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Certain Results on Subclasses of Analytic and Bi-Univalent Functions Associated with Coefficients Estimates and Quasi-Subordination

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Abstract: The purpose of the present paper is to introduce and investigate new subclasses of analytic function class of bi univalent function defined in open unit disk connected with a linear q-convolution operator, which are associated with the quasi-subordination. We find coefficients estimate $|h_2|, |h_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out.

Keywords: analytic functions, univalent function; fractional derivatives; convolution (q-derivatives); quasi–subordination; coefficient estimate

Mathematics Subject Classification: 30C45; 30C50

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1. Introduction

The theory of q-calculus plays an important rôle in many areas of mathematical physical and engineering sciences. Jackson (see [11] and [10]) was the first to have some applications of the q-calculus and introduced the q-analogue of the classical derivative and integral operators (see also [30]).

Let $\mathcal{A}$ be the class of analytic functions $\mathcal{T}$ in an open unit disk $\mathfrak{U} = \{\varepsilon \in \mathbb{C} : |\varepsilon| < 1\}$ of the form:

$$\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} a_j \varepsilon^j, \quad (\varepsilon \in \mathfrak{U}). \qquad (1.1)$$

and satisfying the normalization conditions (see [1]): $\mathcal{T}(0) = \mathcal{T}'(0) - 1 = 0$

Assume that $\sum_{\mathfrak{U}}$ denotes the class of all functions in $\mathcal{A}$ defined by (1.1), which are univalent in $\mathfrak{U}$.

The well-known Koebe-One Quarter Theorem [5] states that the image of the open unit disk $\mathfrak{U}$ under each univalent function in a disk with the radius $\frac{1}{4}$. Thus, every univalent function $\mathcal{T}$ has an inverse $\mathcal{T}^{-1}$, such that

$$\mathcal{T}^{-1}(\mathcal{T}(\varepsilon)) = \varepsilon \quad , (z \in \mathfrak{U}),$$

and

$$\mathcal{T}(\mathcal{T}^{-1}(\varsigma)) = \varsigma \quad (|\varsigma| < r_0(\mathcal{T}); r_0(\mathcal{T}) \geq \frac{1}{4}).$$

In fact, the inverse function $\xi = \mathcal{T}^{-1}$ is given by

$$\xi(\varsigma) = \varsigma - a_2\varsigma^2 + (2a_2^2 - a_3)\varsigma^3 - (5a_2^3 - 5a_2a_3 + a_4)\varsigma^4 + \cdots. \qquad (1.3)$$
$$= \varsigma + \sum_{n=2}^{\infty} A_n \varsigma^n$$

The function $\mathcal{T} \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ are univalent functions in $\mathfrak{U}$ given by (1.1).

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The class of bi-univalent functions was introduced by Lewin [14] and proved that $|a_2| \leq 1.51$ for the function of the form (1.1). Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Later Netanyahu *in* [17] proved that $\max_{\mathcal{T}\in\sum}|a_2| = \frac{4}{3}$. Also several authors studied classes of bi-univalent analytic functions and found estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in these classes [For two analytic functions $\mathcal{T}$ and ξ, $\mathcal{T}$ is quasi-subordinate to ξ, written as follows:

$$\mathcal{T}(\varepsilon) \prec_q \xi(\varepsilon) \quad (\varepsilon \in \mathfrak{U}) \tag{1.3}$$

if there exist analytic functions $h(\varepsilon)$ and $\hbar(\varepsilon)$, with $|h(z)| \leq 1$, $\hbar(0) = 0$ and $|\hbar(\varepsilon)| < 1$, $(\varepsilon \in \mathfrak{U})$, such that

$$\mathcal{T}(\varepsilon) = h(\varepsilon)\xi(\hbar(\varepsilon)), (\varepsilon \in \mathfrak{U}).$$

Note that if ($h(\varepsilon) = 1$), then $\mathcal{T}(\varepsilon) = \xi(\hbar(\varepsilon))$, hance $\mathcal{T}(\varepsilon) \prec \xi(\varepsilon)$ $(z \in \mathfrak{U})$. If ξ *be* univalent in $\mathfrak{U}$, *then* $\mathcal{T} \prec \xi$ if and only if $\mathcal{T}(0) = \xi(0)$ and $\mathcal{T}(\mathfrak{U}) \subset \xi(\mathfrak{U})$.

