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Article

Total Edge Irregularity Strength of Star Snake Graphs

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Abstract: In different fields in our life, like physics, coding theory and computer science, graph labeling dramas an vital role and appears in many applications. A labeling of a graph $M(V,E)$ is a map which assign each element in G with a positive integer number. An edge irregular total $\mathfrak{z}$ -labeling is a function $\mho: V(M) \cup E(M) \rightarrow \{1,2,3,\ldots.,\mathfrak{z}\}$ such that $W_{\mho}(h) \neq W_{\mho}(z)$ where $W_{\mho}(h)$ and $W_{\mho}(z)$ are weights for any two distinct edges. In this case, M has total edge irregularity strength (TEIS) if $\mathfrak{z}$ is minimum. In our paper, we defined a new type of graphs called a triple star snake graph $PS_{3,n}$and $m-$ star snake graph $PS_{m,n}$. Also, we investigated TEIS for a triple star snake graph $PS_{3,n}$. After that, we generalized the results for $m-$ star snake graph $PS_{m,n}$.

Keywords: edge labeling; irregularity strength; irregular labelling; total edge irregularity strength; star snake graph

Mathematics Subject Classification: 05C78; 05C38

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1. Introduction

For a graph $M(V,E)$, which is connected and simple, an edge irregular total $\mathfrak{z}$ -labeling was introduced by Baca et al. in [1] as a map $\mho: V(M) \cup E(M) \rightarrow \{1,2,3,\ldots.,\mathfrak{z}\}$ such that $W_{\mho}(h) \neq W_{\mho}(z)$ where $W_{\mho}(h)$ and $W_{\mho}(z)$ are weights for any two distinct edges. Also, the inequality of TEIS a graph G , with the maximum degree of vertices ΔG , was deduced in the form

$$tes(M) \geq max\left\{\left\lceil\frac{|E(M)|+2}{3}\right\rceil,\left\lceil\frac{\Delta M+1}{2}\right\rceil\right\} \quad (1)$$

Since then, many authors have begun to find TEIS for many families of graphs. Ivanĉo and Jendroî in [2] determined TEIS for a tree as

$$tes(T) = max\left\{\left\lceil\frac{k+1}{3}\right\rceil,\left\lceil\frac{\Delta M+1}{2}\right\rceil\right\}.$$

Ahmad et al. [3–9] have investigated TEIS for zigzag graphs, helm and sun graphs, the categorical product of two cycles, the categorical product of two paths, the generalized Petersen graph, certain families of graphs and some classes of plane graphs. Therefore, TEIS has been determined for hexagonal grid graphs in Al-Mushayt and Ahmad [10], planar graphs in Yang et al. [11], for some classes of plane graphs in Tarawneh et al. [12], for fan, wheel, triangular book, and friendship graphs in Tilukay et al. [13], for subdivision of star in Siddiqui [14], for some Cartesian product graphs in Ramdan and Salman [15], for trees in Amar and Togn [16], for generalized web graphs and related graphs in Indriat et al. [17], for generalized prism in Bača and Siddiqui [18], for complete graph and complete bipartite graphs in Jendroî et al. [19], for the disjoint union of wheel graphs in Jeyanth and Sudhai [20], for dense graphs in Majersk et al. [21], for the grids in Miškuf and Jendroî [22], for, disjoint union of isomorphic copies of generalized Petersen graph in Naeem and

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Siddiqui [23], for large graphs in Pfender [24], for centralized uniform theta graphs in Putra and Susanti [25], for series parallel graphs in Rajasingh et al. [26].

Salama [27–31] has determined TEIS for polar grid graph, special families of graphs, heptagonal snake graph, uniform theta snake graphs and quintet snake graph.

In our paper, we defined new types of graphs called a triple star snake graph $PS_{3,n}$ and $m-$ star snake graph $PS_{m,n}$. Also, we investigated TEIS for a triple star snake graph $PS_{3,n}$. After that, we generalized the results for $m-$ star snake graph $PS_{m,n}$.

2. Main results:

In this section, we will define a star snake graph and some related graphs and determine TEIS for these graphs.

