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Posted Date: 6 November 2023

doi: 10.20944/preprints202311.0272.v1

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Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 November 2023 doi:10.20944/preprints202311.0272.v1

Article

Almost Automorphic Solutions to Nonlinear Difference Equations

Marko Kostić [1,*,‡], Halis Can Koyuncuoğlu [2,‡] and Vladimir E. Fedorov [3,‡]

1 University of Novi Sad, Faculty of Technical Sciences, Trg D. Obradovića 6, 21125 Novi Sad, Serbia; marco.s@verat.net
2 Izmir Katip Celebi University, Department of Engineering Sciences, 35620, Izmir, Turkey; haliscan.koyuncuoglu@ikcu.edu.tr
3 Chelyabinsk State University, Kashirin Brothers St. 129, Chelyabinsk, 454001 Russia; kar@csu.ru
* Correspondence: marco.s@verat.net
‡ These authors contributed equally to this work.

Abstract: In the present work, we concentrate on a certain class of nonlinear difference equations and obtain sufficient conditions for the existence of almost automorphic solutions by employing fixed point theory. Also, we investigate the relationship between the existence of bounded solutions and the existence of almost automorphic solutions for the proposed difference equation type. Thus, we present a Bohr-Neugebauer type theorem for difference equations.

Keywords: discrete almost automorphic; discrete bi-almost automorhic; fixed point; contraction; Bohr-Neugebauer

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1. Introduction

In the theory of dynamic equations, investigation of the existence and uniqueness of periodic solutions has become a very popular research topic for mathematicians, and there is a vast literature on this research direction which focuses on the real life models constructed on continuous, discrete or hybrid time domains with periodic structures. Indeed, analysis of difference equations has taken a prominent attention as much as differential equations, and the studies based on periodicity for the solutions of differential equations have been carried on to discrete domains. Consequentially, the literature on differential and difference equations has grown simultaneously.

Conventional periodicity is a strong but a relaxable condition for some classes of functions. The studies concentrating on the existence of conventionally periodic solutions of dynamic equations may not cover many mathematical models which involve not exactly periodic but nearly periodic arguments in roughly speaking. It is possible to see such real life models in signal processing or in astrophysics (see [1–3]). As a relaxation of the conventional periodicity, the almost periodicity notion was first introduced by H. Bohr ([4]), and the theory of almost periodic functions has been developed by the contributions of several scientists including A.S. Besicovitch, S. Bochner, J. von Neumann, and W. Stepanoff who are very well known in the mathematics community (see [5–8]). The first definition of an almost periodic function was introduced as a topological property; that is a continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if the set

$$E(\varepsilon, f(t)) := \{\tau : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$ for all $\varepsilon > 0$. Subsequently, Bochner proposed normality condition as an almost periodicity criterion, i.e., a continuous function $f(\cdot)$ is called almost periodic if for every real sequence $\{v'_n\}$, there exists a subsequence $\{v_n\}$ of $\{v'_n\}$ such that $\lim_{n\to\infty} f(t+v_n) = \bar{f}(t)$ uniformly for all t (see [6]). Afterwards, the theory of almost automorphic functions was introduced by S. Bochner ([9]) by relaxing the uniform convergence from the normality condition. That is, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost automorphic if for every real sequence $\{v'_n\}$, one can extract a subsequence $\{v_n\}$ of $\{v'_n\}$ such that $\lim_{n\to\infty} \lim_{m\to\infty} f(t+v_n-v_m) = f(t)$ for each $t \in \mathbb{R}$. Thus, the almost automorphy notion can be regarded as a weaker version of almost periodicity. It is obvious that the following relationship holds between the periodicity notions

$$\text{conventional periodicity} \Rightarrow \text{almost periodicity} \Rightarrow \text{almost automorphy},$$

while the inverse of the implication may not be correct. For example, the function

$$f(t) = \sin(2\pi t) + \sin\left(2\sqrt{2}\pi t\right),\ t \in \mathbb{R},$$

is almost periodic but not conventionally periodic, and

$$g(t) = \frac{2 + \exp(it) + \exp\left(i\sqrt{2}t\right)}{\left|2 + \exp(it) + \exp\left(i\sqrt{2}t\right)\right|},\ t \in \mathbb{R},$$

is an almost automorphic function which is not almost periodic (see [10] and [11]). In the recent past, the theories of almost periodic and almost automorphic functions have taken prominent attention from scholars, and the existence of almost periodic and almost automorphic solutions of dynamic equations has become a hot research topic on time domains with continuous, discrete and hybrid structures. We refer to readers the monographs ([10,12–15]), papers ([16–27]), and references therein.