For the functions $\mathcal{T}, \rho \in \sum_{\mathfrak{U}}$ defined by

$$\mathcal{T}(\varepsilon) = \sum_{j=1}^{\infty} a_j\varepsilon^j \text{ and } \rho(\varepsilon) = \sum_{j=1}^{\infty} h_j\varepsilon^j \ (\varepsilon \in \mathfrak{U}),$$

the convolution of $\mathcal{T}$ and ρ denoted by $\mathcal{T} * \rho$ is

$$(\mathcal{T} * \rho)(\varepsilon) = \sum_{j=1}^{\infty} a_j h_j \varepsilon^j = (\rho * \mathcal{T})(\varepsilon) \qquad (\varepsilon \in \mathfrak{U}).$$

To start with, we recall the following differential and integral operators. For $0 < q < 1$, El-Deeb et al. [8,24] defined the q-convolution operator (see also [10]) for $\mathcal{T} * \rho$ by

$$\mathfrak{Q}_q(\mathcal{T} * \rho)(\varepsilon) = \mathfrak{Q}_q\left(\varepsilon + \sum_{j=2}^{\infty} a_j h_j \varepsilon^j\right)$$

$$\frac{(\mathcal{T} * \rho)(\varepsilon) - (\mathcal{T} * \rho)(q\varepsilon)}{\varepsilon(1-q)} = 1 + \sum_{j=2}^{\infty}[j]_q\, a_j h_j \varepsilon^{j-1}, \varepsilon \in \mathfrak{U},$$

where

$$[j]_q = \frac{1-q^j}{1-q} = 1 + \sum_{j=1}^{j-1} q^j, [0]_q = 0. \tag{1.4}$$

We used the linear operator $\mathcal{Y}_\rho^{\zeta,q}: \mathcal{A} \to \mathcal{A}$ according to El-Deeb et al. [8] (see also [25]) for and $\zeta > -1, 0 < q < 1$. If

$$\mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon) * \mathbb{I}_q^{\zeta+1}(\varepsilon) = \varepsilon\mathfrak{Q}_q(\mathcal{T} * \rho)(\varepsilon), \varepsilon \in \mathfrak{U},$$

where $\mathbb{I}_q^{\zeta+1}$ is given by

$$\mathbb{I}_q^{\zeta+1}(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} \frac{[\zeta+1]_{q,\varepsilon-1}}{[\varepsilon-1]_q!}\varepsilon^j, \varepsilon \in \mathfrak{U},$$

then

$$\mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} a_j h_j\, \varepsilon^j (\zeta > -1, 0 < q < 1, \varepsilon \in \mathfrak{U}). \tag{1.5}$$

Using the operator $\mathcal{Y}_\rho^{\zeta,q}$, we define a new operator as follows:

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,0}\mathcal{T}(\varepsilon) = \mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon)$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,1}\mathcal{T}(\varepsilon) = (\sigma-\vartheta)\varepsilon^3\left(\mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon)\right)''' + (1+2(\sigma-\vartheta))\varepsilon^2\left(\mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon)\right)'' + \varepsilon\left(\mathcal{Y}_\rho^{\zeta,q}\mathcal{T}(\varepsilon)\right)' \tag{1.6}$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon) =$$

$$(\sigma-\vartheta)\varepsilon^3\left(\mathcal{Y}_\rho^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)''' + (1+2(\sigma-\vartheta))\varepsilon^2\left(\mathcal{Y}_\rho^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)'' + \varepsilon\left(\mathcal{Y}_\rho^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)'$$

$$= \varepsilon + \sum_{j=2}^{\infty} j^{2n}((\sigma-\vartheta)(j-1)+1)^n \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} a_j h_j\, \varepsilon^j$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} \psi_j h_j\, \varepsilon^j \left(\begin{matrix}\zeta > -1, 0 < q < 1, \vartheta \geq 0, \sigma > 0, \sigma \neq \vartheta, \\ n \in \mathbb{N}_0 \cup \{0\} \text{and } \varepsilon \in \mathfrak{U}\end{matrix}\right), \tag{1.7}$$

where

$$\psi_j = j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} a_j\,,$$

and by [10] ,let $0 < q < 1$ and $[j]_q$ is defined by $[j]_q = \frac{1-q^j}{1-q} = 1+\sum_{j=1}^{j-1} q^j$, $[0]_q = 0$.

The q - number shift factorial is given by

$$[j]_q! = \begin{cases}[j]_q\,[j-1]_q \ldots [2]_q\,[1]_q, & \text{if } j = 1,2,3,\ldots,\\ 1, & if\ j = 0,\end{cases}$$

From the definition relation (1.5), we get

$$(i)[\zeta+1]_q\ \mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon) = [\zeta]_q\ \mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta+1,q,n}\mathcal{T}(\varepsilon) + q^\zeta \varepsilon\, \mathfrak{D}_q\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta+1,q,n}\mathcal{T}(\varepsilon)\right), \varepsilon \in \mathfrak{U}; \qquad (1.8)$$

$$(ii)\mathcal{R}_{\rho,\sigma,\vartheta}^{\zeta,n}\mathcal{T}(\varepsilon) = \lim_{q\to 1^-} \mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \frac{[j]_q!}{[\zeta]_{q,\epsilon-1}} a_j\, h_j\, \varepsilon^j \quad (1.9)$$

The q –generalized Pochhammersymbol is defined by $[\zeta]_{q,\epsilon-1} = \frac{\Gamma_q(\zeta+\epsilon-1)}{\Gamma_q(\zeta)}$, $\epsilon - 1 \in \mathbb{N}$, $\zeta \in \mathbb{N}$.

