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Article

On a Generalized Gagliardo-Nirenberg Inequality with Radial Symmetry and Decaying Potentials

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Abstract: We establish a new Gagliardo-Nirenberg inequality characterized by radial symmetry and involving potentials exhibiting pure power polynomial behaviour. As an application of our result, we investigate the existence of extremals for this inequality, which also correspond to stationary solutions for the nonlinear Schrödinger equation with inhomogeneous nonlinearity, competing with H^s -subcritical nonlinearities, either of local or non-local nature.

Keywords: Fractional Laplacian; radially symmetric potential; non-homogeneous potential; Gagliardo-Nirenberg inequality; non-local nonlinearity

MSC: 30L15; 35A23; 35R11; 46B50; 46E35

1. Introduction

Consider the Cauchy problem associated to the fractional NLS, posed on $\mathbb{R}^d$ with $d \geq 2$:

$$\begin{cases} i\partial_t u - (-\Delta)^s u - |x|^{-\gamma} |u|^{\tilde{q}-2} u = f(x, u), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = u_0(x), \end{cases} \quad (1)$$

here $\tilde{q} \geq 0$ and γ are nonlinear parameters, the fractional Laplacian is defined, via Fourier transform, by $\widehat{(-\Delta)^s u}(\xi) = (2\pi\xi)^s \widehat{u}(\xi)$, provided $s \leq \frac{d}{2}$, $u = u(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, $u_0(x)$ is an initial data assumed to be in some function space and $f(x, u)$ denotes a general nonlinearity. The stationary points of the above evolution equation satisfy the following non-linear fractional Laplacian equation.

$$(-\Delta)^s u + |x|^{-\gamma} |u|^q u = f(x, u). \quad (2)$$

We consider the nonlinearities of type

$$f(x, u) = |u(x)|^{\tilde{p}} u(x), \quad \text{for } 0 < \tilde{p} < \frac{4s}{d-2s} \quad (3)$$

and

$$f(x, u) = \int_{\mathbb{R}^d} \frac{|u(x)|^{p-2} u(x) |u(y)|^p}{|x-y|^{d-\alpha}} dy, \quad \text{for } 0 < p < \frac{d+\alpha}{d-2s}, \quad 0 < \alpha < d. \quad (4)$$

A substantial body of literature exists regarding the radial symmetry of solutions to elliptic equations of type (2), with the research tradition dating back to the seminal work [4]. As a result, it is not feasible to provide an exhaustive list of works in this context. We concentrate our attention on [3], [7] and [14], addressing the references therein for a comprehensive overview of the topics. In [3], it is

investigated the phenomenon of symmetry breaking for (2) with nonlinearity of type (3) and compact embedding theorems for Sobolev-type spaces involving radial functions with polynomial-weight are established. In [7] is demonstrated the existence of radial ground states of (2) in the case (3) with $q = 2$ and $\gamma = 1$. Finally in [14], a set of embeddings for the fractional space in the presence of a radial potential is proved by using Lions-type theorems using a refined Sobolev inequality with the Morrey norm. Subsequently, they utilize the results obtained to inspect the existence of ground state solutions for (2) in the case (3) with $q = 2$ and $\gamma \neq 0$. Motivated by that, we generalize the above outcomes extending the range of the parameters p, q, γ and s associated to the corresponding embeddings for function spaces. In addition, we improve the Gagliardo-Nirenberg type inequalities with symmetry related to (2), generalizing them to the non-local frame and, as a direct consequence, we shed lights on the extremals of the corresponding minimization problems (see various Remarks 1.1, 1.2, 1.3, 1.4, 1.5, 4.1 and 6.1 for a complete overview of the details). Before stating our main results, we introduce some notations. We say that a function u is rapidly decreasing, that is $u \in \mathcal{S}(\mathbb{R}^d)$ with

$$\mathcal{S}(\mathbb{R}^d) = \left\{ u \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta u(x)| < +\infty \right\},$$

for all multi-indices $\alpha, \beta \in \mathbb{N}$. The Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ with $L^1_{loc}(\mathbb{R}^d)$ Fourier transform endowed with the norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

We recall also that the fractional Laplacian, for $0 < s < 1$, can be defined by

$$(-\Delta)^s u(x) = C_{d,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy,$$

with $C_{d,s}$ a normalization constant (see [1], [10] and [11]). Thus, in this regime, we have

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy. \quad (5)$$

We denote by $L_\gamma^q(\mathbb{R}^d)$ the weighted Lebesgue space with the norm as

$$\|u\|_{L_\gamma^q(\mathbb{R}^d)}^q = \int_{\mathbb{R}^d} \frac{|u|^q}{|x|^\gamma} dx.$$

Moreover, we introduce also

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) := \dot{H}^s(\mathbb{R}^d) \cap L_\gamma^q(\mathbb{R}^d),$$

with the norm

$$\|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}^2 := \|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 + \|u\|_{L_\gamma^q(\mathbb{R}^d)}^2.$$

In addition, let $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ be the set of radial functions in $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$. We start with the following

Theorem 1.1 (Continuous Embedding I). *Let $d \geq 2$ and $\frac{1}{2} < s < \frac{d}{2}$ and $q > 1$. Then we have that*

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (6)$$

with

$$p \in [p_{s,\gamma}^*, p_s^*], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma} \quad (7)$$

or

$$p \in (p_s^*, p_{s,\gamma}^*], \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma}, \quad (8)$$

where

$$p_{s,\gamma}^* := q + \frac{((2s-1)q+2)\gamma}{2s(d-1)-(2s-1)\gamma}, \quad p_s^* := \frac{2d}{d-2s} \quad (9)$$

and

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right). \quad (10)$$

In addition,

Theorem 1.2 (Compact Embedding I). *Let $d \geq 2$, $\frac{1}{2} < s < \frac{d}{2}$ and $q > 1$. Then we have that*

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (11)$$

for $p \neq p_{s,\gamma}^*$ and $p \neq p_s^*$, where $p_{s,\gamma}^*, p_s^*, q$ as in (7) or (8), with $p_{s,\gamma}^*, p_s^*$ defined as in (9) and $0 < \gamma < d$ as in (10).

Remark 1.1. The embeddings (6) and (11) in the case (7) were available in [3], we improved here the lower bound of the range of admissibility for p . The embeddings in the case (8) were given in [14] with $q = 2$, we extended them to $q > 1$.

We prove also

Theorem 1.3 (Continuous Embedding II). *Let $d \geq 2$ and $0 < s \leq \frac{1}{2}$ and $q > 1$. Then we have that*

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (12)$$

$$p \in [p_{s,\gamma}^*, p_s^*], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma} \quad (13)$$

or

$$p \in (p_s^*, p_{s,\gamma}^*], \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma}. \quad (14)$$

With also

Theorem 1.4 (Compact Embedding II). *Let $d \geq 2$, $0 < s < \frac{d}{2}$, $0 < \gamma < d$ and $q > 1$ such that*

$$q > 2 - \frac{\gamma}{d-1}. \quad (15)$$

Then we have that

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (16)$$

for $p \neq p_{s,\gamma}^*$ and $p \neq p_s^*$, where $p_{s,\gamma}^*, p_s^*, q$ as in (7), (8) or (13), (14), with $p_{s,\gamma}^*, p_s^*$ defined as in (9).

Remark 1.2. The embeddings (12) and (16) in the case (7) were obtained in [14] with $q = 2$, we generalized them to $q > 1$. Let us underline that Theorem 1.4 is new in the literature and breaks down the dichotomy $s > \frac{1}{2}$ and $s \leq \frac{1}{2}$. In addition, we bypass the application of Proposition 2.5 which is mandatory to achieve the crucial

equicontinuity property in order to apply the method appearing in [3] and [12]. This property, which is based on the representations of a radial function with Fourier transform in $L^1_{loc}(\mathbb{R}^d)$ by means of the Jost functions (see [13]), relies on the fact that $s > \frac{1}{2}$ (see the proof of Lemma 4.1 in [3]). We pay only the extra restriction (15). However, it perfectly handles the embedding in the case (7) of the work [3] and extend sit to the case (8).

