-) - \mathcal{P}^-(c^-)$, $k^+ = \frac{2\sigma^+}{R_0^+} - \mathcal{P}^+(c^+) + \mathcal{P}^-(c^-) \equiv \mathcal{P}^+(\rho^+) - \mathcal{P}^+(c^+) + \mathcal{P}^-(c^-) - \mathcal{P}^-(\rho^-)$ in view of (2);

$$\begin{aligned} f^- &= \sigma^- \Big(\delta_0 \big(\mathcal{H}^-(x) + \frac{2}{R_0^-}\big) - \big(\mathcal{H}^-(x) + \frac{2}{R_0^-}\big)\Big) - \mathcal{P}^-\big(\frac{\omega^2}{2}|x'|^2 + c^-\big) \\ &\qquad + \mathcal{P}^-(c^-), \\ f^+ &= \sigma^+ \Big(\delta_0 \big(\mathcal{H}^+(x) + \frac{2}{R_0^+}\big) - \big(\mathcal{H}^+(x) + \frac{2}{R_0^+}\big)\Big) - \mathcal{P}^+\big(\frac{\omega^2}{2}|x'|^2 + c^+\big) \\ &\qquad + \mathcal{P}^-\big(\frac{\omega^2}{2}|x'|^2 + c^-\big) + \mathcal{P}^+(c^+) - \mathcal{P}^-(c^-). \end{aligned} \tag{17}$$

We integrate equations (16) over S_1 by parts, then we get

$$\frac{2\sigma^\pm}{R_0^{\pm 2}} \int_{S_1} r^\pm \, \mathrm{d}\xi = \int_{S_1} (k^\pm + f^\pm) \, \mathrm{d}\xi. \tag{18}$$

The integrals $\int_{S_1} r^{\pm}\,\mathrm{d}\xi$ can be expressed in terms of the differences of the volumes $|\mathcal{F}| - |B_{R_0^-}|$ and $|\mathcal{F}^+| - |B_{R_0^+}|$, $|B_{R_0^{\pm}}| = \frac{4\pi}{3} R_0^{\pm 3}$. For example,

$$|\mathcal{F}| - |B_{R_0^-}| = \frac{1}{3}\int_{S_1} (R^{-3} - R_0^{-3})\,\mathrm{d}\xi = R_0^{-2}\int_{S_1} r^-\,\mathrm{d}\xi + R_0^- \int_{S_1} r^{-2}\,\mathrm{d}\xi + \frac{1}{3}\int_{S_1} r^{-3}\,\mathrm{d}\xi,$$

and hence

$$\begin{aligned} \int_{S_1} r^+\,\mathrm{d}\xi &= \frac{1}{R_0^{+2}}\big(|\mathcal{F}^+| - |B_{R_0^+}|\big) + Q[r^+], \\ \int_{S_1} r^-\,\mathrm{d}\xi &= \frac{1}{R_0^{-2}}\big(|\mathcal{F}| - |B_{R_0^-}|\big) + Q[r^-], \end{aligned} \tag{19}$$

where $Q[r^{\pm}] \equiv -\frac{1}{R_0^{\pm}}\int_{S_1} r^{\pm 2}\,\mathrm{d}\xi - \frac{1}{3R_0^{\pm 2}}\int_{S_1} r^{\pm 3}\,\mathrm{d}\xi$.

We rewrite (12) as follows

$$\begin{aligned} m^+ &= \int_{S_1}\mathrm{d}\xi \int_0^{R^+(\xi)} \frac{\omega^2}{2}|\xi'|^2 s^4\,\mathrm{d}s + c^+|\mathcal{F}^+| = \frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2\,\mathrm{d}\xi + c^+|\mathcal{F}^+|, \\ m^- &= \int_{S_1}\mathrm{d}\xi \int_{R^+(\xi)}^{R^-(\xi)} \frac{\omega^2}{2}|\xi'|^2 s^4\,\mathrm{d}s + c^-|\mathcal{F}^-| = \frac{\omega^2}{10}\int_{S_1} (R^{-5} - R^{+5})|\xi'|^2\,\mathrm{d}\xi + c^-|\mathcal{F}^-|, \end{aligned}$$

where $|\xi'|^2 = \xi_1^2 + \xi_2^2$. On the other hand, $m^+ = \rho^+|B_{R_0^+}|$, $m^- = \rho^-(|B_{R_0^-}| - |B_{R_0^+}|)$. That's why

$$\begin{aligned} 0 &= \frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2\,\mathrm{d}\xi + \rho^+\big(|\mathcal{F}^+| - |B_{R_0^+}|\big) + |\mathcal{F}^+|(c^+ - \rho^+), \\ 0 &= \frac{\omega^2}{10}\int_{S_1} (R^{-5} - R^{+5})|\xi'|^2\,\mathrm{d}\xi + \rho^-\big(|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|\big) + |\mathcal{F}^-|(c^- - \rho^-). \end{aligned} \tag{20}$$

We express $|\mathcal{F}^{\pm}| - |B_{R_0^{\pm}}|$ from (20) and substitute in (19). Then equalities (18) imply that

$$\frac{2\sigma^+}{R_0^{+2}}\Big\{\frac{-1}{\rho^+ R_0^{+2}}\Big(\frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2\,\mathrm{d}\xi + |\mathcal{F}^+|(c^+ - \rho^+)\Big) + Q[r^+]\Big\} = \int_{S_1} (k^+ + f^+)\,\mathrm{d}\xi \tag{21}$$

and

$$\frac{2\sigma^-}{R_0^{-2}}\Big(\frac{1}{R_0^{-2}}\big(|\mathcal{F}^+| + |\mathcal{F}^-| - |B_{R_0^-}| \pm |B_{R_0^+}|\big) + Q[r^-]\Big) = \int_{S_1} (k^- + f^-)\,\mathrm{d}\xi.$$