For, $q \to 1-$, then $[\zeta]_{q,\epsilon-1}$ reduces to $(\zeta)_{\epsilon-1} = \frac{\Gamma(\zeta+\epsilon-1)}{\Gamma(\zeta)}$.

Remark(1.1): We find the following special cases for the operator $\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}$ by considering several particular cases for the coefficients a_j and n :

(i)Putting $a_j = 1, \vartheta = 0$ and $n = 0$ into this operator, we obtain the operator $QTRcalB_q^\alpha$defined by Srivastava et al. [23];

(ii) Putting $a_j = \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)}$ $(\rho > 0)$, $\vartheta = 0$ and n = 0 in this operator, we obtain the operator $\mathcal{N}_{p,q}^{\sigma}$defined by El-Deeb and Bulboac˘a [9] and El-Deeb [8];

(iii) Putting $a_j = \left(\frac{\tau+1}{\tau+j}\right)^r$ $(r > 0, \tau \geq 0)$, $\vartheta = 0$ and $n = 0$ in this operator, we obtain the operator $\mathcal{M}_{\tau,q}^{\sigma,r}$defined by El-Deeb and Bulboac˘a [24] and Srivastava and El-Deeb [25];

(iv) Putting $a_j = \frac{\varsigma^{j-1}}{(j-1)!}\varrho^{-\varsigma}$ $(\varsigma > 0)$ and $n = 0$ in this operator, we obtain the q-analogue of Poisson operator $\mathrm{I}_q^{\vartheta,\varsigma}$ defined by El-Deeb et al. [8];

(v) Putting $a_j = 1, \vartheta = 0$ in this operator, we obtain the operator $QTRcalB_{\vartheta,\sigma}^{\delta,q,n}$defined as follows:

$$B_{\vartheta,\sigma}^{\delta,q,n}\mathrm{F}(\mathcal{E}) = \varepsilon + \sum_{j=2}^{\infty} j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} h_j\, \varepsilon^j; \qquad (1.10)$$

(vi) Putting $a_j = \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)}$ $(\rho > 0)$ in this operator, we obtain the operator$\mathcal{N}_{\varsigma,p,q}^{\sigma,n}$ defined as follows:

$$\mathcal{N}_{\varsigma,p,q}^{\sigma,n} F(\mathcal{E}) = \varepsilon + \sum_{j=2}^{\infty} j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \frac{[j]_q!}{[\zeta+1]_{q,\varepsilon-1}} \frac{(-1)^j\,\Gamma(\varrho+1)}{4^{j-1}(j-1)!\,\Gamma(r+\varrho)} \hbar_j\, \varepsilon^j$$

$$= \varepsilon + \sum_{j=2}^{\infty} \varphi_j \hbar_j\, \varepsilon^j\,, \qquad (1.11)$$

where

$$\varphi_j = j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\,\Gamma(r+\rho)}, \qquad (1.12)$$

(vii) Putting $a_j = \left(\frac{\tau+1}{\tau+j}\right)^r$ $(r > 0, \tau \geq 0)$ in this operator, we obtain the operator $\mathcal{M}_{\tau,\theta,q}^{\sigma,n,r}$ defined as follows:

$$\mathcal{M}_{\tau,\theta,q}^{\sigma,n,r}\mathrm{F}(\mathcal{E}) = \varepsilon + \sum_{j=2}^{\infty} j^{2n}\left((\sigma-\vartheta)(j-1)+1\right)^n \left(\frac{\tau+1}{\tau+j}\right)^r \frac{[j]_q!}{[\zeta+1]_{q,\varepsilon-1}} h_j\, \varepsilon^j.$$

Ma and Minda have given a unified treatment of various subclass consisting of starlike and convex functions for either one of the quantities $\frac{\varepsilon\mathcal{T}'(\varepsilon)}{\mathcal{T}(\varepsilon)}$ or $1+\frac{\varepsilon\mathcal{T}''(\varepsilon)}{\mathcal{T}(\varepsilon)}$ is subordinate to a more general superordinate function. The $S^*(\phi)$ introduced by Ma and Minda [15] consists of function $\mathcal{T} \in \mathcal{A}$ satisfying $\frac{\varepsilon\mathcal{T}'(\varepsilon)}{\mathcal{T}(\varepsilon)} \prec \phi(z), z \in \mathfrak{U}$ and corresponding class $k(\phi)$ of convex functions $\mathcal{T} \in \mathcal{A}$ satisfying $1+\frac{\varepsilon\mathcal{T}''(\varepsilon)}{\mathcal{T}(\varepsilon)} \prec \phi(z), z \in \mathfrak{U}$, Ma and Minda [15], where ϕ is analytic and univalent