Definition1. In a path P_n if we replace every edge in it with a star S_3 then have a new graph called a triple star snake graph, denoted $PS_{3,n}$, see Figure (1).

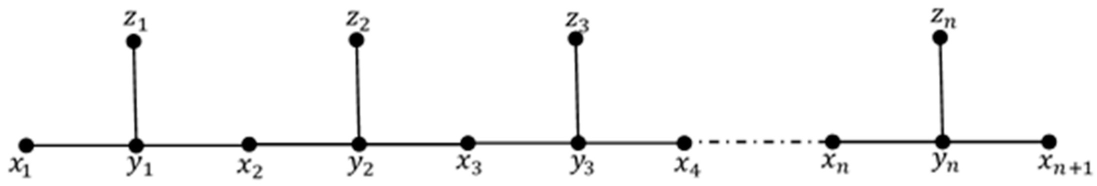

Figure (1) A triple star snake graph $PS_{3,n}$

Theorem1. If $PS_{3,n}$ is a triple star snake graph with $3n+1$ vertices, then TEIS is :

$$tes(PS_{3,n}) = n+1 \qquad (1)$$

Proof. Since $|E(PS_{3,n})| = 3n$ *and* $\Delta(PS_{3,n}) = 3$. Then inequality (1) becomes:

$$tes(PS_{3,n}) \geq n+1$$

To complete the proof, we will prove the inverse inequality. Let $ʒ = n+1$ and $Ʊ : V(PS_{3,n}) \cup E(PS_{3,n}) \rightarrow \{1,2,3,4,\ldots,ʒ\}$ is a total $ʒ$ -labeling defined as:

$$Ʊ(x_\varrho) = \varrho \qquad for \quad \varrho \in \{1,2,3,4\ldots,n+1\}$$
$$Ʊ(y_\varrho) = Ʊ(z_\varrho) = \varrho \qquad for \quad \varrho \in \{1,2,3,4,\ldots,n\}$$
$$Ʊ(x_ъ y_\varrho) = \varrho \qquad for \ \varrho \in \{1,2,3,4,\ldots,n\}$$
$$Ʊ(y_\varrho x_{\varrho+1}) = Ʊ(y_\varrho z_\varrho) = \varrho+1 \quad for \quad \varrho \in \{1,2,3,4,\ldots,n\}$$

the above equations means that $ʒ = n+1$ is the greatest label of edges and vertices. The weights of edges are given by:

$$w_Ʊ(x_\varrho y_\varrho) = 3\varrho \qquad for \quad \varrho \in \{1,2,3,4,\ldots,n\}$$
$$w_Ʊ(y_\varrho x_{\varrho+1}) = 3\varrho+2 \qquad for \quad \varrho \in \{1,2,3,4,\ldots,n\}$$
$$w_Ʊ(y_\varrho z_\varrho) = 3\varrho+1 \qquad for \ \varrho \in \{1,2,3,4,\ldots,n\}$$

It is clear that: the edges weights are dissimilar. Then

$$tes(PS_{3,n}) = n+1$$

Definition2. An $m-$ star snake graph $PS_{m,n}$ is a path P_n in which we replace each edge with a star S_m , see Figure (2-a), (2-b).

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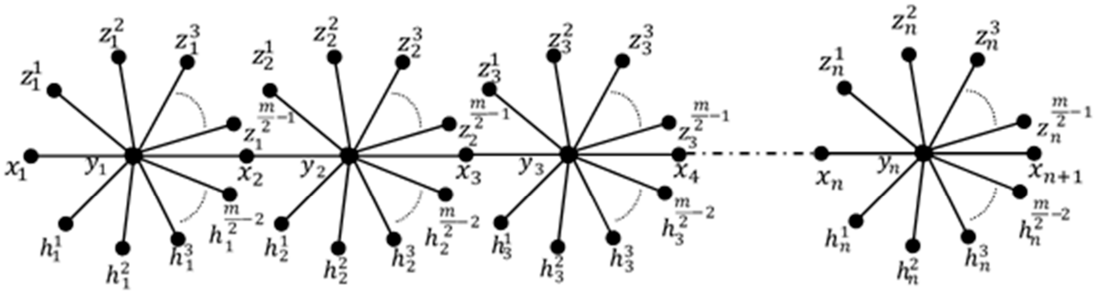