Finally, we have

$$\begin{aligned} \frac{2\sigma^-}{R_0^{-2}}\Big\{\frac{-1}{\rho^- R_0^{-2}}\Big\{&\frac{\omega^2}{10}\int_{S_1} (R^{-5} - R^{+5})|\xi'|^2\,\mathrm{d}\xi + |\mathcal{F}^-|(c^- - \rho^-) \\ &+ \frac{\rho^-}{\rho^+}\Big(\frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2\,\mathrm{d}\xi + |\mathcal{F}^+|(c^+ - \rho^+)\Big)\Big\} \\ + Q[r^-]\Big\} &= \int_{S_1} (k^- + f^-)\,\mathrm{d}\xi. \end{aligned} \tag{22}$$

In order to prove the solvability of system (16), (13), (21), (22), we use implicit function theorem. We represent this system as a nonlinear vector equation:

$$\mathbf{\Phi}(\boldsymbol{\varphi}) = 0, \tag{23}$$

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where $\mathbf{\Phi}(\boldsymbol{\varphi}) = (\Phi_1^{\pm}, \Phi_2, \Phi_3^{\pm}), \boldsymbol{\varphi} \equiv (r^{\pm}, \omega, \lambda^{\pm}), \lambda^{\pm} \equiv c^{\pm} - \rho^{\pm}$,

$$\begin{aligned}
\Phi_1^{\pm}(\boldsymbol{\varphi}) &= \frac{\sigma^{\pm}}{R_0^{\pm 2}}(\Delta_{S_1} r^{\pm} + 2r^{\pm}) - k^{\pm} - f^{\pm},\\
\Phi_2(\boldsymbol{\varphi}) &= \omega \int_{S_1} |\xi'|^2 \, d\xi \int_0^{R_0^+ + r^+} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + c^+\Big) s^4 \, ds \\
&\quad + \omega \int_{S_1} |\xi'|^2 \, d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + c^-\Big) s^4 \, ds - \beta,\\
\Phi_3^{+}(\boldsymbol{\varphi}) &= \frac{2\sigma^+}{R_0^{+2}}\Big\{\frac{1}{\rho^+ R_0^{+2}}\Big(\frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2 \, d\xi + |\mathcal{F}^+|\lambda^+\Big) - Q[r^+]\Big\}\\
&\quad + \int_{S_1}(k^+ + f^+) \, d\xi,\\
\Phi_3^{-}(\boldsymbol{\varphi}) &= \frac{2\sigma^-}{R_0^{-2}}\Big\{\frac{1}{\rho^- R_0^{-2}}\Big\{\int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \frac{\omega^2}{2}|\xi'|^2 s^4 \, ds + |\mathcal{F}^-|\lambda^-\\
&\qquad + \frac{\rho^-}{\rho^+}\Big(\frac{\omega^2}{10}\int_{S_1} R^{+5}(\xi)|\xi'|^2 \, d\xi + |\mathcal{F}^+|\lambda^+\Big)\Big\}\\
&\qquad - Q[r^-]\Big\} + \int_{S_1}(k^- + f^-) \, d\xi.
\end{aligned}$$

We linearize (23) at zero and show that the derivative of $\mathbf{\Phi}'$ at this point is not equal to zero. Thus,

$$\mathbf{\Phi}'(0)\boldsymbol{\varphi} + \boldsymbol{\psi}(\boldsymbol{\varphi}) = 0, \tag{24}$$

where $\mathbf{\Phi}'(0) = (\Phi_1^{\pm\prime}, \Phi_2', \Phi_3^{\pm\prime})(0)$ is the Fréchet derivative of $\mathbf{\Phi}$, that is,

$$\begin{aligned}
\Phi_1^{+\prime}(0)\boldsymbol{\varphi} &= \frac{\sigma^+}{R_0^{+2}}(\Delta_{S_1} r^+ + 2r^+) + \rho^+\lambda^+ - \rho^-\lambda^-,\\
\Phi_1^{-\prime}(0)\boldsymbol{\varphi} &= \frac{\sigma^-}{R_0^{-2}}(\Delta_{S_1} r^- + 2r^-) + \rho^-\lambda^-,\\
\Phi_2'(0)\boldsymbol{\varphi} &= \omega\rho^+ \int_{S_1} |\xi'|^2 \, d\xi \int_0^{R_0^+} s^4 \, ds + \omega\rho^- \int_{S_1} |\xi'|^2 \, d\xi \int_{R_0^+}^{R_0^-} s^4 \, ds\\
&= \frac{8\pi}{15}\omega\Big\{\rho^+ R_0^{+5} + \rho^-(R_0^{-5} - R_0^{+5})\Big\},\\
\Phi_3^{+\prime}(0)\boldsymbol{\varphi} &= \frac{2\sigma^+}{R_0^{+2}}\frac{|B_{R_0^+}|\lambda^+}{\rho^+ R_0^{+2}} - 4\pi\rho^+\lambda^+ + 4\pi\rho^-\lambda^-\\
&= 4\pi\Big\{\Big(\frac{\mathcal{P}^+(\rho^+) - \mathcal{P}^-(\rho^-)}{3\rho^+} - \rho^+\Big)\lambda^+ + \rho^-\lambda^-\Big\},\\
\Phi_3^{-\prime}(0)\boldsymbol{\varphi} &= \frac{2\sigma^-}{R_0^{-2}}\frac{1}{\rho^- R_0^{-2}}\Big\{(|B_{R_0^-}| - |B_{R_0^+}|)\lambda^- + \frac{\rho^-}{\rho^+}|B_{R_0^+}|\lambda^+\Big\} - 4\pi\rho^-\lambda^-\\
&= 4\pi\Big\{\frac{\mathcal{P}^-(\rho^-)R_0^{+3}}{3\rho^+ R_0^{-3}}\lambda^+ + \Big(\frac{R_0^{-3} - R_0^{+3}}{3\rho^- R_0^{-3}}\mathcal{P}^-(\rho^-) - \rho^-\Big)\lambda^-\Big\},
\end{aligned}$$