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function with positive real part in the unit disc U, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathfrak{U})$is a starlike region with the respect to 1 and symmetric with the respect to the real axis. The functions in the classes $S^*(\phi)$ and $K(\phi)$, are called starlike of Ma-Minda type or convex of Ma-Minda type respectively. By $S^*_{\Sigma_{\mathfrak{U}}}(\phi)$ and $K_{\Sigma_{\mathfrak{U}}}(\phi)$, we denote to bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type respectively. [15]. In this investigation, we assume that

$$\phi(\varepsilon) = 1 + B_1\varepsilon + B_2\varepsilon^2 + B_3\varepsilon^3 + \cdots, \quad B_1 > 0. \tag{1.13}$$

and

$$h(\varepsilon) = h_0 + h\varepsilon + h_2\varepsilon^2 + h_3\varepsilon^3 + \cdots. \tag{1.14}$$

The aim of this paper is to introduce new subclasses of the class $\sum_{\mathfrak{U}}$ and determine estimates of bounds on the coefficient$|h_2|$ and $|h_3|$ and for the functions in above subclasses.

In [3] (see also [35,3–38]), certain subclasses of the bi-univalent analytic functions class B were introduced and non-sharp estimates on the first two coefficients $|h_2|$ and $|h_3|$ were found. The object of the present paper is to introduce two new subclasses as in Definitions 2.1 and 3.1 of the function class B using the lin e ar q-convolution operat o r and determine estimat e s of the coefficients $|h_2|$ and $|h_3|$ for the functions in these new subclasses of the function class.

Lemma (1.1)[8]: Let $p(\varepsilon) \in \mathcal{P}$,then $|pi| \leq 2$ for each i where $\mathcal{P}$ is the family of all functions p, analytic in $\mathfrak{U}$, for which $Re(p(\varepsilon)) > 0, (\varepsilon \in \mathfrak{U})$, where

$$p(z) = 1 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \cdots.$$

2. Coefficients Estimates for the Class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$.

Definition (2.1): For a function $\mathcal{T} \in \sum_{\mathfrak{U}}$ defined by (1.1) is said to be in the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$ if the following quasi-subordination conditions are satisfied:

$$\left[\frac{\varepsilon\left[\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \gamma\varepsilon\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon(\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \delta\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'') + (1-\gamma)((1-\delta)Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon) + \delta\varepsilon\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}\right] - 1 \prec_q (\varphi(\varepsilon) - 1), \tag{2.1}$$

and

$$\left[\frac{\varsigma\left[\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \gamma\varsigma\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma(\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \delta\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'') + (1-\gamma)((1-\delta)Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma) + \delta\varsigma\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'}\right] - 1 \prec_q (\varphi(\varsigma) - 1), \tag{2.2}$$

where $\gamma, \delta, \mu \in [0,1]$ and $Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)$ is defined in (1.7) and $(\varepsilon, \varsigma \in \mathfrak{U})$.

For special values to parameters $\mu, \delta, \gamma, \zeta, n, \rho, \sigma, \vartheta$ and $\varphi(\varepsilon)$, leads to get Known and new classes.

Remark (3.1): For $\delta = 0$, a function $\mathcal{T} \in \sum_{\mathfrak{U}}$ define by (1.7) is said to be in the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$if the following quasi-subordination conditions are satisfied:

$$\left[\frac{\varepsilon\left[\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \gamma\varepsilon\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + +(1-\gamma)Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)}\right] - 1 \prec_q (\varphi(\varepsilon) - 1),$$

and

$$\left[\frac{\varsigma\left[\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \gamma\varsigma\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + (1-\gamma)Q^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)}\right] - 1 \prec_q (\varphi(\varsigma) - 1),$$

where ξ is the inverse function of $\mathcal{T}$and $(\varepsilon, \varsigma \in \mathfrak{U})$.

Remark (3.3): For $\delta = 1$, a functi o n $\mathcal{T} \in \sum_{\mathfrak{U}}$ define by (1.7) is said to be in the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$if the following quasi-subordination conditions are satisfied:

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''+(1-\gamma)\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right]-1\prec_q(\varphi(\varepsilon)-1),$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''+(1-\gamma)\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right]-1\prec_q(\varphi(\varsigma)-1),$$

where ξ is the inverse function of $\mathcal{T}$and $(\varepsilon,\varsigma\in\mathfrak{U})$.