Analysis of the linkage between the existence of bounded and periodic solutions of dynamic equations has always been an interesting research topic in the applied mathematics. Massera's theorem is the primary result for the qualitative theory of differential equations since it commentates boundedness and periodicity of the solutions (see [28]). Since then, various versions of Massera's theorem have been studied for linear and nonlinear dynamic equations over the last five decades. Undoubtfully, when the dynamic equation contains almost periodic or almost automorphic arguments, it becomes a gruelling task to relate the existence of bounded and almost periodic (almost automorphic) solutions. In [29], Bohr and Neugebauer concentrated on the linear system

$$x'(t) = Ax(t) + f(t),$$

and showed that all bounded solutions of almost periodic system of this form are almost periodic on $\mathbb{R}$. Actually, this crucial result can be regarded as an almost periodic analogue of the Massera's theorem. Besides, it should be noted that when $A = A(t)$, and A is conventionally periodic, then it is possible to pursue a similar approach in the light of Floquet theory ([30]). On the other hand, the nonautonomous linear system with almost periodic coefficients

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R},$$

is handled by Favard ([31]), and it is shown that the linear system has at least one almost periodic solution if it has a bounded solution under a separation assumption; that is, each bounded nontrivial solution of the system

$$x'(t) = B(t)x(t), \quad t \in \mathbb{R},$$

satisfies $\inf_{t\in\mathbb{R}} |x(t)| > 0$ where B is in the hull of A. This conception is known as Favard's theory in the existing literature. These milestone results have motivated researchers remarkably, and it is possible to find a detailed literature providing Massera, Bohr-Neugebauer, and Favard type theorems for various kind of dynamic equations based on conventional periodicity, almost periodicity, or almost automorphy notions. We refer to ([21,32–40]) as pioneering studies. However, we shall point out that there is a poor research backlog on Massera or Bohr-Neugebauer type theorems on the almost automorphic solutions of difference equations unlike the enormous literature on differential equations. Thus, one of the main objectives of this research is to make a new contribution to the qualitative theory of difference equations by filling the above-mentioned gap.

In this paper, we are inspired by the recent work [21] of A. Chávez, M. Pinto and U. Zavaleta. We introduce a certain kind of nonlinear summation equation, namely a difference equation,

$$x(t+1) = a(t)x(t) + \sum_{j=-\infty}^{t-1} \Lambda_1(t, j, x(j)) + \sum_{j=t}^{\infty} \Lambda_2(t, j, x(j))$$

with discrete almost automorphic arguments. As the initial task of the study, we focus on the existence and uniqueness of discrete almost automorphic solutions of the nonlinear difference equation by employing fixed point theory. Then, we propose a Bohr-Neugebauer type theorem which relates the existence of bounded and discrete almost automorphic solutions. To the best of our knowledge, our study is the first of its kind since it introduces a discrete counterpart of Bohr-Neugebauer theorem which has not been considered so far, and consequently, it contributes the ongoing theory of difference equations.

2. Background Material

In this section, we aim to give a precise review on discrete almost automorphic functions, and their basic characteristics. For the presentation of the preliminary content, we will first assume that $\mathcal{X}$ stands for a real (or complex) Banach space endowed with the norm $\|\cdot\|_{\mathcal{X}}$.

Definition 1 (Discrete almost automorphy ([19])). *A function $f : \mathbb{Z} \to \mathcal{X}$ is said to be discrete almost automorphic if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} f(t+v_n) =: \bar{f}(t) \tag{1}$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} \bar{f}(t-v_n) = f(t) \tag{2}$$

for each $t \in \mathbb{Z}$.