and $\boldsymbol{\psi}(\boldsymbol{\varphi}) = \boldsymbol{\Phi}(\boldsymbol{\varphi}) - \boldsymbol{\Phi}'(0)\boldsymbol{\varphi} \equiv (\psi_1^{\pm}, \psi_2, \psi_3^{\pm})$ with

$$\begin{aligned}
\psi_1^+(\boldsymbol{\varphi}) =& \mathcal{P}^+(c^+) - \mathcal{P}^+(\rho^+) - \mathcal{P}^-(c^-) + \mathcal{P}^-(\rho^-) - \rho^+\lambda^+ + \rho^-\lambda^- - f^+,\\
\psi_1^-(\boldsymbol{\varphi}) =& \mathcal{P}^-(c^-) - \mathcal{P}^-(\rho^-) - \rho^-\lambda^- - f^-,\\
\psi_2(\boldsymbol{\varphi}) =& \omega \int_{S_1} |\xi'|^2 \,d\xi \int_0^{R_0^+} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \lambda^+\Big) s^4 \,ds\\
&+ \omega \int_{S_1} |\xi'|^2 \,d\xi \int_{R_0^+}^{R_0^+ + r^+} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + c^+\Big) s^4 \,ds\\
&+ \omega \int_{S_1} |\xi'|^2 \,d\xi \int_{R_0^+}^{R_0^-} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \lambda^-\Big) s^4 \,ds\\
&+ \omega \int_{S_1} |\xi'|^2 \,d\xi \int_{R_0^-}^{R_0^- + r^-} \Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + c^-\Big) s^4 \,ds - \beta,\\
\psi_3^+(\boldsymbol{\varphi}) =& \frac{2\sigma^+}{R_0^{+2}} \Big\{ \frac{1}{\rho^+ R_0^{+2}} \Big(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi)|\xi'|^2 \,d\xi + (|\mathcal{F}^+| - |B_{R_0^+}|)\lambda^+ \Big) - Q[r^+] \Big\}\\
&+ \int_{S_1} (k^+ - \rho^+\lambda^+ + \rho^-\lambda^- + f^+) \,d\xi,\\
\psi_3^-(\boldsymbol{\varphi}) =& \frac{2\sigma^-}{R_0^{-2}} \Big\{ \frac{1}{\rho^- R_0^{-2}} \Big\{ \frac{\omega^2}{10} \int_{S_1} (R^{-5} - R^{+5})|\xi'|^2 \,d\xi + \frac{\rho^-}{\rho^+} \Big(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi)|\xi'|^2 \,d\xi\\
&\quad + (|\mathcal{F}^+| - |B_{R_0^+}|)\lambda^+ \Big) + (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|)\lambda^- \Big\} - Q[r^-] \Big\}\\
&+ \int_{S_1} (k^- - \rho^-\lambda^- + f^-) \,d\xi.
\end{aligned}$$

Let $\mathcal{X} = \tilde{C}^{2+\alpha} \times \tilde{C}^{2+\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{Y} = \tilde{C}^{\alpha} \times \tilde{C}^{\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Using fixed point theorem, we will show that equation (23) is solvable in $\mathcal{X}$.

In [16] (see the corollary to Theorem 2.1), it was proved that for any $f \in \tilde{C}^{\alpha}(S_1)$ the equation

$$\Delta_{S_1} r + 2r = f \tag{25}$$

has a unique solution $r \in \tilde{C}^{2+\alpha}(S_1)$ and

$$|r|_{C^{2+\alpha}(S_1)} \le c_1 |f|_{C^{\alpha}(S_1)}.$$

Moreover, due to the assumptions of the theorem, the determinant of the matrix in the left hand sides of two equations $\Phi_3^{\pm\prime}(0)\boldsymbol{\varphi} = -\psi_3^{\pm}(\boldsymbol{\varphi})$ is not equal to zero:

$$\Big(\frac{2\sigma^+}{3\rho^+ R_0^+} - \rho^+\Big)\Big(\frac{2\sigma^-(R_0^{-3} - R_0^{+3})}{3\rho^- R_0^{-4}} - \rho^-\Big) - \frac{2\sigma^-\rho^- R_0^{+3}}{3\rho^+ R_0^{-4}} \ne 0. \tag{26}$$

(Here we have taken into account relations (8).) Hence, this matrix is invertible. Thus, the whole operator $\boldsymbol{\Phi}'(0) : \mathcal{X} \to \mathcal{Y}$ is also invertible and

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le c_2 \|\boldsymbol{\Phi}'(0)\boldsymbol{\varphi}\|_{\mathcal{Y}},$$

where $\|\boldsymbol{\varphi}\|_{\mathcal{X}} = |r^+|_{C^{2+\alpha}(S_1)} + |r^-|_{C^{2+\alpha}(S_1)} + |\omega| + |\lambda^+| + |\lambda^-|$ and
$\|\boldsymbol{\Phi}'(0)\boldsymbol{\varphi}\|_{\mathcal{Y}} = |\Phi_1^{+\prime}(0)\boldsymbol{\varphi}|_{C^{\alpha}(S_1)} + |\Phi_1^{-\prime}(0)\boldsymbol{\varphi}|_{C^{\alpha}(S_1)} + |\Phi_2'(0)\boldsymbol{\varphi}| + |\Phi_3^{+\prime}(0)\boldsymbol{\varphi}| + |\Phi_3^{-\prime}(0)\boldsymbol{\varphi}|$.
Therefore, equation (24) can be written in the form:

$$\boldsymbol{\varphi} + \mathcal{A}\boldsymbol{\psi}(\boldsymbol{\varphi}) = 0, \qquad \mathcal{A} = \boldsymbol{\Phi}'(0)^{-1}. \tag{27}$$