Theorem (2.1): If the function $\mathcal{T}$ belongs to the class $\mathscr{f}_{q,\Sigma}^{\mu}(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$, then, we have

$$|h_2|\leq\frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{(1+2\gamma)(3\mu-2\delta-1)A_0B_1^2\psi_3-(1+\gamma)^2[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2]}},\tag{3.3}$$

and

$$|h_3|\leq\frac{B_1(|A_0|+|A_1|)}{(1+2\gamma)(3\mu-2\delta-1)\psi_3}+\frac{A_0^2B_1^2}{4(1+\gamma)^2(2\mu-\delta-1)^2\psi_2^2},\quad B_1>1,\tag{2.4}$$

Proof : Let $\mathcal{T}\in\mathscr{f}_{q,\Sigma}^{\mu}(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$,there exist two analytic functions u,v and $u,v:\mathfrak{U}\longrightarrow\mathfrak{U}$ with $u(0)=v(0)=0,|u(\varepsilon)|<1$ and $|v(\varsigma)|\leq1$,for all $\varepsilon,\ \varsigma\in\mathfrak{U}$ satisfying the following conditions.

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'')+(1-\gamma)((1-\delta)\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)+\delta\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right]-1\prec_q\hbar(\varepsilon)(\varphi(u(\varepsilon)-1)),\ \varepsilon\in\mathfrak{U}\tag{2.5}$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'')+(1-\gamma)((1-\delta)\,\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)+\delta\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right]-1\prec_q\hbar(\varsigma)(\varphi(v(\varsigma)-1)),\ \varsigma\in\mathfrak{U}\ ,\tag{2.6}$$

where ξ is the inverse function of $\mathcal{T}$and $(\varepsilon,\varsigma\in\mathfrak{U})$.

Determine the definition of the functions $p(\varepsilon)$ and $q(\varsigma)$ by

$$p(\varepsilon)=\frac{1+u(\varepsilon)}{1-u(\varepsilon)}=1+c_1\varepsilon^2+c_2\varepsilon^2+\cdots\tag{2.7}$$

and

$$q(\varsigma)=\frac{1+v(\varsigma)}{1-v(\varsigma)}=1+d_1\varsigma^2+d_2\varsigma^2+\cdots.\tag{2.8}$$

Equivalently,

$$u(\varepsilon):=\frac{p(\varepsilon)-1}{p(\varepsilon)+1}=\frac{1}{2}\left\{c_1\varepsilon+\left(c_2-\frac{c_1^2}{2}\right)\varepsilon^2+\cdots\right\},\tag{2.9}$$

and

$$v(\varsigma):=\frac{q(\varsigma)-1}{q(\varsigma)+1}=\frac{1}{2}\left\{b_1\varsigma+\left(b_2-\frac{b_1^2}{2}\right)\varsigma^2+\cdots\right\}.\tag{2.10}$$

Applying (3.9) ,(3.10) in (3.5) and (3.6),respectively, we have

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'')+(1-\gamma)((1-\delta)\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)+\delta\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right]-1=h(\varepsilon)\left(\varphi\left(\frac{p(\varepsilon)-1}{p(\varepsilon)+1}\right)-1\right),\tag{2.11}$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'')+(1-\gamma)((1-\delta)\,\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)+\delta\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right]-1=h(\varsigma)\left(\varphi\left(\frac{q(\varsigma)-1}{q(\varsigma)+1}\right)-1\right).$$

(2.12)

Utilize (2.8) and (2.9)in the right hands RH of the relations (3.1) and (3.13),we obtain

$$\hbar(\varepsilon)\left(\varphi\left(\frac{p(\varepsilon)-1}{p(\varepsilon)+1}\right)-1\right)=\frac{1}{2}A_0B_1c_1\varepsilon+\left\{\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\right\}\varepsilon^2+\cdots.$$

(2.13)

and

$$h(\varsigma)\left(\varphi\left(\frac{q(\varsigma)-1}{q(\varsigma)+1}\right)-1\right)=\frac{1}{2}A_0B_1d_1\varsigma+\left\{\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2\right\}\varsigma^2+\cdots.$$

(2.14)