Figure (2-b) $m-$ star snake graph $PS_{m,n}$, (m odd)

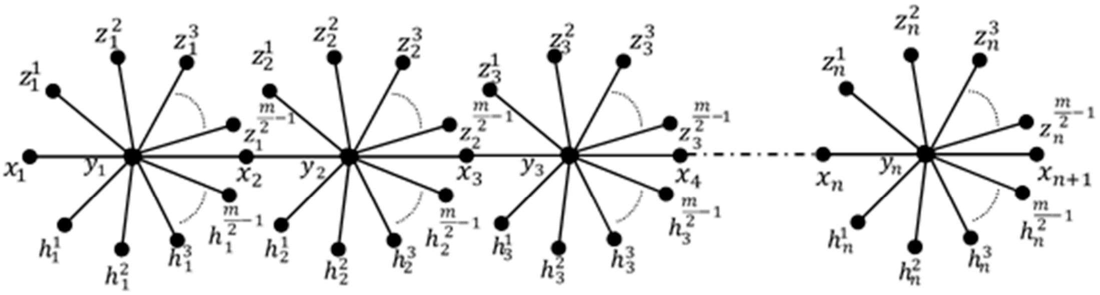

Figure (2-a) $m-$ star snake graph $PS_{m,n}$, (m $even$)

Theorem2. An $m-$ star snake graph $PS_{m,n}$ with $mn+1$ vertices $\boldsymbol{n>3}$, has TEIS given by:

$$tes(PS_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil \qquad (2)$$

Proof. By substituting with $|E(PS_{4,n})| = mn$ and $\Delta(PS_{m,n}) = m$ in inequality(2) we find

$$tes(PS_{m,n}) \geq \left\lceil \frac{mn+2}{3} \right\rceil.$$

An edge irregular total $ƺ$ -labeling will be shown assuming that a map $\text{Ʊ}: E(PS_{m,n}) \cup V(PS_{m,n}) \rightarrow \{1,2,3,4,\dots,ƺ\}$ is a total $ƺ$ -labeling defines in two cases:

Case 1: $\boldsymbol{m}$ is even Ʊ is defined as:

$$\text{Ʊ}(x_{\varrho}) = \begin{cases} \frac{m}{2}(\varrho-1)+1 & for\ \varrho \in \left\{1,2,3,\dots,\left\lfloor \frac{ƺ-1}{\frac{m}{2}} \right\rfloor + 1\right\} \\ ƺ & for\ \varrho \in \left\{\left\lfloor \frac{ƺ-1}{\frac{m}{2}} \right\rfloor + 2,\dots,n+1\right\} \end{cases}$$

$$(y_{\varrho}) = \begin{cases} \frac{m}{2}(\varrho-1)+1 & for\ \varrho \in \left\{1,2,3\dots,\left\lfloor \frac{ƺ-1}{\frac{m}{2}} \right\rfloor + 1\right\} \\ ƺ & for\ \varrho \in \left\{\left\lfloor \frac{ƺ-1}{\frac{m}{2}} \right\rfloor + 2,\dots,n\right\} \end{cases}$$

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$$\mho\left(h_{\varrho}^{s}\right)=\mho\left(z_{\varrho}^{s}\right)=\begin{cases}\left(\frac{m}{2}-1\right)(\varrho-1)+s & \text{for } \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+1\right\}\\ \left(\frac{m}{2}-1\right)(\varrho-1)+s & \text{for }\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2\\ s=\left\{1,2,3,\ldots,\mathfrak{z}-\left(\frac{m}{2}-1\right)(\varrho-1)\right\}\end{cases}\\ \mathfrak{z} & \text{for }\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2\\ s=\left\{\mathfrak{z}-\left(\frac{m}{2}-1\right)(\varrho-1)+1,\ldots,\frac{m}{2}-1\right\}\end{cases}\\ \mathfrak{z} & \text{for } \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+3,\ldots,n\right\}\end{cases}$$