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Let us now estimate the non-linear operator $\boldsymbol{\psi}$. The term f^- given by (17) can be written as

$$f^- = -\sigma^- \int_0^1 (1-s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{H}^-[sr]\,\mathrm{d}s - \mathcal{P}^{-\prime}(\bar{c}^-)\frac{\omega^2}{2}R^{+2}(\xi)|\xi'|^2, \tag{28}$$

where $\mathcal{H}^-[sr]$ is twice the mean curvature of the surface $\mathcal{G}_s^-$; $\mathcal{G}_s^\pm = \{x = R_0^\pm\xi + sr^\pm(\xi)\boldsymbol{N},$ $\xi \in S_1\}$, $\boldsymbol{N} \equiv \frac{\xi}{|\xi|}$; $\bar{c}^-$ is some point of the interval $[c^-, \frac{\omega^2}{2}R^{+2}(\xi)|\xi'|^2 + c^-]$. One can write a similar formula for f^+. The terms $f^\pm$ have the second order of smallness with respect to $\boldsymbol{\varphi}$. After simple calculations for two values of the arguments $\boldsymbol{\varphi} = (r^\pm, \omega, \lambda^\pm)$ and $\boldsymbol{\varphi}' = (r^{\pm\prime}, \omega', \lambda^{\pm\prime}) \in \mathcal{X}$ such that

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}}, \|\boldsymbol{\varphi}'\|_{\mathcal{X}} \le \eta, \qquad \eta \le \min\{1, R_0^+/2, (R_0^- - R_0^+)/2\},$$

it can be shown that the estimates

$$\|\boldsymbol{\psi}(\varphi)\|_{\mathcal{Y}} \le c\|\boldsymbol{\varphi}\|_{\mathcal{X}}^{1+\alpha} + |\beta| \le c_3\eta\|\boldsymbol{\varphi}\|_{\mathcal{X}}^{\alpha} + |\beta|,$$
$$\|\boldsymbol{\psi}(\boldsymbol{\varphi}) - \boldsymbol{\psi}(\boldsymbol{\varphi}')\|_{\mathcal{Y}} \le c_4\eta^\alpha\|\boldsymbol{\varphi} - \boldsymbol{\varphi}'\|_{\mathcal{X}}$$

hold. This implies that $\mathcal{A}\boldsymbol{\psi}(\boldsymbol{\varphi})$ maps the ball $\{\boldsymbol{\varphi} \in \mathcal{X} : \|\boldsymbol{\varphi}\|_{\mathcal{X}} \le \eta\}$ into itself and it is a contraction operator if

$$c_2(c_3\eta^{1+\alpha} + |\beta|) \le \eta, \qquad c_2c_4\eta^\alpha < 1.$$

These inequalities are satisfied for η such that

$$2c_2|\beta| \le \eta \le \min\{(2c_2c_3)^{-1/\alpha}, (c_2c_4)^{-1/\alpha}\}. \tag{29}$$

So if

$$|\beta| \le \frac{1}{2c_2}\min\{(2c_2c_3)^{-1/\alpha}, (c_2c_4)^{-1/\alpha}\}, \tag{30}$$

then, by fixed point theorem, equation (27) has a unique solution in the ball $\{\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le \eta\}$ with η satisfying (29). Thus, equation (23) is also uniquely solvable for β obeying inequality (30), which gives the estimate of ε in condition (14).

We now prove estimate (15). Since

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le \|\mathcal{A}\|\|\boldsymbol{\psi}(\boldsymbol{\varphi})\|_{\mathcal{Y}} \le c_2(c_3\eta^\alpha\|\boldsymbol{\varphi}\|_{\mathcal{X}} + |\beta|),$$

then for $c_2c_3\eta^\alpha \le \frac{1}{2}$

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le 2c_2|\beta|,$$

which coincides with inequality (15). □

Thus, we have shown that, for certain data of the problem, there are equilibrium figures for a two-layer compressible fluid, pressure function being $\mathcal{P}^\pm = \frac{\rho^2(x)}{2} + p^\pm$ and the density ρ being determined by formulas (10).

3. General case

Now, we study the general case of pressure function. We assume only that $\mathcal{P}'(\rho)$ is positive and may have a jump across $\mathcal{G}^+$.

Velocity vector field remains the same as above:

$$\boldsymbol{\mathcal{V}}(x) = \omega(-x_2, x_1, 0).$$

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It satisfies (4) together with pressure function gradient $\nabla\mathcal{P}(\rho)=\rho\frac{\omega^2}{2}\nabla|x'|^2$. We introduce $\mathcal{Q}(\rho)$ such that

$$\nabla\mathcal{Q}(\rho)\equiv\frac{\mathcal{P}'(\rho)\nabla\rho}{\rho}=\nabla\Big(\frac{\omega^2}{2}|x'|^2\Big). \tag{31}$$

The function $\mathcal{Q}(\rho)=\int_{\rho_1}^{\rho}\frac{\mathcal{P}'(s)}{s}\,\mathrm{d}s$, $\rho_1\geq 0$. Since $\mathcal{Q}'(\rho)=\frac{\mathcal{P}'(\rho)}{\rho}>0$, there is an inverse function $\mathcal{Q}^{-1}$. And (31) implies

$$\rho=\mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|x'|^2+C^{\pm}\Big)\ \text{in}\ \mathcal{F}^{\pm} \tag{32}$$

with arbitrary constants $C^{\pm}$.