Finally,

Theorem 1.5. Let $d \geq 2$, $\frac{1}{2} < s < \frac{d}{2}$ and $1 \leq q, p < \infty$, $-\infty < \gamma < 0$. Then we have that:

$$\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \quad (17)$$

with

$$p \in [p_{s,\gamma}^*, p_s^*], \quad \frac{1}{q} > \frac{d-2s}{2d+2|\gamma|} \quad (18)$$

or

$$p \in (p_s^*, p_{s,\gamma}^*], \quad \frac{1}{q} < \frac{d-2s}{2d+2|\gamma|}, \quad |\gamma| < d(q-p). \quad (19)$$

Moreover the embedding is compact for $p \neq p_{s,\gamma}^*$ and $p \neq p_s^*$ with $p_{s,\gamma}^*, p_s^*$ defined as in (9).

Remark 1.3. The compact embedding (17) in the case (18) was available in [3], we extended also here the lower bound of the range of admissibility for p . The compact embedding in the case (19) was proven in [14] with $q = 2$. We improved it to $q > 1$.

As a consequence of the above results we get

Theorem 1.6. Let $d \geq 2$, $0 < s < \frac{d}{2}$, $-d(q-1) < \gamma < d$, $1 \leq q, p < \infty$. There exists a constant $C = C(d, s, \gamma, q, p) > 0$ such that the scaling-invariant inequality

$$\int_{\mathbb{R}^d} |u(x)|^p dx \leq C \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2p(d-\gamma)-2dq}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{2d-p(d-2s)}{2d-2\gamma-q(d-2s)}} \quad (20)$$

holds for all functions $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ if

$$\begin{aligned} p \in [p_{s,\gamma}^*, p_s^*], \quad \frac{1}{q} &> \frac{d-2s}{2d-2\gamma}, \\ p \in (p_s^*, p_{s,\gamma}^*], \quad s &> \frac{1}{2}, \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma}, \end{aligned}$$

so that (8) or (19) are fulfilled and with the extra condition,

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right).$$

Furthermore, the inequality (20) remains valid if

$$p \in [p_s^*, p_{s,\gamma}^*], \quad s \leq \frac{1}{2}, \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma},$$

with $\gamma > 0$ and $p_{s,\gamma}^*, p_s^*$ defined as in (9).

as well as

Corollary 1.1. Let $d \geq 2, 0 < s < \frac{d}{2}, -d(q-1) < \gamma < d, 1 \leq q, p < \infty$ and $0 < \alpha < d$. There exists a constant $C = C(d, s, \gamma, q, p) > 0$ such that the scaling-invariant inequality

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{d-\alpha}} dx dy \leq C \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{4p(d-\gamma)-2q(d+\alpha)}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{2(d+\alpha)-2p(d-2s)}{q(2d-2\gamma-q(d-2s))}} \quad (21)$$

holds for all functions $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ if

$$p \in [p_{s,\alpha,\gamma}^*, p_s^*], \quad \frac{1}{q} > \frac{d-2s}{2d-2\gamma},$$

$$p \in [p_{s,\alpha}^*, p_{s,\alpha,\gamma}^*], \quad s > \frac{1}{2}, \quad \frac{1}{q} < \frac{d-2s}{2d-2\gamma}, \quad (|\gamma| < d(q-p), \gamma < 0),$$

are fulfilled with the extra condition,

$$\gamma \in \left(0, \frac{2s(d-1)}{2s-1}\right).$$

Furthermore, the inequality (21) remains valid if $\gamma > 0$,

$$p \in [p_{s,\alpha}^*, p_{s,\alpha,\gamma}^*], \quad s \leq \frac{1}{2}, \quad \frac{1}{2} - s < \frac{1}{q} < \frac{d-2s}{2d-2\gamma}.$$

Remark 1.4. The inequality (20) in the cases (7) and (18) was available in [3] (and seminally in [7], for $q = 2$ and $\gamma = 1$), we improved the lower bound of the domain of admissibility for p . Moreover, we extended it in the ranges given in (8) and (19), respectively. The inequality (21) appears for the first time in the literature.

Let us introduce now the Weinstein-type functionals

$$W_1^{p,q,s,\gamma}(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p+2)(d-\gamma)-2dq}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^{q+2}}{|x|^\gamma} dx \right)^{\frac{2d-(p+2)(d-2s)}{q(2d-2\gamma-q(d-2s))}}}{\int_{\mathbb{R}^d} |u|^{p+2} dx} \quad (22)$$

and

$$W_2^{p,q,s,\alpha,\gamma}(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{4p(d-\gamma)-2q(d+\alpha)}{2d-2\gamma-q(d-2s)}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^{q+2}}{|x|^\gamma} dx \right)^{\frac{2(d+\alpha)-2p(d-2s)}{q(2d-2\gamma-q(d-2s))}}}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{d-\alpha}} dx dy}. \quad (23)$$

Finally, by concentration-compactness arguments, we are in position to show also

Theorem 1.7. Let $\frac{1}{2} < s < 1, p_{s,\gamma}^* < p < \frac{2d}{d-2s}$, with $p_{s,\gamma}^*, q$ and γ as in Theorem 1.6. Then, there exists a function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ such that $W_1^{p,q,s,\gamma}(u) = m$ with $m > 0$ and so that

$$m = \inf \left\{ W_1^{p,q,s,\gamma}(u); u \neq 0, u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \right\}.$$

Analogously, it is possible to prove the following

Corollary 1.2. Let $\frac{1}{2} < s < 1, p_{s,\alpha,\gamma}^* < p < \frac{d+\alpha}{d-2s}$, with $p_{s,\alpha,\gamma}^*, q, \alpha$ and γ as in Corollary 1.1. Then, there exists a function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ such that $W_2^{p,q,s,\alpha,\gamma}(u) = m$ with $m > 0$ and so that

$$m = \inf \left\{ W_2^{p,q,s,\alpha,\gamma}(u); u \neq 0, u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \right\}.$$

Remark 1.5. Theorems 1.7 and Corollary 1.2 are new in the literature.

Outline of the paper. After introducing some preliminaries and auxiliary results in Section 2, through Section 3 we prove, in Theorem 1.1 and Theorem 1.3, the continuous embedding of the function spaces $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ into the Lebesgue spaces $L^p(\mathbb{R}^d)$. The principal target of Section 4 is to unveil that the previous embeddings are compact. This is done in Theorem 1.2, Theorem 1.4 and Theorem 1.5. We underline that in Theorem 1.4 we introduce a new method to prove the compactness of the embedding of $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$. This approach allows us to handle both $s > \frac{1}{2}$ and $s \leq \frac{1}{2}$ avoiding the use of Proposition 2.5. In Section 5 we give the proof of the Gagliardo-Nirenberg inequalities (20) and (21). Finally, in Section 6, we prove Theorem 1.7 and Corollary 1.2 and thus the existence of positive radial solutions in $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ for (2).

2. Preliminaries

In this section, we collect some notations as well as several useful results. We define $\mathcal{B}_R(0) = \{x \in \mathbb{R}^d | |x| < R\}$. Let be a set $E \subset \Omega \subseteq \mathbb{R}^d$, we denote by $E^c = \Omega \setminus E$ the complement of E in Ω . For any two positive real numbers a, b , we write $a \lesssim b$ (resp. $a \gtrsim b$) to denote $a \leq Cb$ (resp. $Ca \geq b$), with $C > 0$, disclosing the constant only when it is essential. For what it concerns compactness, we have (see [17]):

Proposition 2.1 (Riesz-Kolmogorov). *Let Ω be an open subset of $\mathbb{R}^d$, $1 \leq p < \infty$, and let $S \subset L^p(\Omega)$ be such that*

1. $\sup_{u \in S} \|u\|_{L^p(\Omega)} < \infty$;
2. for every $\varepsilon > 0$, there exists compact $K \subset \Omega$ such that $\sup_{u \in S} \int_{K^c} |u|^p dx \leq \varepsilon^p$;
3. for every compact $K \subset \Omega$, $\lim_{y \rightarrow 0} \sup_{u \in S} \|u(\cdot + y) - u(\cdot)\|_{L^p(K)} = 0$.