By equalizing (2.11) , (2.12) ,(3.13) and (3.14),respectively ,we get

$$(1+\gamma)(2\mu-\delta-1)\hbar_2\psi_2=\frac{1}{2}A_0B_1c_1, \quad (2.15)$$

$$[(1+2\gamma)(3\mu-2\delta-1)h_3\psi_3+(1+\gamma)^2[2\mu(\mu-1)-(1+\delta)(2\mu-\delta-1)]h_2^2\psi_2^2]$$
$$=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2. \quad (2.16)$$

and

$$-(1+\gamma)(2\mu-\delta-1)h_2\psi_2=\frac{1}{2}A_0B_1b_1 \quad (2.17)$$

$$\left[\begin{matrix}[(1+\gamma)^2[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]+2(1+2\gamma)(3\mu-2\delta-1)]h_2^2\psi_2^2\\-(1+2\gamma)(3\mu-2\delta-1)h_3\psi_3\end{matrix}\right]$$
$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1+\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2. \quad (2.18)$$

From (2.15) and (2.17), we have

$$\hbar_2=\frac{A_0B_1c_1}{2(1+\gamma)(2\mu-\delta-1)\psi_2}=-\frac{A_0B_1d_1}{2(1+\gamma)(2\mu-\delta-1)\psi_2} \quad (2.19)$$

It follows that

$$c_1=-d_1\ , \quad (2.20)$$

and

$$8(1+\gamma)^2(2\mu-\delta-1)^2\ h_2^2\psi_2^2=A_0^2B_1^2(d_1^2+c_1^2). \quad (2.21)$$

Now , by summing (3.19) and (3.31), in light of (2.19) and (2.20),we obtain

$$8\left[(1+\gamma)^2[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]A_0B_1^2\psi_2^2+(1+2\gamma)(3\mu-2\delta-1)\psi_3A_0B_1^2\right]\hbar_2^2$$
$$=2A_0^2B_1^3(c_2+d_2)+\left(8(1+\gamma)^2(2\mu-\delta-1)^2\,(B_2-B_1)\hbar_2^2\psi_2^2\right), \quad (2.22)$$

which implies

$$\hbar_2^2=\frac{2A_0^2B_1^3(c_2+d_2)}{8\{(1+2\gamma)(3\mu-2\delta-1)A_0B_1^2\psi_3-(1+\gamma)^2[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]A_0B_1^2]\}}. \quad (2.23)$$

Applying lemma(1.1) $|c_i|\le 2, |d_i|\le 2,$ in (3.33) ,we get the desired result (3.3).
Next, for the bound on $|a_3|$, by subtracting (3.18) from (3.16), we obtain

$$4\left\{(1+2\gamma)(3\mu-2\delta-1)\psi_3\hbar_3-(1+2\gamma)(3\mu-2\delta-1)\psi_3\ \hbar_2^2\right\}=2A_1B_1c_1+A_0B_1(c_2-d_2)$$

(2.24)

By substituting (3.18) from (3.16), further computation using (3.30) and (3.31), we obtain

$$h_3=\frac{2A_1B_1c_1}{4(1+2\gamma)(3\mu-2\delta-1)\psi_3}+\frac{A_0B_1(c_2-d_2)}{4(1+2\gamma)(3\mu-2\delta-1)\psi_3}+\frac{A_0^2B_1^2(c_1^2+d_1^2)}{8(1+\gamma)^2(2\mu-\delta-1)^2\psi_2^2}\ . \quad (3.35)$$

Applying Lemma (1.1) $|c_i|\le 2, |d_i|\le 2,$ in (3.34), we get (3.4). This completes the proof of the Theorem (2.1).

By putting $\delta=0$ in Theorem (3.1), we obtain the following Corollary:

Corollary (3.1): If the functi o n $\mathcal{T}(\varepsilon)$ giv e n by (1.1) belong to th e class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta,n,\rho,\sigma,\vartheta,\gamma,0,\varphi)$, then

$$|h_2| \le \frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{(1+2\gamma)(3\mu-1)A_0B_1^2\psi_3-(1+\gamma)^2[(2\mu-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-1)]\psi_2^2A_0B_1^2]}},$$

and

$$|\hbar_3| \le \frac{B_1(|A_0|+|A_1|)}{(1+2\gamma)(3\mu-1)\psi_3}+\frac{A_0^2B_1^2}{4(1+\gamma)^2(2\mu-1)^2\psi_2^2}.$$

By putting $\delta=1$ in Theorem (3.1), we obtain the following Corollary:

Corollary (3.3): Let $\mathcal{T}(\varepsilon)$ given by (1.1) belongs to the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta,n,\rho,\sigma,\vartheta,\gamma,1,\varphi)$. Then

$$|\hbar_2| \le \frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{3(1+2\gamma)(\mu-1)A_0B_1^2\psi_3-(1+\gamma)^2[(2\mu-2)^2(B_2-B_1)-2[\mu(\mu-1)-(2\mu-2)]\psi_2^2A_0B_1^2]}},$$

and

$$|\hbar_3| \le \frac{B_1(|A_0|+|A_1|)}{3(1+2\gamma)(\mu-1)\psi_3}+\frac{A_0^2B_1^2}{8(1+\gamma)^2(\mu-1)^2\psi_2^2}.$$