Substituting ρ into equations (9), we have

$$\begin{aligned}&\sigma^-\mathcal{H}^-(x)+\mathcal{P}^-\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^-\big)\Big)=0,\quad x\in\mathcal{G}^-,\\&\sigma^+\mathcal{H}^+(x)+\mathcal{P}^+\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^+\big)\Big)-\mathcal{P}^-\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^-\big)\Big)=0,\quad x\in\mathcal{G}^+.\end{aligned} \tag{33}$$

We prescribe the masses of fluids and apply (32):

$$m^{\pm}=\int_{\mathcal{F}^{\pm}}\rho(x)\,\mathrm{d}x\equiv\int_{\mathcal{F}^{\pm}}\mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|x'|^2+C^{\pm}\Big)\,\mathrm{d}x. \tag{34}$$

Then one has the equations for determining the constants $C^{\pm}$.

Similar, a given angular momentum β defines the angular velocity ω:

$$\beta\equiv\int_{\mathcal{F}^+\cup\mathcal{F}^-}\rho(x)\boldsymbol{\mathcal{V}}\cdot\boldsymbol{\eta}_3\,\mathrm{d}x=\omega\int_{\cup\mathcal{F}^{\pm}}\mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|x'|^2+C^{\pm}\Big)|x'|^2\,\mathrm{d}x. \tag{35}$$

Let us state the main theorem.

Theorem 2. *Let $\alpha>0$, $\alpha\notin\mathbb{N}$, and let $\mathcal{P}^{\pm}(\rho)\in C^{\alpha}(\mathbb{R}_+)$ be positive increasing functions such that equalities (2) are satisfied for it. Here $\mathbb{R}_+\equiv\{x\in\mathbb{R}|x>0\}$. We assume also that the data of problem (4), (5) are subjected to condition (47). Then for an arbitrary β satisfying the estimate*

$$|\beta|<\varepsilon \tag{36}$$

with ε small enough, there exists a unique solution $(R^{\pm},\omega,C^{\pm})\in\tilde{C}^{2+\alpha}(S_1)\times\tilde{C}^{2+\alpha}(S_1)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ to system (33)–(35), and the inequality

$$\sum_{\pm}\{|R^{\pm}-R_0^{\pm}|_{C^{2+\alpha}(S_1)}+|\mathcal{Q}^{-1}(C^{\pm})-\rho^{\pm}|\}+|\omega|<c|\beta| \tag{37}$$

holds.

Proof. After linearisation, system (33) takes the form of (16) with

$$\begin{aligned}k^-&=\frac{2\sigma^-}{R_0^-}-\mathcal{P}^-(\mathcal{Q}^{-1}(C^-))\equiv\mathcal{P}^-(\rho^-)-\mathcal{P}^-(\mathcal{Q}^{-1}(C^-)),\\k^+&=\frac{2\sigma^+}{R_0^+}-\mathcal{P}^+(\mathcal{Q}^{-1}(C^+))+\mathcal{P}^-(\mathcal{Q}^{-1}(C^-))\\&\equiv\mathcal{P}^+(\rho^+)-\mathcal{P}^+(\mathcal{Q}^{-1}(C^+))+\mathcal{P}^-(\mathcal{Q}^{-1}(C^-))-\mathcal{P}^-(\rho^-)\end{aligned} \tag{38}$$

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due to (2) and

$$\begin{aligned} f^- &= \sigma^-\Big(\delta_0\big(\mathcal{H}^-(x)+\frac{2}{R_0^-}\big)-\big(\mathcal{H}^-(x)+\frac{2}{R_0^-}\big)\Big)-\mathcal{P}^-\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^-\big)\Big) \\ &\quad+\mathcal{P}^-\big(\mathcal{Q}^{-1}(C^-)\big), \\ f^+ &= \sigma^+\Big(\delta_0\big(\mathcal{H}^+(x)+\frac{2}{R_0^+}\big)-\big(\mathcal{H}^+(x)+\frac{2}{R_0^+}\big)\Big)-\mathcal{P}^+\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^+\big)\Big) \\ &\quad+\mathcal{P}^-\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|x'|^2+C^-\big)\Big)+\mathcal{P}^+\big(\mathcal{Q}^{-1}(C^+)\big)-\mathcal{P}^-\big(\mathcal{Q}^{-1}(C^-)\big). \end{aligned} \tag{39}$$

Following Sect. 2, we obtain (18) by integrating equations (16) by parts but now with new $k^\pm$ (38) and $f^\pm$ (39). We substitute (19) in equality (18):

$$\begin{aligned} &\frac{2\sigma^+}{R_0^{+2}}\Big(\frac{1}{R_0^{+2}}\big(|\mathcal{F}^+|-|B_{R_0^+}|\big)+\widetilde{Q}[r^+]\Big)=\int_{S_1}(k^++f^+)\,d\xi, \\ &\frac{2\sigma^-}{R_0^{-2}}\Big(\frac{1}{R_0^{-2}}\big(|\mathcal{F}^+|+|\mathcal{F}^-|-|B_{R_0^-}|\pm|B_{R_0^+}|\big)+\widetilde{Q}[r^-]\Big)=\int_{S_1}(k^-+f^-)\,d\xi, \end{aligned} \tag{40}$$

where $\widetilde{Q}[r^\pm]\equiv-\frac{1}{R_0^\pm}\int_{S_1}r^{\pm2}\,d\xi-\frac{1}{3R_0^{\pm2}}\int_{S_1}r^{\pm3}\,d\xi$.