Then S is precompact in $L^p(\Omega)$.

We need the following generalization of Strauss Lemma (see [3], Theorem 3.1):

Proposition 2.2. *Let $d \geq 2$, $s > \frac{1}{2}$, $q > 1$, and*

$$-d(q-1) \leq \gamma < (d-1).$$

Then

$$|x|^\sigma |u(x)| \leq C \|u\|_{\dot{H}^s(\mathbb{R}^d)}^\eta \|u\|_{L_\gamma^q(\mathbb{R}^d)}^{1-\eta}, \quad (24)$$

for any $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, where

$$\sigma = \frac{2s(d-1) - (2s-1)\gamma}{(2s-1)q+2}, \quad \eta = \frac{2}{(2s-1)q+2}.$$

Notice that a particular case of the previous (24) is the inequality

$$\sup_{|x|>0} |x|^{\frac{d-2s}{2}} |u(x)| \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}, \quad (25)$$

valid for all $u \in \dot{H}_{rad}^s(\mathbb{R}^d)$. We have also (see [1], [3] and [15])

Proposition 2.3. *Let $d \geq 2$ and $0 < s < d/2$. Then*

$$\left(\int_{\mathbb{R}^d} |u(x)|^r |x|^{-\beta r} dx \right)^{\frac{1}{r}} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^d)}, \quad (26)$$

for any $u \in \dot{H}_{rad}^s(\mathbb{R}^d)$, where $r \geq 2$ and

$$-(d-1) \left(\frac{1}{2} - \frac{1}{r} \right) \leq \beta < \frac{d}{r}, \quad \frac{1}{r} = \frac{1}{2} + \frac{\beta-s}{d}. \quad (27)$$

A particular case of the above inequality (26) is the following estimate contained in [1].

Proposition 2.4. Assume $d \geq 2, 0 < s \leq 1/2$ and $\frac{1}{2} - s \leq \frac{1}{p} \leq \frac{1}{2} - \frac{s}{d}$. Then for $R > 0$, the inequality

$$\int_{B_R^c(0)} |u(x)|^p dx \leq C R^{d-p(\frac{d}{2}-s)} \|u\|_{\dot{H}_{rad}^s(\mathbb{R}^d)}^p, \quad (28)$$

with $C = C(d, s, p) > 0$, if fulfilled for any $u \in \dot{H}_{rad}^s(\mathbb{R}^d)$.

The following result is about the local Hölder continuity property of functions in $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ (see [3]).

Proposition 2.5. Let $B_R^c(0)$, with $R > 0$ and $s > \frac{1}{2}$. Then the continuous representation of $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ is Hölder continuous in $B_R^c(0)$, and moreover there exists a constant $C > 0$ such that

$$|u(x_1) - u(x_2)| \leq C |x_1 - x_2|^{\frac{2qs-q}{2qs+2-q}} \|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}. \quad (29)$$

Moreover (see [17])

Proposition 2.6. Let $1 < p < \infty$, and let $u_j, j \in \mathbb{N}$, be a sequence weakly convergent to u in $L^p(\Omega)$, with $\Omega \subseteq \mathbb{R}^d$. Then $u_j \in L^p(\Omega)$ is bounded and

$$\|u\|_{L^p(\Omega)} \leq \lim_{j \rightarrow \infty} \|u_j\|_{L^p(\Omega)}.$$

Let us recall the following generalized Leibnitz fractional rule (see [6]).

Proposition 2.7. Suppose $1 < p < \infty, s \geq 0$ and

$$\frac{1}{\ell} = \frac{1}{\ell_i} + \frac{1}{\tilde{\ell}_i},$$

with $i = 1, 2, 1 < \ell_1 \leq \infty, 1 < \tilde{\ell}_2 \leq \infty$. Then

$$\left\| (-\Delta)^{\frac{s}{2}}(fg) \right\|_{L^\ell(\mathbb{R}^d)} \leq C \left(\left\| (-\Delta)^{\frac{s}{2}}(f) \right\|_{\ell_1} \|g\|_{L^{\tilde{\ell}_1}(\mathbb{R}^d)} + \|f\|_{\ell_2} \left\| (-\Delta)^{\frac{s}{2}}g \right\|_{L^{\tilde{\ell}_2}(\mathbb{R}^d)} \right), \quad (30)$$

where the constants $C > 0$ depend on all of the parameters above but not on f and g .

We have the following Hardy-Littlewood- Sobolev inequality (see Lemma 2.4 in [8]):

Proposition 2.8. For $0 < \alpha < d$ and $p > 1$, there exists a sharp constant $C = C(d, p, \alpha) > 0$ such that

$$\left\| \int_{\mathbb{R}^d} |x|^{\alpha-d} * u(x) dy \right\|_{L^q(\mathbb{R}^d)} \leq C \|u\|_{L^p(\mathbb{R}^d)}, \quad (31)$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ and $p < \frac{d}{\alpha}$.

and the Hausdorff-Young inequality (see for example [5])

Proposition 2.9. Assume that f in $L^p(\mathbb{R}^d)$ we have then

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}, \quad (32)$$

with $1 \leq p \leq 2$.

The next tool is a Brezis-Lieb lemma for the nonlocal term (see Theorem in [9]).

Lemma 2.1. Let $d > 1, 0 < \alpha < d, 1 \leq p \leq \frac{2d}{d+\alpha}$ and $u_j, j \in \mathbb{N}$, be a bounded sequence in $L^{\frac{2dp}{d+\alpha}}(\mathbb{R}^d)$. If $u_j \rightarrow u$ almost everywhere on $\mathbb{R}^d$ as $j \rightarrow \infty$, then

$$\begin{aligned} \lim_{j \rightarrow \infty} \left(\int_{\mathbb{R}^d} (|x|^{\alpha-d} * |u_j|^p) |u_j|^p dx - \int_{\mathbb{R}^d} (|x|^{\alpha-d} * |u_j - u|^p) |u_j - u|^p dx \right) \\ = \int_{\mathbb{R}^d} (|x|^{\alpha-d} * |u|^p) |u|^p dx. \end{aligned} \quad (33)$$

3. Embedding in function spaces: continuity

We provide the proof of the Theorems 1.1 and 1.3. We start with

Proof of Theorem 1.1. Let us choose $R > 0$, we shall estimate the L^p norm of $u \in \dot{H}_{rad}^{s,q,\gamma}$ separately in $\mathcal{B}_R(0)$ and in $\mathcal{B}_R^c(0)$, respectively. Since $p < \frac{2d}{d-2s}$, in $B_R(0)$ we have, by using the Sobolev embedding

$$\int_{B_R(0)} |u(x)|^p dx \lesssim R^{1-p(\frac{1}{2}-\frac{s}{d})} \|u\|_{L^{p^*}(\mathbb{R}^d)}^p \lesssim R^{1-p(\frac{1}{2}-\frac{s}{d})} \|u\|_{\dot{H}^s(\mathbb{R}^d)}^p. \quad (34)$$

To handle the estimate in $\mathcal{B}_R^c(0)$ we follow the lines of the one given in [3] by using now the inequality (24). More precisely, we have

$$\begin{aligned} \int_{\mathcal{B}_R^c(0)} |u(x)|^p dx &\leq \sup_{|x|>R} \left(|u(x)| |x|^{\frac{\gamma}{p-q}} \right)^{p-q} \int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(1-\eta)(p-q)}{q}} \int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2} + 1}. \end{aligned} \quad (35)$$