$$\mho\left(x_{\varrho}y_{\varrho}\right)=\begin{cases}1 & \text{for } \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+1\right\}\\ (\varrho-1)m-2\mathfrak{z}+3 & \text{for } \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2,\ldots,n\right\}\end{cases}$$

$$\mho\left(y_{\varrho}x_{\varrho+1}\right)=\begin{cases}\frac{m}{2} & \text{for } \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor\right\}\\ \frac{m}{2}(\varrho+1)-\mathfrak{z}+1 & \text{for } \varrho=\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+1\\ \varrho m-2\mathfrak{z}+2 & \text{for } \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2,\ldots,n\right\}\end{cases}$$

$$\mho\left(y_{\varrho}z_{\varrho}^{s}\right)=\begin{cases}\varrho+s & \text{for } \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+1\right\}\\ \frac{m}{2}(\varrho-1)-\mathfrak{z}+\varrho+s+1 & \text{for }\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2\\ s=\left\{1,2,3,\ldots,\mathfrak{z}-\left(\frac{m}{2}-1\right)(\varrho-1)\right\}\end{cases}\\ (\varrho-1)m-2\mathfrak{z}+2s+2 & \text{for }\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+2\\ s=\left\{\mathfrak{z}-\left(\frac{m}{2}-1\right)(\varrho-1)+1,\ldots,\frac{m}{2}-1\right\}\end{cases}\\ (\varrho-1)m-2\mathfrak{z}+2s+2 & \text{for } \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\frac{m}{2}}\right\rfloor+3,\ldots,n\right\}\end{cases}$$

$$\mho(y_\varrho h_\varrho^s) = \begin{cases} \varrho + s + 1 & \text{for } \varrho \in \left\{1, \dots, \left\lfloor \frac{\zeta - 1}{\frac{m}{2}} \right\rfloor + 1\right\} \\ \frac{m}{2}(\varrho - 1) - \zeta + \varrho + s + 2 & \text{for } \begin{cases} \varrho = \left\lfloor \frac{\zeta - 1}{\frac{m}{2}} \right\rfloor + 2 \\ s = \left\{1, 2, 3, \dots, \zeta - \left(\frac{m}{2} - 1\right)(\varrho - 1)\right\} \end{cases} \\ (\varrho - 1)m - 2\zeta + 2s + 3 & \text{for } \begin{cases} \varrho = \left\lfloor \frac{\zeta - 1}{\frac{m}{2}} \right\rfloor + 2 \\ s = \left\{\zeta - \left(\frac{m}{2} - 1\right)(\varrho - 1) + 1, \dots, \frac{m}{2} - 1\right\} \end{cases} \\ (\varrho - 1)m - 2\zeta + 2s + 3 & \text{for } \varrho \in \left\{\left\lfloor \frac{\zeta - 1}{\frac{m}{2}} \right\rfloor + 3, \dots, n\right\} \end{cases}$$

From the previous equations, we can say ζ is the maximum number which labels vertices and edges. The edges weights of $PS_{m,n}$ are given by:

$$w_\mho(x_\varrho y_\varrho) = (\varrho - 1)m + 3$$
$$w_\mho(y_\varrho x_{\varrho+1}) = m\varrho + 2$$
$$w_\mho(y_\varrho z_\varrho^s) = m(\varrho - 1) + 2(s + 1)$$
$$w_\mho\left(y_\varrho h_\varrho^s\right) = m(\varrho - 1) + 2s + 3$$