We can write (34) as follows

$$\begin{aligned} m^+ &= \int_{S_1}d\xi\int_0^{R^+(\xi)}\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2s^2+C^+\big)-\mathcal{Q}^{-1}(C^+)\Big)s^2\,ds+\mathcal{Q}^{-1}(C^+)|\mathcal{F}^+|, \\ m^- &= \int_{S_1}d\xi\int_{R^+(\xi)}^{R^-(\xi)}\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2s^2+C^-\big)-\mathcal{Q}^{-1}(C^-)\Big)s^2\,ds+\mathcal{Q}^{-1}(C^-)|\mathcal{F}^-|. \end{aligned}$$

In view of (3), we have

$$\begin{aligned} 0 &= \int_{S_1}d\xi\int_0^{R^+(\xi)}\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2s^2+C^+\big)-\mathcal{Q}^{-1}(C^+)\Big)s^2\,ds \\ &\qquad+\rho^+\big(|\mathcal{F}^+|-|B_{R_0^+}|\big)+|\mathcal{F}^+|\big(\mathcal{Q}^{-1}(C^+)-\rho^+\big), \\ 0 &= \int_{S_1}d\xi\int_{R^+(\xi)}^{R^-(\xi)}\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2s^2+C^-\big)-\mathcal{Q}^{-1}(C^-)\Big)s^2\,ds \\ &\qquad+\rho^-\big(|\mathcal{F}^-|-|B_{R_0^-}|+|B_{R_0^+}|\big)+|\mathcal{F}^-|\big(\mathcal{Q}^{-1}(C^-)-\rho^-\big). \end{aligned} \tag{41}$$

By expressing the differences $|\mathcal{F}^+|-|B_{R_0^+}|$ and $|\mathcal{F}^-|-|B_{R_0^-}|+|B_{R_0^+}|$ from equalities (41) and substituting them in (40), we arrive at

$$\begin{aligned} &\frac{2\sigma^+}{R_0^{+2}}\Big\{\frac{-1}{\rho^+R_0^{+2}}\Big(\int_{S_1}d\xi\int_0^{R^+(\xi)}\Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2s^2+C^+\big)-\mathcal{Q}^{-1}(C^+)\Big)s^2\,ds \\ &\qquad+|\mathcal{F}^+|\big(\mathcal{Q}^{-1}(C^+)-\rho^+\big)\Big)+\widetilde{Q}[r^+]\Big\}=\int_{S_1}(k^++f^+)\,d\xi \end{aligned} \tag{42}$$

and

$$\frac{2\sigma^-}{R_0^{-2}}\bigg\{\frac{-1}{\rho^- R_0^{-2}}\bigg\{\int_{S_1} d\xi \int_{R^+(\xi)}^{R^-(\xi)} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-\big) - \mathcal{Q}^{-1}(C^-)\Big)s^2\, ds$$
$$+ |\mathcal{F}^-|\big(\mathcal{Q}^{-1}(C^-) - \rho^-\big)$$
$$+ \frac{\rho^-}{\rho^+}\Big(\int_{S_1} d\xi \int_0^{R^+(\xi)} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\big) - \mathcal{Q}^{-1}(C^+)\Big)s^2\, ds$$
$$+ |\mathcal{F}^+|\big(\mathcal{Q}^{-1}(C^+) - \rho^+\big)\Big)\bigg\} + \widetilde{\mathbb{Q}}[r^-]\bigg\} = \int_{S_1} (k^- + f^-)\, d\xi. \quad (43)$$

Next, we represent system (16), (35), (42), (43) in the form of vector equation (23) with $\mathbf{\Phi}(\boldsymbol{\varphi}) = (\Phi_1^\pm, \Phi_2, \Phi_3^\pm)$, $\boldsymbol{\varphi} \equiv (r^\pm, \omega, \lambda^\pm)$, $\lambda^\pm \equiv \mathcal{Q}^{-1}(C^\pm) - \rho^\pm$,

$$\Phi_1^\pm(\boldsymbol{\varphi}) = \frac{\sigma^\pm}{R_0^{\pm 2}}(\Delta_{S_1} r^\pm + 2r^\pm) - k^\pm - f^\pm,$$

$$\Phi_2(\boldsymbol{\varphi}) = \omega \int_{S_1} |\xi'|^2\, d\xi \int_0^{R_0^+ + r^+} \mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\Big)s^4\, ds$$
$$+ \omega \int_{S_1} |\xi'|^2\, d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-\Big)s^4\, ds - \beta, \quad (44)$$

$$\Phi_3^+(\boldsymbol{\varphi}) = \frac{2\sigma^+}{R_0^{+2}}\bigg\{\frac{1}{\rho^+ R_0^{+2}}\Big(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\big) - \mathcal{Q}^{-1}(C^+)\Big)s^2\, ds$$
$$+ |\mathcal{F}^+|\lambda^+\Big) - \widetilde{\mathbb{Q}}[r^+]\bigg\} + \int_{S_1} (k^+ + f^+)\, d\xi,$$

$$\Phi_3^-(\boldsymbol{\varphi}) = \frac{2\sigma^-}{R_0^{-2}}\bigg\{\frac{1}{\rho^- R_0^{-2}}\bigg\{\int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-\big) - \mathcal{Q}^{-1}(C^-)\Big)s^2\, ds + |\mathcal{F}^-|\lambda^-$$
$$+ \frac{\rho^-}{\rho^+}\Big(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\big) - \mathcal{Q}^{-1}(C^+)\Big)s^2\, ds$$
$$+ |\mathcal{F}^+|\lambda^+\Big)\bigg\} - \widetilde{\mathbb{Q}}[r^-]\bigg\} + \int_{S_1} (k^- + f^-)\, d\xi.$$

Then we apply again implicit function theorem to (23). To this end, we calculate the Fréchet derivative of $\mathbf{\Phi}$ at zero and linearize (23) at this point. As a result, one has

$$\mathbf{\Phi}'(0)\boldsymbol{\varphi} + \boldsymbol{\psi}(\boldsymbol{\varphi}) = 0 \quad (45)$$