Note that, in order to apply (24), one needs that

$$\frac{\gamma}{p-q} \leq \sigma = \frac{2s(d-1-\gamma) + \gamma}{(2s-1)q+2}, \quad (36)$$

which is fulfilled since $q < p_{s,\gamma}^* < p$ and $q < \frac{2d-2\gamma}{d-2s}$. We shall look now at the embedding (6) in the case (8). On $\mathcal{B}_R^c(0)$, for any $p > \frac{2d}{d-2s}$ we can estimate

$$\int_{\mathcal{B}_R^c(0)} |u(x)|^p dx \lesssim \left\| |u(x)| |x|^{\frac{(d-2s)}{2}} \right\|_{L^\infty(\mathcal{B}_R^c(0))}^p \int_{\mathcal{B}_R^c(0)} |x|^{-\frac{p(d-2s)}{2}} dx \lesssim CR^{d-p(\frac{d}{2}-s)} \|u\|_{\dot{H}^s(\mathbb{R}^d)}^p, \quad (37)$$

by an application of the Hölder inequality together with (25). To achieve a bound in $\mathcal{B}_R(0)$, we observe that $p_s^* < q < p_{s,\gamma}^*$ due to $q > \frac{2d-2\gamma}{d-2s}$ and hence we can assume that $q < p < p_{s,\gamma}^*$. Then we get

$$\begin{aligned} \int_{\mathcal{B}_R(0)} |u(x)|^p dx &\leq \left\| |u(x)| |x|^{\frac{\gamma}{p-q}} \right\|_{L^\infty(\mathcal{B}_R(0))}^{p-q} \int_{\mathcal{B}_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2} + 1}. \end{aligned} \quad (38)$$

Bear in mind that in this framework, to apply the inequality (24), we need the elementary bound $|x|^{\frac{\gamma}{p-q}} \lesssim |x|^\sigma$, for $|x| \lesssim 1$, which is guaranteed if

$$\frac{\gamma}{p-q} \geq \sigma = \frac{2s(d-1-\gamma) + \gamma}{(2s-1)q+2}. \quad (39)$$

This completes the proof. $\square$

Our next target is the following.

Proof of Theorem 1.3. Let it be $R > 0$, we will control the L^p norm of $u \in \dot{H}_{rad}^{s,q,\gamma}$ in $\mathcal{B}_R(0)$ in the same way that we did in the proof of Theorem 1.1 because of $p < \frac{2d}{d-2s}$. The estimate in $\mathcal{B}_R^c(0)$ can be handled by using now the inequality (26). In fact we achieve, by selecting $q < p_{s,\gamma}^* < p < r$ and by a direct application of the Hölder inequality,

$$\begin{aligned} \int_{\mathcal{B}_R^c(0)} |u|^p dx &\leq \left(\int_{\mathcal{B}_R^c(0)} |u(x)|^r |x|^{\frac{r-p}{p-q}} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \left(\int_{\mathcal{B}_R^c(0)} |u(x)|^r |x|^{-r\beta} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{p-q}{r-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}}, \end{aligned} \quad (40)$$

where in the second line of the above inequality we applied (26), with r and β solution of the system

$$\frac{1}{r} = \frac{1}{2} + \frac{\beta-s}{d}, \quad \frac{r-p}{p-q} = \frac{\beta}{\gamma},$$

that is

$$r = \frac{2(\gamma p - d(p-q))}{2\gamma - (d-2s)(p-q)}, \quad \beta = \frac{1}{2} \frac{\gamma(2d - p(d-2s))}{\gamma p - d(p-q)}, \quad (41)$$

because of the relations (27). It is easy to see that $\beta < 0$ because $q < \frac{2d-2\gamma}{d-2s}$ and $p < \frac{2d}{d-2s}$. In addition, we require also that

$$\frac{1}{2} \frac{\gamma(2d - p(d-2s))}{\gamma p - d(p-q)} \geq \frac{1-d}{2} \frac{\gamma(p-2) - 2s(p-q)}{\gamma p - d(p-q)}, \quad (42)$$

due to the second of the conditions in (41), which is satisfied when $p \geq p_{s,\gamma}^*$. Notice that we can rewrite

$$p_{s,\gamma}^* = \begin{cases} \frac{2}{1-2s} + \frac{(2s(d-1)((1-2s)q-2))}{(1-2s)(2s(d-1) + \gamma(1-2s))}, & s \neq \frac{1}{2} \\ \frac{q(d-1) + 2\gamma}{d-1}, & s = \frac{1}{2}. \end{cases} \quad (43)$$

Let us examine now the case

$$\frac{(1-2s)(2d-2\gamma)}{d-2s} < (1-2s)q < 2, \quad q < p \leq p_{s,\gamma}^*. \quad (44)$$

In this regime we bound the L^p norm of $u \in \dot{H}_{rad}^{s,q,\gamma}$ in $\mathcal{B}_R^c(0)$ by using the inequality (28), because of $p > \frac{2d}{d-2s}$. For what it concerns the region $\mathcal{B}_R(0)$, we will argue exactly as in (40), that is

$$\begin{aligned} \int_{\mathcal{B}_R(0)} |u|^p dx &\leq \left(\int_{\mathcal{B}_R(0)} |u(x)|^r |x|^{\gamma \frac{r-p}{p-q}} dx \right)^{\frac{p-q}{r-q}} \left(\int_{\mathcal{B}_R(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}} \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{p-q}{r-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r-p}{r-q}}, \end{aligned}$$

by taking notice now that $\beta < 0$ since $q > \frac{2d-2\gamma}{d-2s}$ and that the second of the conditions in (41) is fulfilled if one has

$$\frac{1}{2} \frac{\gamma(2d-p(d-2s))}{\gamma p - d(p-q)} \leq \frac{1-d}{2} \frac{\gamma(p-2) - 2s(p-q)}{\gamma p - d(p-q)},$$

which means

$$\frac{1}{p} \geq \frac{2s(d-1) + \gamma(1-2s)}{2qs(d-1) + 2\gamma} = \frac{1}{p_{s,\gamma}^*}.$$

The proof is then completed. $\square$

4. Embedding in function spaces: compactness

This section is divided into two parts. The first concerns the compactness results for functions in $\dot{H}^s(\mathbb{R}^d)$, with $s > \frac{1}{2}$. The second is devoted to shed lights on the compact embeddings for $s \leq \frac{1}{2}$.

4.1. Compactness: higher regularity.

Let us focus now on the proof of the compactness results given in Theorem 1.2 and in Theorem 1.5. To show compactness, we will follow the classical argument introduced in [12] and lately extended in [3], with some refinements. More precisely

Proof of Theorem 1.2. Observe that the space $\dot{H}^{s,q,\gamma}(\mathbb{R}^d)$ is reflexive, then it suffices to show that every given sequence u_j converging weakly to 0 in $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, converges strongly in $L^p(\mathbb{R}^d)$, that is $\|u_j\|_{L^p(\mathbb{R}^d)} \rightarrow 0$. Given $\varepsilon > 0$, we split $\mathbb{R}^d$ in three parts and thus:

$$\|u_j\|_{L^p(\mathbb{R}^d)}^p = \int_{|x|>R} |u_j(x)|^p dx + \int_{|x|<R^{-1}} |u_j(x)|^p dx + \int_{R \leq |x| \leq R^{-1}} |u_j(x)|^p dx, \quad (45)$$

where $R = R(\varepsilon)$ will be chosen later. Assume now that conditions (8) are satisfied. We have, arguing as in the proof of (37),

$$\int_{|x|>R} |u_j(x)|^p dx \leq CR^{d-p(\frac{d}{2}-s)} \leq \frac{\varepsilon}{3}, \quad (46)$$

for $R \geq R_1(\varepsilon)$, given that $p > \frac{2d}{d-2s}$. We have also, by using the inequality (24),

$$\begin{aligned} \int_{|x|<R^{-1}} |u_j(x)|^p dx &= \int_{|x|<R^{-1}} \frac{|u_j(x)|^q}{|x|^\gamma} |u_j(x)|^{p-q} |x|^\gamma dx \\ &\leq C \int_{|x|<R^{-1}} \frac{|u_j(x)|^q}{|x|^\gamma} |x|^{\gamma-\sigma(p-q)} dx \leq CR^{\sigma(p-q)-\gamma} \int_{\mathbb{R}^d} \frac{|u_j(x)|^q}{|x|^\gamma} dx \leq CR^{\sigma(p-q)-\gamma} < \frac{\varepsilon}{3}, \end{aligned} \quad (47)$$

for $R \geq R_2(\varepsilon)$ and $\gamma > \sigma(p - q)$ which is fulfilled for $p < p_{s,\gamma}^*$ once $q > \frac{2d-2\gamma}{d-2s}$. Finally, by choosing $R = \max\{R_1(\varepsilon), R_2(\varepsilon)\}$, we observe that according to the Hölder continuity property (29) of Proposition, we have

$$\begin{aligned} \int_{R \leq |x| \leq R^{-1}} |u_j(x+y) - u_j(x)|^p dx &\leq \int_{R \leq |x| \leq R^{-1}} |u_j(x+y) - u_j(x)|^p dx \\ &\leq \|u_n\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}^p \int_{R \leq |x| \leq R^{-1}} |y|^{p\alpha} dx, \end{aligned}$$

with $y \in \mathbb{R}^d$ and

$$\alpha = \frac{2qs - q}{2qs + 2 - q}.$$

By Proposition 2.1, the sequence u_j , $j \in \mathbb{N}$, admits a subsequence u_{j_k} which converges almost everywhere to 0 on the compact set

$$\{x \in \mathbb{R}^d \mid R \leq |x| \leq R^{-1}\}.$$

By taking $j \in \mathbb{N}$ large enough one obtains

$$\int_{R \leq |x| \leq R^{-1}} |u_j(x)|^p dx < \frac{\varepsilon}{3}. \quad (48)$$