As it is underlined in [19, Remark 2.2], if the convergence in Definition 1 is uniform, then the concept of discrete almost automorphy turns into a more specific notion, namely discrete almost periodicity. It is clear that every discrete almost periodic function is discrete almost automorphic, however the inverse of the assertion may not be true. In the existing literature, it is possible to find some studies which propose examples of discrete almost automorphic functions that are not discrete almost periodic. For example, Bochner gave an example of discrete almost automorphic function which is not discrete almost periodic

$$f(t) =: sgn(\sin(2\pi t\Omega)), \; t \in \mathbb{Z},$$

for an irrational number Ω in his pioneering work [9] (see also [41]).

Definition 2 (([19])). *A function $g : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ is said to be discrete almost automorphic in t for each $x \in \mathcal{X}$, if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$, there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} g(t+v_n, x) =: \bar{g}(t, x)$$

is well defined for each $t \in \mathbb{Z}$, $x \in \mathcal{X}$, and

$$\lim_{n\to\infty} \bar{g}(t-v_n, x) =: g(t, x)$$

for each $t \in \mathbb{Z}$, and $x \in \mathcal{X}$.

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We refer to [19, Theorem 2.4 and Theorem 2.9] (see also [14]) for reviewing the well-known properties of discrete almost automorphic functions.

Next, we give the notion of discrete bi-almost automorphy in the light of [21, Definition 2.7] for multivariable functions.

Definition 3 (Discrete bi-almost automorphy)**.** *A function $\Lambda : \mathbb{Z} \times \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ is called discrete bi-almost automorphic in $(t, s) \in \mathbb{Z} \times \mathbb{Z}$ uniformly for x on bounded subsets of $\mathcal{X}$ if given any integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ and a bounded set $B \subset \mathcal{X}$, then there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} \Lambda (t + v_n, s + v_n, x) = \bar{\Lambda} (t, s, x)$$

is well defined for each $(t, s) \in \mathbb{Z} \times \mathbb{Z}$, $x \in B$, and

$$\lim_{n\to\infty} \bar{\Lambda} (t - v_n, s - v_n, x) = \Lambda (t, s, x)$$

for each $(t, s) \in \mathbb{Z} \times \mathbb{Z}$, and $x \in B$.

Let $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ denotes the set of all discrete almost automorphic functions defined on $\mathbb{Z}$. Then, $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ is a Banach space when it is endowed with the norm

$$\|f\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} := \sup_{t\in\mathbb{Z}} \|f(t)\|_{\mathcal{X}} . \tag{3}$$

The next result is crucial for the setup of the main outcomes.

Theorem 1 ([19])**.** *Let $g : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ be discrete almost automorphic in t, for each $x \in \mathcal{X}$, and suppose that it satisfies the Lipschitz condition in x uniformly in t, that is*

$$\|g(t, x) - g(t, y)\|_{\mathcal{X}} \le L \|x - y\|_{\mathcal{X}} , \ x, y \in \mathcal{X}.$$

Then, the function $g (t, \varphi (t))$ is discrete almost automorphic function whenever $\varphi : \mathbb{Z} \to \mathcal{X}$ is discrete almost automorphic.

For more details about multi-dimensional almost automorphic sequences and their applications, we also refer the reader to our recent research paper [24].

3. Setup and Main Results

Consider the following abstract nonlinear difference equation

$$x (t + 1) = a (t) x (t) + \sum_{j=-\infty}^{t-1} \Lambda_1 (t, j, x (j)) + \sum_{j=t}^{\infty} \Lambda_2 (t, j, x (j)) , \tag{4}$$

where $a : \mathbb{Z} \to \mathbb{C}$, $a (t) \neq 0$ for all $t \in \mathbb{Z}$, and $\Lambda_{1,2} : \mathbb{Z} \times \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$.