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with $\mathbf{\Phi}'(0) = (\Phi_1^{\pm\prime}, \Phi_2', \Phi_3^{\pm\prime})(0)$,

$$\begin{aligned}
\Phi_1^{+\prime}(0)\boldsymbol{\varphi} &= \frac{\sigma^+}{R_0^{+2}}(\Delta_{S_1} r^+ + 2r^+) + \mathcal{P}^{+\prime}(\rho^+)\lambda^+ - \mathcal{P}^{-\prime}(\rho^-)\lambda^-,\\
\Phi_1^{-\prime}(0)\boldsymbol{\varphi} &= \frac{\sigma^-}{R_0^{-2}}(\Delta_{S_1} r^- + 2r^-) + \mathcal{P}^{-\prime}(\rho^-)\lambda^-,\\
\Phi_2'(0)\boldsymbol{\varphi} &= \omega\rho^+ \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_0^{R_0^+} s^4 \,\mathrm{d}s + \omega\rho^- \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_{R_0^+}^{R_0^-} s^4 \,\mathrm{d}s\\
&= \frac{8\pi}{15}\omega\Big\{\rho^+ R_0^{+5} + \rho^-(R_0^{-5} - R_0^{+5})\Big\},\\
\Phi_3^{+\prime}(0)\boldsymbol{\varphi} &= \frac{2\sigma^+}{R_0^{+2}}\frac{|B_{R_0^+}|\lambda^+}{\rho^+ R_0^{+2}} - 4\pi\mathcal{P}^{+\prime}(\rho^+)\lambda^+ + 4\pi\mathcal{P}^{-\prime}(\rho^-)\lambda^- \\
&= 4\pi\Big\{\Big(\frac{2\sigma^+}{3\rho^+ R_0^+} - \mathcal{P}^{+\prime}(\rho^+)\Big)\lambda^+ + \mathcal{P}^{-\prime}(\rho^-)\lambda^-\Big\},\\
\Phi_3^{-\prime}(0)\boldsymbol{\varphi} &= \frac{2\sigma^-}{R_0^{-2}}\frac{1}{\rho^- R_0^{-2}}\Big\{(|B_{R_0^-}| - |B_{R_0^+}|)\lambda^- + \frac{\rho^-}{\rho^+}|B_{R_0^+}|\lambda^+\Big\} - 4\pi\mathcal{P}^{-\prime}(\rho^-)\lambda^-\\
&= 4\pi\Big\{\frac{2\sigma^- R_0^{+3}}{3\rho^+ R_0^{-4}}\lambda^+ + \Big(\frac{2\sigma^-(R_0^{-3} - R_0^{+3})}{3\rho^- R_0^{-4}} - \mathcal{P}^{-\prime}(\rho^-)\Big)\lambda^-\Big\},
\end{aligned} \tag{46}$$

and $\boldsymbol{\psi}(\boldsymbol{\varphi}) = \mathbf{\Phi}(\boldsymbol{\varphi}) - \mathbf{\Phi}'(0)\boldsymbol{\varphi} \equiv (\psi_1^{\pm}(\boldsymbol{\varphi}), \psi_2(\boldsymbol{\varphi}), \psi_3^{\pm}(\boldsymbol{\varphi}))$, where

$$\begin{aligned}
\psi_1^+(\boldsymbol{\varphi}) =& \mathcal{P}^+(\mathcal{Q}^{-1}(C^+)) - \mathcal{P}^+(\rho^+) - \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) + \mathcal{P}^-(\rho^-) - \mathcal{P}^{+\prime}(\rho^+)\lambda^+\\
&+ \mathcal{P}^{-\prime}(\rho^-)\lambda^- - f^+,\\
\psi_1^-(\boldsymbol{\varphi}) =& \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) - \mathcal{P}^-(\rho^-) - \mathcal{P}^{-\prime}(\rho^-)\lambda^- - f^-,\\
\psi_2(\boldsymbol{\varphi}) =& \omega \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_0^{R_0^+} \Big\{\mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \mathcal{Q}(\rho^+ + \lambda^+)\Big) - \rho^+\Big\} s^4 \,\mathrm{d}s\\
&+ \omega \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_{R_0^+}^{R_0^+ + r^+} \mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \mathcal{Q}(\rho^+ + \lambda^+)\Big) s^4 \,\mathrm{d}s\\
&+ \omega \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_{R_0^+}^{R_0^-} \Big\{\mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \mathcal{Q}(\rho^- + \lambda^-)\Big) - \rho^-\Big\} s^4 \,\mathrm{d}s\\
&+ \omega \int_{S_1} |\xi'|^2 \,\mathrm{d}\xi \int_{R_0^-}^{R_0^- + r^-} \mathcal{Q}^{-1}\Big(\frac{\omega^2}{2}|\xi'|^2 s^2 + \mathcal{Q}(\rho^- + \lambda^-)\Big) s^4 \,\mathrm{d}s - \beta,\\
\psi_3^+(\boldsymbol{\varphi}) =& \frac{2\sigma^+}{R_0^{+2}}\Big\{\frac{1}{\rho^+ R_0^{+2}}\Big(\int_{S_1} \mathrm{d}\xi \int_0^{R_0^+ + r^+} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\big) - \mathcal{Q}^{-1}(C^+)\Big) s^2 \,\mathrm{d}s\\
&+ (|\mathcal{F}^+| - |B_{R_0^+}|)\lambda^+\Big) - \tilde{\mathcal{Q}}[r^+]\Big\}\\
&+ \int_{S_1} (k^+ - \mathcal{P}^{+\prime}(\rho^+)\lambda^+ + \mathcal{P}^{-\prime}(\rho^-)\lambda^- + f^+) \,\mathrm{d}\xi,
\end{aligned}$$

$$\psi_3^-(\boldsymbol{\varphi}) = \frac{2\sigma^-}{R_0^{-2}}\Big\{\frac{1}{\rho^- R_0^{-2}}\Big\{\int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-\big) - \mathcal{Q}^{-1}(C^-)\Big) s^2\, ds$$

$$+ \frac{\rho^-}{\rho^+}\Big(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \Big(\mathcal{Q}^{-1}\big(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+\big) - \mathcal{Q}^{-1}(C^+)\Big) s^2\, ds$$

$$+ (|\mathcal{F}^+| - |B_{R_0^+}|)\lambda^+\Big) + (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|)\lambda^-\Big\} - \widetilde{Q}[r^-]\Big\}$$

$$+ \int_{S_1} (k^- - \mathcal{P}^{-\prime}(\rho^-)\lambda^- + f^-)\, d\xi.$$

We recall the notation: $\mathcal{X} = \tilde{C}^{2+\alpha} \times \tilde{C}^{2+\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{Y} = \tilde{C}^{\alpha} \times \tilde{C}^{\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. By means of fixed point theorem, we prove the solvability of system (23) with (44) in $\mathcal{X}$.