Thus, by (46), (47) and the above inequality, we work out $\|u_j\|_{L^p(\mathbb{R}^d)} \rightarrow 0$ for $p_s^* < p < p_{s,\gamma}^*$, as $j \rightarrow \infty$. The case depicted in (7), can be handled in a similar way as in [3], with the following difference that we argue as in the proof of (34) and exploit the bound,

$$\int_{|x| < R^{-1}} |u_j(x)|^p dx \leq CR^{p(\frac{1}{2} - \frac{s}{d}) - 1}, \quad (49)$$

if one uses again (45). The proof is now completed. $\square$

4.2. Compactness: unified approach.

In this section, inspired by [1], we present a method to show compactness having as main scope to treat in a unified manner both the cases of functions with low and high regularity. Let us consider now

Proof of Theorem 1.4. We select $\varphi \in \mathcal{S}(\mathbb{R}^d)$, by the fractional Leibniz rule (30) and Sobolev embedding, we obtain

$$\left\| (-\Delta)^{\frac{s}{2}} \varphi u \right\|_{L^2(\mathbb{R}^d)} \lesssim \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^d)} \|\varphi\|_{L^\infty(\mathbb{R}^d)} + \left\| (-\Delta)^{\frac{s}{2}} \varphi \right\|_{L^{\frac{2d}{d+2s}}(\mathbb{R}^d)} \|u\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \quad (50)$$

$$\lesssim \|\varphi u\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim \|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}, \quad (51)$$

where the last inequality is provided by Theorem 1.3. For all $\tilde{R} > 0$, we pick a smooth $\varphi(x)$ such that $\varphi(x) = 1$ in $\mathcal{B}_{\tilde{R}}(0)$ and $\varphi(x) = 0$ in $\mathcal{B}_{2\tilde{R}}^c(0)$. Let us set φu_j , with u_n , $n \in \mathbb{N}$ being a bounded sequence in $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$. Furthermore one has that φu_j is bounded also in $H^s(\mathbb{R}^d)$ because the continuous embedding $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$ which is a consequence Theorem 1.3. In fact, if $q < p_{s,\gamma}^* < p < p_s^*$ with $q < \frac{2d-2\gamma}{d-2s}$ as in (15), one can see that

$$p_{s,\gamma}^* := q + \frac{((2s-1)q+2)\gamma}{2s(d-1) - (2s-1)\gamma} > 2,$$

while the case (44), with $p > p_s^*$ is straightforward. This bears to the fact that φu_j converges weakly to some w in $L^2(\mathbb{R}^d)$ with support still in $\mathcal{B}_{2\tilde{R}}(0)$. Notice that we have also that $\widehat{w} \in L^\infty(\mathbb{R}^d)$. By application of the Plancharel's identity we achieve

$$\|\varphi u_j - w\|_{L^2(\mathbb{R}^d)} \leq \|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}_R(0))} + \|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}_R^c(0))} \quad (52)$$

for any $R > 0$. Then

$$\|\widehat{\varphi u_j} - \widehat{w}\|_{L^2(\mathcal{B}_R^c(0))} \leq \frac{1}{R^s} \|\varphi u_j - w\|_{H^s(\mathbb{R}^d)}, \quad (53)$$

which means that the quantity $\widehat{\varphi u_j}(\xi) - \widehat{w}(\xi)$ is uniformly small if $|\xi|$ is sufficiently large. In addition, if one observes that

$$\lim_{n \rightarrow \infty} \langle \varphi u_j - w, e^{ix \cdot \xi} \rangle_{L^2(\mathbb{R}^2)} = \lim_{n \rightarrow \infty} (\widehat{\varphi u_j} - \widehat{w}) = 0,$$

by the definition of Fourier transform and of the weak convergence in $L^2(\mathbb{R}^d)$, we have $\widehat{\varphi u_j}(\xi)$ tends to $\widehat{w}(\xi)$ almost everywhere as $j \rightarrow \infty$. By (52), (53) and Hölder's inequality we have

$$\|\varphi u_j - v\|_{L^1(\mathcal{B}_R(0))} \lesssim \|\varphi u_j - w\|_{L^2(\mathbb{R}^d)} \lesssim R^{\frac{d}{2}} \|\widehat{\varphi u_j} - \widehat{w}\|_{L^\infty(\mathbb{R}^d)} + \frac{1}{R^s} \|\varphi u_j - w\|_{H^s(\mathbb{R}^d)} \quad (54)$$

for a suitable $R > 0$. Additionally, by an application of Young-Hausdorff inequality (32) and again Hölder's inequality, we see that

$$\|\widehat{\varphi u_j} - \widehat{w}\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\varphi u_j - v\|_{L^1(\mathcal{B}_{2R}(0))} \lesssim \tilde{R}^{\frac{d}{2}} \|\varphi u_j - v\|_{L^2(\mathcal{B}_{2R}(0))} \lesssim \tilde{R}^{\frac{d}{2}} \|\varphi u_j - v\|_{H^s(\mathbb{R}^d)}. \quad (55)$$

The bounds (54) (55) allow us to acquire the uniform estimate

$$\|\varphi u_j - v\|_{L^1(\mathcal{B}_R(0))} \lesssim (R\tilde{R})^{\frac{d}{2}} \|\varphi u_j - v\|_{H^s(\mathbb{R}^d)} + \frac{1}{R^s} \|\varphi u_j - w\|_{H^s(\mathbb{R}^d)} \lesssim \|\varphi u_j - w\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}. \quad (56)$$

By an use of Lebesgue's dominated convergence theorem we have that u_j converges to u in the $L^1(\mathcal{B}_R(0))$ and thus almost everywhere, once $j \rightarrow \infty$. This shows that $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ is compactly embedded in $L^1_{loc}(\mathbb{R}^d)$. To deal with the general case we shall use a continuity argument in conjunction with a perturbation argument. Namely, if $0 < s \leq \frac{1}{2}$, $q < p_{s,\gamma}^* < p < p_s^*$ with $q < \frac{2d-2\gamma}{d-2s}$ enjoying (15), we note that the constraint (42) is fulfilled with the strict inequality. We pick a $\gamma(\varepsilon) = \gamma + \varepsilon$, with $\varepsilon > 0$, that gives rise to a new set of parameters $(p, q, \gamma(\varepsilon), p_{s,\gamma(\varepsilon)}^*, \beta(\varepsilon), r(\varepsilon))$. We have that

$$\lim_{\varepsilon \rightarrow 0} (p_{s,\gamma(\varepsilon)}^*, \beta(\varepsilon), r(\varepsilon)) = (p_{s,\gamma}^*, \beta, r).$$