We can say that from the previous equations, the weights are distinct for any two edges so

$$tes(PS_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil.$$

Case 2: m is odd $\mho$ is defined as:

$$\mho(x_\varrho) = \begin{cases} 1 + \left\lfloor \frac{m}{2} \right\rfloor(\varrho - 1) & \text{for } \varrho \in \left\{1, 2, 3, \dots, \left\lfloor \frac{\zeta - 1}{\left\lfloor \frac{m}{2} \right\rfloor} \right\rfloor + 1\right\} \\ \zeta & \text{for } \varrho \in \left\{\left\lfloor \frac{\zeta - 1}{\left\lfloor \frac{m}{2} \right\rfloor} \right\rfloor + 2, \dots, n\right\} \end{cases}$$

$$\mho(y_\varrho) = \begin{cases} 1 + \left\lfloor \frac{m}{2} \right\rfloor(\varrho - 1) & \text{for } \varrho \in \left\{1, 2, 3, \dots, \left\lfloor \frac{\zeta - 1}{\left\lfloor \frac{m}{2} \right\rfloor} \right\rfloor + 1\right\} \\ \zeta & \text{for } \varrho \in \left\{\left\lfloor \frac{\zeta - 1}{\left\lfloor \frac{m}{2} \right\rfloor} \right\rfloor + 2, \dots, n + 1\right\} \end{cases}$$

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$$\mho(z_{\varrho}^{s})=\begin{cases} s+\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1) & for\ \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor\right\} \\ s+\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1) & for\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1\\ s=\left\{1,2,3,\ldots,\mathfrak{z}-\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1)\right\}\end{cases} \\ \mathfrak{z} & for\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1\\ s=\left\{\mathfrak{z}-\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1)+1,\ldots,\left\lfloor\frac{m}{2}\right\rfloor\right\}\end{cases} \\ \mathfrak{z} & for\ \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

$$\mho(h_{\varrho}^{s})=\begin{cases} s+\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1)+1 & for\ \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor\right\} \\ s+\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1)+1 & for\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1\\ s=\left\{1,2,3,\ldots,\mathfrak{z}-\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1)-1\right\}\end{cases} \\ \mathfrak{z} & for\begin{cases}\varrho=\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1\\ s=\left\{\mathfrak{z}-\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1),\ldots,\left\lfloor\frac{m}{2}\right\rfloor-1\right\}\end{cases} \\ \mathfrak{z} & for\ \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

$$\mho(x_{\varrho}y_{\varrho})=\begin{cases} \varrho & for\ \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1\right\} \\ (\varrho-1)m-2\mathfrak{z}+3 & for\ \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

$$\mho(y_{\varrho}x_{\varrho+1})=\begin{cases} \left\lfloor\frac{m}{2}\right\rfloor+\varrho & for\ \varrho\in\left\{1,2,3,\ldots,\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor\right\} \\ \varrho m+1-\mathfrak{z}-\left\lfloor\frac{m}{2}\right\rfloor(\varrho-1) & for\ \varrho=\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1 \\ \varrho m-2\mathfrak{z}+2 & for\ \varrho\in\left\{\left\lfloor\frac{\mathfrak{z}-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

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$$Ʊ\left(y_{ϱ}z_{ϱ}^{s}\right)=\begin{cases} ϱ+s & \text{for } ϱ\in\left\{1,2,3,\ldots,\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor\right\} \\ ϱ+s & \text{for } \begin{cases} ϱ=\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1 \\ s=\left\{1,2,3,\ldots,ʒ-\left\lfloor\frac{m}{2}\right\rfloor(ϱ-1)\right\} \end{cases} \\ \left\lfloor\frac{m}{2}\right\rfloor(ϱ-1)-ʒ+2s+ϱ & \text{for } \begin{cases} ϱ=\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1 \\ s=\left\{ʒ-\left\lfloor\frac{m}{2}\right\rfloor(ϱ-1)+1,\ldots,\left\lfloor\frac{m}{2}\right\rfloor\right\} \end{cases} \\ (ϱ-1)m-2ʒ+2s+2 & \text{for } ϱ\in\left\{\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