By putting $\gamma=1$ in Theorem (3.1), we have the following Corollary:

Corollary (3.3): Let $\mathcal{T}(\varepsilon)$ given by (1.1) belongs to the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta,n,\rho,\sigma,\vartheta,1,\delta,\varphi)$. Then

$$|\hbar_2| \le \frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{3(3\mu-2\delta-1)A_0B_1^2\psi_3-4[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2]}},$$

and

$$|\hbar_3| \le \frac{B_1(|A_0|+|A_1|)}{3(3\mu-2\delta-1)\psi_3}+\frac{A_0^2B_1^2}{16(2\mu-\delta-1)^2\psi_2^2},\quad B_1>1.$$

By putting $\gamma=0$ in Theorem (3.1), we have the following Corollary:

Corollary(3.4): Let $\mathcal{T}(\varepsilon)$ given by (1.1) belongs to the class $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta,n,\rho,\sigma,\vartheta,0,\delta,\varphi)$.

Then $$|\hbar_2| \le \frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{(3\mu-2\delta-1)A_0B_1^2\psi_3-[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2]}},$$

and

$$|h_3| \le \frac{B_1(|A_0|+|A_1|)}{(3\mu-2\delta-1)\psi_3}+\frac{A_0^2B_1^2}{4(2\mu-\delta-1)^2\psi_2^2},\quad B_1>1.$$

3. Coefficients Estimates for the Subclass $\aleph^{q,\delta}_{\Sigma}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$.

Definition (3.1): A function $\mathcal{T}\in\sum_{\mathfrak{U}}$ defined b y (1.1) is said to be in th e class $\aleph^{q,\delta}_{\Sigma}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ if the following quasi-subordination conditions are satisfied:

$$1+\frac{1}{\gamma}\left[(1-\delta)\frac{\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}{(1-\lambda)\varepsilon+\lambda\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)}+\delta\left(\frac{\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''+\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}{\lambda\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''+\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}\right)-1\right]\prec_q(\varphi(\varepsilon)-1)\qquad(3.1)$$

and

$$1+\frac{1}{\gamma}\left[(1-\delta)\frac{\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'}{(1-\lambda)\varsigma+\lambda\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)}+\delta\left(\frac{\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''+\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'}{\lambda\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''+\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'}\right)-1\right]\prec_q(\varphi(\varsigma)-1),\qquad(3.2)$$

where $(0\le\lambda<1, 0\le\delta\le1, \gamma\in\mathbb{C}\setminus\{0\}, \varepsilon,\in\mathfrak{U})$.

For special values to parameters λ *and* δ, we get new and well-known classes.

Remark (3.1): For $\lambda = 0$, a function $\mathcal{T} \in \sum_{\mathfrak{U}}$ define by (1.1) is said to be in the class $\aleph_{\Sigma}^{q,\delta}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$ if the following quasi-subordination conditions are satisfied:

$$1 + \frac{1}{\gamma}\left[(1-\delta)\frac{\varepsilon\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}{\varepsilon} + \delta\left(\frac{\varepsilon\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'' + \left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}{\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right) - 1\right] \prec_q (\varphi(z) - 1)$$

and

$$1 + \frac{1}{\gamma}\left[(1-\delta)\frac{\varsigma\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{\varsigma} + \delta\left(\frac{\varsigma\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'' + \left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{\left(Q_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right) - 1\right] \prec_q (\varphi(w) - 1)$$

Theorem (3.1.): If the function $\mathcal{T}$ belongs to the class $\aleph_{\Sigma}^{q,\delta}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$, then we have

$$|\hbar_2| \leq \frac{\gamma|A_0|\, B_1\sqrt{B_1}}{\sqrt{2(1-\lambda)(1+2\delta)A_0B_1^2\psi_3 - (1-\lambda)^2[(1+3\delta)A_0B_1^2 - (1+\delta)^2(B_2 - B_1)]\psi_2^2}} \quad (3.3)$$

and

$$|\hbar_3| \leq \frac{\gamma B_1(|A_0| + |A_1|)}{(1-\lambda)(1+2\delta)\psi_3} + \frac{A_0^2B_1^2\gamma^2}{(1+\delta)^2(1-\lambda)^2\psi_2^2}, B_1 > 1, \quad (3.4)$$

where $0 \leq \delta \leq 1, 0 \leq \lambda \leq 1, \gamma \in \mathfrak{U} - \{0\}$.