By (43), one can readily see that $p_{s,\gamma(\varepsilon)}^*$ approaches to $p_{s,\gamma}^*$ since is a decreasing function of ε . Moreover, by (41), we earn

$$\beta(\varepsilon) = \frac{d - p(d - 2s)}{p} \left(1 + \frac{d(p - q)}{\gamma(\varepsilon)p - d(p - q)} \right)$$

and that $\beta(\varepsilon) \nearrow \beta < 0$, $r(\varepsilon) \searrow r$ as $\varepsilon \rightarrow 0$. In conclusion, we can choose ε suitably small that still ensures $q < \frac{2d-2\gamma}{d-2s}$, $p > p_{s,\gamma}^*$ and that one can proceed as for (40) and deduce by Hölder inequality the following

$$\begin{aligned} \int_{\mathcal{B}_R^c(0)} |u|^p dx &\leq \left(\int_{\mathcal{B}_R^c(0)} |u(x)|^r |x|^{\gamma(\varepsilon) \frac{r(\varepsilon)r-p}{p-q}} dx \right)^{\frac{p-q}{r(\varepsilon)-q}} \left(\int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^{\gamma+\varepsilon}} dx \right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}} \\ &\lesssim \frac{1}{R^{\varepsilon \frac{r(\varepsilon)-p}{r(\varepsilon)-q}}} \left(\int_{\mathcal{B}_R^c(0)} |u(x)|^r |x|^{-r(\varepsilon)r\beta(\varepsilon)} dx \right)^{\frac{p-q}{r(\varepsilon)-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}} \\ &\lesssim \frac{1}{R^{\varepsilon \frac{r(\varepsilon)-p}{r(\varepsilon)-q}}} \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{p-q}{r(\varepsilon)-q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{r(\varepsilon)-p}{r(\varepsilon)-q}}. \end{aligned} \quad (57)$$

Let $s > \frac{1}{2}$. Selecting again $\gamma(\varepsilon) = \gamma + \varepsilon$, we can see that if ε is small enough so that $q < \frac{2d-2\gamma(\varepsilon)}{d-2s}$, so the inequality

$$\frac{\gamma(\varepsilon)}{p-q} < \sigma(\varepsilon) = \frac{2s(d-1) - \gamma(\varepsilon)(2s-1)}{(2s-1)q+2}, \quad (58)$$

is still valid. We get then, similarly for (35),

$$\begin{aligned} \int_{\mathcal{B}_R^c(0)} |u(x)|^p dx &\leq \sup_{|x|>R} \left(|u(x)| |x|^{\frac{\gamma(\varepsilon)}{p-q}} \right)^{p-q} \int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^{\gamma(\varepsilon)}} dx \\ &\lesssim \frac{1}{R^\varepsilon} \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(1-\eta)(p-q)}{q}} \int_{\mathcal{B}_R^c(0)} \frac{|u(x)|^q}{|x|^\gamma} dx \\ &\lesssim \frac{1}{R^\varepsilon} \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2} + 1}, \end{aligned} \quad (59)$$

where in the second inequality we used that $\sigma(\varepsilon) \nearrow \sigma$ for $\varepsilon \rightarrow 0$, with $\sigma(\varepsilon)$ as in (58) and $|x|^{\sigma(\varepsilon)} \lesssim |x|^\sigma$ for $|x| \geq R > 1$. In the case (44), $p > p_s^*$ instead we recall that we have, by (28),

$$\int_{\mathcal{B}_R^c(0)} |u(x)|^p dx \lesssim \frac{1}{R^{p(\frac{d}{2}-s)-d}} \|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}^p, \quad (60)$$

for $0 < s \leq \frac{1}{2}$ and, by (37), one can write the similar inequality

$$\int_{\mathcal{B}_R^c(0)} |u(x)|^p dx \lesssim CR^{d-p(\frac{d}{2}-s)} \|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}^p, \quad (61)$$

when $s > \frac{1}{2}$. The previous (57), (58), (60), and (61) give that

$$\lim_{R \rightarrow \infty} \sup_{j \in \mathbb{N}} \|u_j\|_{L^p(\mathcal{B}_R^c(0))} \rightarrow 0,$$

with $p \geq 1$. The proof of the theorem follows by interpolation with the case $p = 1$ and by the above embedding $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L_{loc}^1(\mathbb{R}^d)$. $\square$

We conclude the section with

Proof of Theorem 1.5. To show (17) if (18) is satisfied, we shall estimate again the L^p norm of $u \in \dot{H}_{rad}^{s,q,\gamma}$ in $\mathcal{B}_R(0)$ and in $\mathcal{B}_R^c(0)$. The bound in $\mathcal{B}_R(0)$, because when $p < \frac{2d}{d-2s}$, is the same as in (34). For what it concerns the bound in $\mathcal{B}_R^c(0)$ we have

$$\begin{aligned} \int_{\mathcal{B}_R^c(0)} |u(x)|^p dx &\leq \sup_{|x|>R} \left(|u(x)| |x|^{-\frac{\gamma'}{p-q}} \right)^{p-q} \int_{\mathcal{B}_R^c(0)} |x|^{\gamma'} |u(x)|^q dx \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} |x|^\gamma |u(x)|^q dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2}} \int_{\mathcal{B}_R^c(0)} |x|^{\gamma'} |u(x)|^q dx \\ &\lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{2(p-q)}{(2s-1)q+2}} \left(\int_{\mathbb{R}^d} |x|^\gamma |u(x)|^q dx \right)^{\frac{(2s-1)(p-q)}{(2s-1)q+2} + 1}, \end{aligned} \quad (62)$$

where in the second line of the above inequality we utilised $|x|^{-\frac{\gamma'}{p-q}} \lesssim |x|^{\frac{\gamma'}{p-q}} \lesssim |x|^{\sigma'}$, for $|x| \gtrsim 1$ and (24), once

$$\frac{\gamma'}{p-q} \leq \sigma' = \frac{2s(d-1) + (2s-1)\gamma'}{(2s-1)q+2} < \sigma, \quad (63)$$

where we took into account that $\gamma' < \gamma$ and $|x|^{\gamma'} \lesssim |x|^\gamma$ for $|x| \geq R > 1$. We observe also that (63) is satisfied for $p_{s,\gamma}^* < q < p$ and $q < \frac{2d+2\gamma}{d-2s}$, with $p_{s,\gamma}^*$ defined as in (9). In the frame of (19) we have again (37) in $B_R^c(0)$, when $p > \frac{2d}{d-2s}$. To estimate in $\mathcal{B}_R(0)$, with $p_s^* < p < p_{s,\gamma}^* < q$ and $q > \frac{2d+2\gamma}{d-2s}$ we catch that, by Hölder inequality,

$$\begin{aligned} \int_{\mathcal{B}_R(0)} |u(x)|^p dx &\lesssim R^{d(q-p)} \int_{\mathcal{B}_R(0)} |u(x)|^q dx \lesssim \int_{\mathcal{B}_R(0)} |x|^{d(q-p)} |u(x)|^q dx \\ &\lesssim \int_{\mathcal{B}_R(0)} |x|^\gamma |u(x)|^q dx \lesssim \|u\|_{\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)}^p, \end{aligned} \quad (64)$$

by the bound $|x|^{d(q-p)} \lesssim |x|^\gamma$, for $|x| \lesssim 1$, if $\gamma \leq d(q-p)$. For what it concerns the compactness, choose $\varepsilon > 0$, then again we take

$$\|u_j\|_{L^p(\mathbb{R}^d)}^p = \int_{|x|>R} |u_j(x)|^p dx + \int_{|x|<R^{-1}} |u_j(x)|^p dx + \int_{R \leq |x| \leq R^{-1}} |u_j(x)|^p dx, \quad (65)$$

where $R = R(\varepsilon)$ will be selected analogously as in the proof of Theorem 1.2. In the regime (18) we estimate the second and the third integrals on the right hand side of the above inequality as in (49) and (48), respectively. For the first one we achieve

$$\begin{aligned} \int_{|x|>R} |u_j(x)|^p dx &= \int_{|x|>R} \frac{|u_j(x)|^{p-q}}{|x|^{\gamma'}} |u_j(x)|^q |x|^{\gamma'} dx \\ &\leq C \int_{|x|>R} |u_j(x)|^q |x|^{\gamma'} |x|^{\gamma'-\sigma'(p-q)} dx \leq CR^{\gamma'-\sigma'(p-q)} \int_{\mathbb{R}^d} \frac{|u_j(x)|^q}{|x|^\gamma} dx \leq CR^{\gamma'-\sigma'(p-q)} < \frac{\varepsilon}{3}, \end{aligned} \quad (66)$$

from (63) if one follows the steps used to prove (62). If one considers now (19), we control the first and the third integrals on the right hand side of (65) as in (46) and (48), respectively. For the second we obtain

$$\int_{|x|<R^{-1}} |u(x)|^p dx \lesssim R^{-d(q-p)+\gamma} \int_{|x|<R^{-1}} |x|^\gamma |u(x)|^q dx < \frac{\varepsilon}{3}, \quad (67)$$

for $\gamma < d(q-p)$, as we did in (64). The proof is thus accomplished. $\square$

Remark 4.1. To demonstrate the compactness of the embedding (17), one can employ the approach illustrated in Theorem 1.4, considering the estimates provided in (62) and (64).