$$Ʊ\left(y_{ϱ}h_{ϱ}^{s}\right)=\begin{cases} ϱ+s+1 & \text{for } ϱ\in\left\{1,\ldots,\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor\right\} \\ ϱ+s+1 & \text{for } \begin{cases} ϱ=\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1 \\ s=\left\{1,2,3,\ldots,ʒ-\left\lfloor\frac{m}{2}\right\rfloor(ϱ-1)-1\right\} \end{cases} \\ \left\lfloor\frac{m}{2}\right\rfloor(ϱ-1)-ʒ+2s+ϱ+1 & \text{for } \begin{cases} ϱ=\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+1 \\ s=\left\{ʒ-\left\lfloor\frac{m}{2}\right\rfloor(ϱ-1),\ldots,\left\lfloor\frac{m}{2}\right\rfloor-1\right\} \end{cases} \\ (ϱ-1)m-2ʒ+2s+3 & \text{for } ϱ\in\left\{\left\lfloor\frac{ʒ-1}{\left\lfloor\frac{m}{2}\right\rfloor}\right\rfloor+2,\ldots,n\right\} \end{cases}$$

From the previous formulas, we can deduce that $ʒ$ is the greatest label of edges and vertices. After calculating the weights of the edges of the graph $PS_{m,n}$ we find:

$$w_{Ʊ}\left(x_{ϱ}y_{ϱ}\right)=(ϱ-1)m+3$$
$$w_{Ʊ}\left(y_{ϱ}x_{ϱ+1}\right)=mϱ+2$$
$$w_{Ʊ}\left(y_{ϱ}z_{ϱ}^{s}\right)=m(ϱ-1)+2(s+1)$$
$$w_{Ʊ}\left(y_{ϱ}h_{ϱ}^{s}\right)=m(ϱ-1)+2s+3$$

From the equations of weights of edges we find them different, so $Ʊ$ is an edge irregular total $ʒ$ labeling. Then:

$$tes\left(PS_{m,n}\right)=\left\lceil\frac{mn+2}{3}\right\rceil.$$

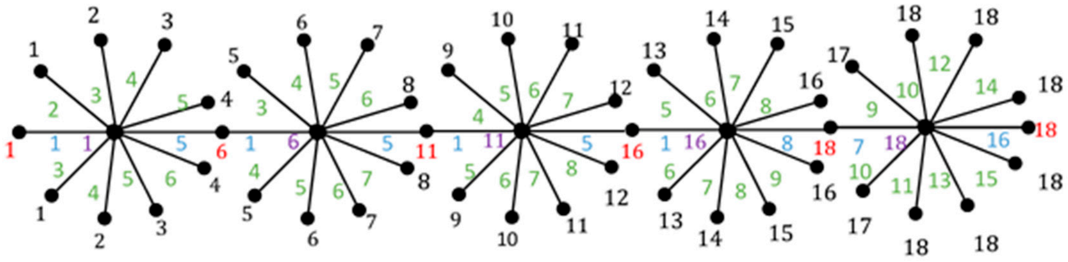

Figure (3) 10 – star snake graph $PS_{10,5}$.

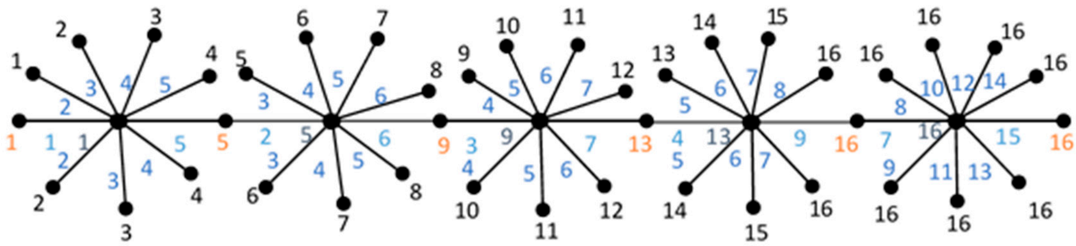

Figure (4) 9 – star snake graph $PS_{9,5}$.

3. Conclusions

In our paper, new types of graphs called a triple star snake graph $PS_{3,n}$and $m-$ star snake graph $PS_{m,n}$ were defied . Also, we have investigated TEIS for a triple star snake graph $PS_{3,n}$,

$$tes(PS_{3,n}) = n + 1.$$

After that, we have generalized the results for $m-$ star snake graph $PS_{m,n}$,

$$tes(PS_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil.$$

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