Proof: Proceeding as in the proof of Theorem (2.1), we can get the relations as follows:

$$\frac{1}{\gamma}(1+\delta)(1-\lambda)\hbar_2\psi_2 = \frac{1}{2}A_0B_1c_1, \quad (3.5)$$

$$\frac{1}{\gamma}[2(1-\lambda)(1+2\delta)h_3\psi_3 - (1-\lambda)(1+\lambda)((1+3\delta))h_2^2\psi_2^2]$$
$$= \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2\} \quad (3.6)$$

and

$$-\frac{1}{\gamma}(1+\delta)(1-\lambda)\hbar_2\psi_2 = \frac{1}{2}A_0B_1b_1, \quad (3.7)$$

$$\frac{1}{\gamma}\left[[4(1-\lambda)(1+2\delta)\psi_3 - (1-\lambda)(1+\lambda)(1+3\delta)]\hbar_2^2\psi_2^2 - 2(1-\lambda)(1+2\delta)\psi_3\hbar_3\right]$$
$$= \frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0B_2}{4}b_1^2\}. \quad (3.8)$$

From (3.5) and (3.7), we obtain

$$c_1 = -d_1 \quad (3.9)$$

and

$$\hbar_2 = \frac{\gamma A_0B_1c_1}{2(1+\delta)(1-\lambda)\psi_2} = -\frac{\gamma A_0B_1b_1}{2(1+\delta)(1-\lambda)\psi_2} \quad (3.10)$$

and

$$8(1+\delta)^2(1-\lambda)^2\,\hbar_2^2\psi_2^2 = A_0^2B_1^2\gamma^2(d_1^2 + c_1^2). \quad (3.11)$$

Now, by summing (3.6) and (3.8) and using (3.11) we obtain

$$\frac{8}{\gamma}\{(2(1-\lambda)(1+2\delta)\psi_3 - (1-\lambda)(1+\lambda)(1+3\delta)\psi_2^2\}\hbar_2^2$$
$$= 2A_0B_1(c_2+d_2) + A_0(B_2-B_1)(c_1^2+d_1^2), \quad (3.12)$$

which implies

$$\hbar_2^2 = \frac{2A_0^2B_1^3(c_2+d_2)}{8\left\{2(1-\lambda)(1+2\delta)A_0B_1^2\psi_3 - (1-\lambda)^2\left[[(1+3\delta)A_0B_1^2 - (1+\delta)^2(B_2 - B_1)]\psi_2^2\right]\right\}}. \quad (3.13)$$

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Applying lemma(1.1) in (3.13) ,we get the desired result (3.3).
Next ,for the bound on $|h_3|$,by subtracting (3.6) from (3.8), we obtain

$$\frac{8}{\gamma}\{(1-\lambda)(1+2\delta)\psi_3 h_3-(1-\lambda)(1+2\delta)\psi_3\, h_2^2\}=2A_1B_1c_1+A_0B_1(c_2-d_2)$$

By substituting (3.18) from (3.16), further computation using (3.9) and (3.10), we obtain

$$h_3=\frac{2\gamma A_1B_1c_1}{4(1-\lambda)(1+2\delta)\psi_3}+\frac{\gamma A_0B_1(c_2-d_2)}{4(1-\lambda)(1+2\delta)}+\frac{A_0^2B_1^2\gamma^2(c_1^2+d_1^2)}{8(1+\delta)^2(1-\lambda)^2\psi_2^2} \qquad (3.14)$$

From (3.14) and (3.13), we get the desired result (3.4). The proof is complete.

Corollary (3.1): If $\mathcal{T}(\varepsilon)\in \aleph_{\Sigma}^{q,\delta}(1,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ defined in (1.1), then we have

$$|h_2|\le\frac{\gamma|A_0|\,B_1\sqrt{B_1}}{\sqrt{2(1+2\delta)A_0B_1^2\psi_3-[(1+3\delta)A_0B_1^2-(1+\delta)^2(B_2-B_1)]\psi_2^2}}$$

and

$$|h_3|\le\frac{\gamma B_1(|A_0|+|A_1|)}{(1+2\delta)\psi_3}+\frac{A_0^2B_1^2\gamma^2}{(1+\delta)^2\psi_2^2}\quad ,B_1>1.$$

Corollary (3.3): If $\mathcal{T}(\varepsilon)\in \aleph_{\Sigma}^{q,1}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ defined in (1.1), then we have

$$|h_2|\le\frac{\gamma|A_0|\,B_1\sqrt{B_1}}{\sqrt{6(1-\lambda)A_0B_1^2\psi_3-(1-\lambda)^2[4A_0B_1^2-4(B_2-B_1)]\psi_2^2}}$$

and

$$|h_3|\le\frac{\gamma B_1(|A_0|+|A_1|)}{3(1-\lambda)\psi_3}+\frac{A_0^2B_1^2\gamma^2}{4(1-\lambda)^2\psi_2^2}\quad ,B_1>1\,.$$

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