In order to have a self contained treatise, we need to prove the following.

Proposition 4.1. Let $d \geq 1$, $s > 0$, and $1 \leq q < \infty$, $-d(q-1) < \gamma < d$, Then the space $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ is complete.

Proof. Assume that $\gamma > 0$ and consider the Cauchy sequence $u_j \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, $j \in \mathbb{N}$, then $(-\Delta)^{\frac{s}{2}} u_j$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$ and thus there exists $f \in L^2(\mathbb{R}^d)$ such that the sequence $(-\Delta)^{\frac{s}{2}} u_j$, converges strongly, as $j \rightarrow \infty$, to f in $L^2(\mathbb{R}^d)$. On the other hand, we have, for every $R > 0$,

$$\int_{\mathcal{B}_R(0)} |u(x)|^q dx \lesssim R^\gamma \int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \quad (68)$$

which gives

$$\lim_{l,j \rightarrow \infty} \int_{\mathcal{B}_R(0)} |u_j(x) - u_l(x)|^q = 0.$$

There exists thus a measurable function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ such that u_j converges, as $j \rightarrow \infty$, to u in $L^q_{\text{loc}}(\mathbb{R}^d)$. By Fatou's lemma, we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u(x)|^q}{|x|^\gamma} dx \leq \lim_{j \rightarrow \infty} \liminf_{l \rightarrow \infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u_l(x)|^q}{|x|^\gamma} dx = 0. \quad (69)$$

We observe that by (68) we can get also

$$\lim_{j \rightarrow \infty} \sup_{R > 0} \frac{1}{R^\gamma} \int_{B_R(0)} |u_j(x) - u(x)|^q \lesssim \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \frac{|u_j(x) - u(x)|^q}{|x|^\gamma} dx = 0, \quad (70)$$

since (69). Furthermore, by the Hölder inequality we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx \\ & \leq \int_{B_R(0)} |(u_j(x) - u(x))\varphi(x)| dx + \int_{B_R^c(0)} |x|^{-\gamma q} |(u_j(x) - u(x))| |x|^{\gamma q} |\varphi(x)| dx \\ & \lesssim \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(B_R(0))} \|u_j - u\|_{L^q(B_R(0))} \\ & \quad + \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(B_R^c(0))} \left(\int_{B_R^c(0)} |x|^{-\gamma} |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \\ & \lesssim \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(\mathbb{R}^d)} \left(\|u_j - u\|_{L^q(B_R(0))} + \left(\int_{\mathbb{R}^d} |x|^{-\gamma} |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \right), \end{aligned} \quad (71)$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $N \geq q\gamma$. An use again of (69) in combination with (70) guarantees

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} |(u_n(x) - u(x))\varphi| dx = 0.$$

For this reason, $u_n(x) - u(x)$, if $n \rightarrow \infty$, converges to 0 as tempered distributions on $\mathbb{R}^d$. Therefore, $(-\Delta)^{\frac{s}{2}} u_j$ converges to $(-\Delta)^{\frac{s}{2}} u$ as distributions on $\mathbb{R}^d$. This fact and the above consideration on the convergence of $(-\Delta)^{\frac{s}{2}} u_j$ in $L^2(\mathbb{R}^d)$ imply that $(-\Delta)^{\frac{s}{2}} u = f$. Let now $\gamma < 0$, $d \geq 2$ and select as above a Cauchy sequence $u_j \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, $j \in \mathbb{N}$, converging strongly, as $j \rightarrow \infty$, to f in $L^2(\mathbb{R}^d)$. One sees that for $R > 0$ and $q \leq \frac{2d}{d-2s}$, by Sobolev embedding,

$$\int_{B_R(0)} |u_j(x) - u_l(x)|^q dx \lesssim R^{1-q(\frac{1}{2}-\frac{s}{d})} \|u_j(x) - u_l(x)\|_{\dot{H}^s(\mathbb{R}^d)}^p$$

and for $q > \frac{2d}{d-2s}$,

$$\int_{B_R(0)} |u_j(x) - u_l(x)|^q dx \lesssim \int_{\mathbb{R}^d} |x|^\gamma |u_j(x) - u_l(x)|^q dx,$$

which enhance to

$$\lim_{l,j \rightarrow \infty} \int_{B_R(0)} |u_j(x) - u_l(x)|^q = 0.$$

Then we can find a measurable function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ such that u_j converges, as $j \rightarrow \infty$, to u in $L^q_{\text{loc}}(\mathbb{R}^d)$. Fatou's lemma shows that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} |x|^\gamma |u_j(x) - u(x)|^q dx \leq \lim_{j \rightarrow \infty} \liminf_{l \rightarrow \infty} \int_{\mathbb{R}^d} |x|^\gamma |u_j(x) - u_l(x)|^q dx = 0. \quad (72)$$

As above

$$\begin{aligned}
& \int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx \quad (73) \\
& \leq \int_{B_R(0)} |(u_j(x) - u(x))\varphi(x)| dx + \int_{B_R^c(0)} |x|^{-\gamma q} |(u_j(x) - u(x))| |x|^{\gamma q} |\varphi(x)| dx \\
& \lesssim \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(B_R(0))} \|u_j - u\|_{L^q(B_R(0))} \\
& \quad + \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(B_R^c(0))} \left(\int_{B_R^c(0)} |x|^\gamma |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \\
& \lesssim \sup_{0 \leq k \leq N} \left\| |x|^k \varphi(x) \right\|_{L^\infty(\mathbb{R}^d)} \left(\|u_j - u\|_{L^q(B_R(0))} + \left(\int_{\mathbb{R}^d} |x|^\gamma |(u_j(x) - u(x))|^q dx \right)^{\frac{1}{q}} \right),
\end{aligned}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $N \geq q\gamma$. The inequality above, a further application of (72) infers

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} |(u_j(x) - u(x))\varphi| dx = 0.$$

The remaining part of the proof is the same as the one carried out above for the case $\gamma > 0$. Then we skip. $\square$

5. Gagliardo-Nirenberg inequalities

This section is addressed to present the proof of the Gagliardo-Nirenberg-type inequalities (20) and (21).