The existence of a unique solution $r \in \tilde{C}^{2+\alpha}(S_1)$ to two first equations in (25) and the estimate for it are discussed in Sec. 2. The operator $\Phi_2'(0)$ is invertible. In addition, we have assumed that the determinant of two equations $\Phi_3^{\pm\prime}(0)\boldsymbol{\varphi} = -\psi_3^{\pm}(\boldsymbol{\varphi})$ with relations (46) is not equal to zero:

$$\Big(\frac{2\sigma^+}{3\rho^+ R_0^+} - \mathcal{P}^{+\prime}(\rho^+)\Big)\Big(\frac{2\sigma^-(R_0^{-3} - R_0^{+3})}{3\rho^- R_0^{-4}} - \mathcal{P}^{-\prime}(\rho^-)\Big) - \frac{2\sigma^- \mathcal{P}^{-\prime}(\rho^-) R_0^{+3}}{3\rho^+ R_0^{-4}} \neq 0. \tag{47}$$

Therefore, the vector value operator $\boldsymbol{\Phi}'(0) : \mathcal{X} \to \mathcal{Y}$ is invertible too and the solution obeys the inequality

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le c_2 \|\boldsymbol{\Phi}'(0)\boldsymbol{\varphi}\|_{\mathcal{Y}},$$

where $\|\boldsymbol{\varphi}\|_{\mathcal{X}} = |r^+|_{C^{2+\alpha}(S_1)} + |r^-|_{C^{2+\alpha}(S_1)} + |\omega| + |\lambda^+| + |\lambda^-|$,
$\|\boldsymbol{\Phi}'(0)\boldsymbol{\varphi}\|_{\mathcal{Y}} = |\Phi_1^{+\prime}(0)\boldsymbol{\varphi}|_{C^{\alpha}(S_1)} + |\Phi_1^{-\prime}(0)\boldsymbol{\varphi}|_{C^{\alpha}(S_1)} + |\Phi_2'(0)\boldsymbol{\varphi}| + |\Phi_3^{+\prime}(0)\boldsymbol{\varphi}| + |\Phi_3^{-\prime}(0)\boldsymbol{\varphi}|$.
Hence, one can write equation (45) in the form (27).

We estimate the operator $\boldsymbol{\psi}$ in the same way as in Sec. 2. Using formulas similar to (28) for the new functions $f^{\pm}$, we conclude again that

$$\|\boldsymbol{\psi}(\varphi)\|_{\mathcal{Y}} \le c\|\boldsymbol{\varphi}\|_{\mathcal{X}}^{1+\alpha} + |\beta| \le c_3\eta\|\boldsymbol{\varphi}\|_{\mathcal{X}}^{\alpha} + |\beta|,$$
$$\|\boldsymbol{\psi}(\boldsymbol{\varphi}) - \boldsymbol{\psi}(\boldsymbol{\varphi}')\|_{\mathcal{Y}} \le c_4\eta^{\alpha}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}'\|_{\mathcal{X}}$$

for two values of the arguments $\boldsymbol{\varphi} = (r^{\pm}, \omega, \lambda^{\pm})$ and $\boldsymbol{\varphi}' = (r^{\pm\prime}, \omega', \lambda^{\pm\prime}) \in \mathcal{X}$ such that

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}}, \|\boldsymbol{\varphi}'\|_{\mathcal{X}} \le \eta, \qquad \eta \le \min\{1, R_0^+/2, (R_0^- - R_0^+)/2\}.$$

By repeating the arguments of Sec. 2, we deduce that $\mathcal{A}\boldsymbol{\psi}(\boldsymbol{\varphi})$ is a contraction operator if inequality (30) holds. Hence, fixed point theorem guarantees the existence of a unique solution to equation (27) in the ball $\{\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le \eta\}$ with η satisfying (29). Therefore (23) with $\boldsymbol{\Phi}(\varphi)$ given by (44) is also uniquely solvable for β such that condition (30) holds. This inequality implies an estimate for ε in (36).

Estimate (37) follows from

$$\|\boldsymbol{\varphi}\|_{\mathcal{X}} \le \|\mathcal{A}\|\|\boldsymbol{\psi}(\boldsymbol{\varphi})\|_{\mathcal{Y}} \le c_2\big(c_3\eta^{\alpha}\|\boldsymbol{\varphi}\|_{\mathcal{X}} + |\beta|\big),$$

if $c_2c_3\eta^{\alpha} \le \frac{1}{2}$. □

Conclusions

Thus, we have shown the existence of axially symmetric equilibrium figures for a two-component compressible fluid when pressure function is given by a smooth growing function of fluid density and the data satisfy some condition. Under a small enough angular momentum these equilibrium figures are close to embedded balls. The next stage of investigation of the problem will consist in proving the existence of a global solution to the nonstationary problem for small initial data and its tendency to the stationary solution $(\mathcal{V}, \rho)$, as well as in studying the stability of equilibrium figures obtained. A new paper will deal with this investigation.

In addition, we note that our analysis has been carried out for the case of neglecting the gravity of liquids. This situation is realized in space, and our two-layer gas cloud can be considered, for example, as a gaseous planet or another cosmic rotating body.

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