Proof of Theorem 1.6. We shall treat only the case $\gamma > 0$, because the proof for $\gamma < 0$ can be carried out in a similar manner, with some minor changes. Let us consider the scaling $u_\chi(x) = \chi^{\frac{d}{p}} u(\chi x)$ such that $\|u_\chi\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p(\mathbb{R}^d)}$. The embedding leads to

$$\|u_\chi\|_{L^p(\mathbb{R}^d)}^2 \leq C \left\| (-\Delta)^{\frac{s}{2}} u_\chi \right\|_{L^2(\mathbb{R}^d)}^2 + C \left(\int_{\mathbb{R}^d} \frac{|u_\chi(x)|^q}{|x|^\gamma} dx \right)^{\frac{2}{q}},$$

which implies the following

$$\begin{aligned}
\|u\|_{L^p(\mathbb{R}^d)}^2 & \leq C \chi^{\frac{2d}{p} - d + 2s} \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^d)}^2 + C \chi^{\frac{2d}{p} - \frac{2(d-\gamma)}{q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{2}{q}} \\
& := C \chi^{\frac{2d}{p} - d + 2s} A + C \chi^{\frac{2d}{p} - \frac{2(d-\gamma)}{q}} B. \quad (74)
\end{aligned}$$

By optimizing the sum on the left hand side of the above inequality (74) one obtains that the minimum of the above sum is attained at

$$\tilde{\chi} = \left(A^{-1} B \frac{2p(d-\gamma) - 2dq}{q(2d - p(d-2s))} \right)^{\frac{q}{2d-2\gamma-q(d-2s)}} = C(p, q, d, \gamma, s) \left(A^{-1} B \right)^{\frac{q}{2d-2\gamma-q(d-2s)}}, \quad (75)$$

with $C = C(p, q, d, \gamma, s) > 0$. By plugging the pervious (75) into (74) we arrive at

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^d)}^2 &\leq C\chi^{\frac{2d}{p}-d+2s} \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^d)}^2 + C\chi^{\frac{2d}{p}-\frac{2(d-\gamma)}{q}} \left(\int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x|^\gamma} dx \right)^{\frac{2}{q}} \\ &\leq CA^{\frac{2d-dp+2ps}{q(d-2s)-(2d-2\gamma)}+1} B^{\frac{q(2d-p(d-2s))}{p(2d-2\gamma-q(d-2s))}} + CA^{\frac{(2d-2\gamma)p-2dq}{p(2d-2\gamma-q(d-2s))}} B^{\frac{2dq-(2d-2\gamma)p}{p(2d-2\gamma-q(d-2s))}+1} \\ &\leq CA^{\frac{(2d-2\gamma)p-2dq}{p(2d-2\gamma-q(d-2s))}} B^{\frac{q(2d-p(d-2s))}{p(2d-2\gamma-q(d-2s))}}, \end{aligned} \quad (76)$$

which gives (20) with $p \neq p_{s,\gamma}^*$ and $p \neq p_s^*$, where $p_{s,\gamma}^*, p_s^*, \gamma, q$ as in (7), (8) or (18), (19), with $p_{s,\gamma}^*, p_s^*$ defined as in (9). $\square$

We are in position now to give

Proof of Corollary 1.1. The proof is a direct consequence of the scaling invariant inequality

$$\int_{\mathbb{R}^d} (|x|^{(\alpha-d)} * |u|^p) |u|^p dx \leq C(d, p, \alpha) \|u\|_{L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)}^{2p}, \quad (77)$$

arising from (31) in Proposition 2.8 and of (20) for $p_{s,\gamma}^* \leq \frac{2pd}{d+\alpha} \leq p_s^*$. $\square$

6. Minimization problems

In this section, we go over the proofs of the theorems connected to the minimization problems (1.7) and (1.2).

Proof of Theorem 1.7. The fact that $m > 0$ follows by Theorem 1.6. We will prove now that there is a function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, such that $W_1^{p,q,s,\gamma}(u) = m$ with $W_1^{p,q,s,\gamma}(u)$ as in (22). For this propose, pick up a minimizing sequence $u_j \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, $j \in \mathbb{N}$ converging weakly to $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ and such that

$$\int_{\mathbb{R}^d} u_j^p(x) dx = 1, \quad \int_{\mathbb{R}^d} \frac{u_j^q(x)}{|x|^\gamma} dx = 1, \quad W_1^{p,q,s,\gamma}(u_j) \rightarrow m,$$

for $j \rightarrow \infty$. We may assume also $u_j \geq 0$ because of the bound

$$\begin{aligned} \left\| (-\Delta)^{\frac{s}{2}} |u| \right\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{||u(x)| - |u(y)||^2}{|x-y|^{d+2s}} dx dy \\ &\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dx dy = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (78)$$

By Proposition 2.6, we have

$$\left\| (-\Delta)^{s/2} u \right\| \leq m, \quad \int_{\mathbb{R}^d} \frac{u^q(x)}{|x|^\gamma} dx \leq 1, \quad \int_{\mathbb{R}^d} u^p(x) dx \leq 1.$$

By the compact embedding $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ of Theorems 1.2 and 1.5 we have that $u_j \rightarrow u$ almost everywhere and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \frac{u_j^q(x)}{|x|^\gamma} dx = \int_{\mathbb{R}^d} \frac{u^q(x)}{|x|^\gamma} dx = 1.$$

This will imply $W_1^{p,q,s,\gamma}(u) \leq m$. Nevertheless, by the definition of m , we arrive at $W_1^{p,q,s,\gamma}(u) = m$. Then, $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ is the required minimizer and the proof is complete. $\square$

Proof of Corollary 1.2. We know that $d > 0$ by Corollary 1.1. Choose as above a non-negative minimizing sequence $u_j \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, $j \in \mathbb{N}$ converging weakly to $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ and such that

$$\int_{\mathbb{R}^d} (|x|^{(\alpha-d)} * |u_j|^p) |u_j|^p = 1, \quad \int_{\mathbb{R}^d} \frac{u_j^q(x)}{|x|^\gamma} dx = 1, \quad W_2^{p,q,s,\alpha,\gamma}(u_j) \rightarrow m,$$

with $W_2^{p,q,s,\alpha,\gamma}(u)$ as in (23), for $j \rightarrow \infty$. Proposition 2.6 and inequality (77) bring to

$$\|(-\Delta)^{s/2} u\| \leq m, \quad \int_{\mathbb{R}^d} \frac{u^q(x)}{|x|^\gamma} dx \leq 1.$$

The compact embedding $\dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d) \hookrightarrow L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)$ of Theorems 1.2 and 1.5 guarantees that $u_j \rightarrow u$, almost everywhere, with $u_j, u \in L^{\frac{2pd}{d+\alpha}}(\mathbb{R}^d)$, and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} u_j^{\frac{2pd}{d+\alpha}}(x) dx = \int_{\mathbb{R}^d} u^{\frac{2pd}{d+\alpha}}(x) dx = 1.$$

Then (33) in Lemma 2.1 we obtain

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} (|x|^{(\alpha-d)} * |u_j|^p) |u_j|^p = \int_{\mathbb{R}^d} (|x|^{(\alpha-d)} * |u|^p) |u|^p = 1.$$

This gives $W_2^{p,q,s,\alpha,\gamma}(u) \leq m$. We conclude as above that $W_2^{p,q,s,\alpha,\gamma}(u) = m$. Then, we found a minimizer function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$. The proof is completed. $\square$

We obtain

Corollary 6.1. Let $\frac{1}{2} < s < 1$, $p_{s,\gamma}^* - 2 < \tilde{p} < \frac{4s}{d-2s}$, with $p_{s,\gamma}^*$, $q = \tilde{q} + 2$ and γ as in Theorem 1.6. Then, there exists a positive function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$, solution to (2) with $f(x, u)$ as in (3) and such that

$$W_1^{p,q,s,\gamma}(u) = \min_{v \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)} W_1^{p,q,s,\gamma}(v).$$

We get also

Corollary 6.2. Let $\frac{1}{2} < s < 1$, $p_{s,\alpha,\gamma}^* < p < \frac{d+\alpha}{d-2s}$, with $p_{s,\alpha,\gamma}^*$, $q = \tilde{q} + 2$, α and γ as in Corollary 1.1. Then, there exists a positive function $u \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)$ solution to (2) with $f(x, u)$ as in (4) and such that

$$W_2^{p,q,s,\alpha,\gamma}(u) = \min_{v \in \dot{H}_{rad}^{s,q,\gamma}(\mathbb{R}^d)} W_2^{p,q,s,\alpha,\gamma}(v).$$

Remark 6.1. In Corollary 6.1 we improve the result in [7]. To be more precise, we extend the lower bound of the domain of admissibility for p from $\frac{2}{d-2s}$ to $\frac{4s\gamma}{2s(d-1)-(2s-1)\gamma}$. We generalize it then to the case $q > 1$ and $\gamma \neq 1$. Corollary 6.2 is instead new in the literature.

Remark 6.2. We emphasize that the existence of positive minimizer solutions for (2) plays a fundamental role in the study of the dynamics of certain nonlinear evolution equations. To have a full insight into the argument and its association with stability and scattering analysis we cite, for instance, [2] and [16], along with the references provided therein.

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