In the sequel, we give the following fundamental result which is essential for the outcomes of the manuscript:

Lemma 1. *The function $x(\cdot)$ is a solution of (4) with the initial data $x (t_0) = x_0$ if and only if*

$$x (t) = x_0 \prod_{s=t_0}^{t-1} a (s) + \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a (s) \right) \left(\sum_{j=-\infty}^{k} \Lambda_1 (k, j, x (j)) + \sum_{j=k+1}^{\infty} \Lambda_2 (k, j, x (j)) \right) . \tag{5}$$

Proof. We multiply both sides of (4) with $\prod_{s=t_0}^{t-1} a^{-1}(s)$, and get

$$x(t+1)\prod_{s=t_0}^{t-1} a^{-1}(s) - a(t)\,x(t)\prod_{s=t_0}^{t-1} a^{-1}(s) = \prod_{s=t_0}^{t-1} a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j)) + \sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right).$$

By writing the above expression as in the following form

$$\begin{aligned} &x(t+1)\,a(t)\prod_{s=t_0}^{t} a^{-1}(s) - a(t)\,x(t)\prod_{s=t_0}^{t-1} a^{-1}(s) \\ &= \prod_{s=t_0}^{t-1} a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j)) + \sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right), \end{aligned}$$

we obtain

$$\Delta\left(x(t)\prod_{s=t_0}^{t-1} a^{-1}(s)\right) = \prod_{s=t_0}^{t-1} a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j)) + \sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right),$$

where Δ stands for the forward difference operator. Next, we take the summation from t_0 to $t-1$

$$\sum_{k=t_0}^{t-1}\Delta\left(x(k)\prod_{s=t_0}^{k-1} a^{-1}(s)\right) = \sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{k} a^{-1}(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,x(j)) + \sum_{j=k+1}^{\infty}\Lambda_2(k,j,x(j))\right).$$

This yields to

$$x(t)\prod_{s=t_0}^{t-1} a^{-1}(s) - x_0 = \sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{k} a^{-1}(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,x(j)) + \sum_{j=k+1}^{\infty}\Lambda_2(k,j,x(j))\right),$$

and one may easily obtain (5). Since every step is reversible, the proof is complete. □

Henceforth, we assume that the following conditions are satisfied throughout the manuscript:

C1 The function $a(\cdot)$ is discrete almost automorphic.
C2 $\Lambda_{1,2}$ are discrete bi-almost automorphic in t and s, uniformly for x.
C3 For $u_{1,2} \in \mathcal{X}$, the Lipschitz inequalities

$$\|\Lambda_1(t,s,u_1) - \Lambda_1(t,s,u_2)\|_{\mathcal{X}} \le m_1(t,s)\,\|u_1 - u_2\|_{\mathcal{X}}$$

and

$$\|\Lambda_2(t,s,u_1) - \Lambda_2(t,s,u_2)\|_{\mathcal{X}} \le m_2(t,s)\,\|u_1 - u_2\|_{\mathcal{X}}$$

hold together with

$$\sup_{t\in\mathbb{Z}}\sum_{j=-\infty}^{t-1} m_1(t,j) = M_1 < \infty,$$
$$\sup_{t\in\mathbb{Z}}\sum_{j=t}^{\infty} m_2(t,j) = M_2 < \infty.$$

Subsequently, we introduce the mapping $H : \mathcal{X} \to \mathcal{X}$ given by

$$(Hx)(t) := x_0\prod_{s=t_0}^{t-1} a(s) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s)\right)\left(S_1(k,x(k)) + S_2(k,x(k))\right), \tag{6}$$

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where

$$S_1(k, x(k)) := \sum_{j=-\infty}^{k} \Lambda_1(k, j, x(j)), \tag{7}$$

and

$$S_2(k, x(k)) := \sum_{j=k+1}^{\infty} \Lambda_2(k, j, x(j)). \tag{8}$$

Lemma 2. *If $x \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$, then $S_1(\cdot, x(\cdot))$ and $S_2(\cdot, x(\cdot))$ are discrete almost automorphic.*

Proof. Suppose that $\xi,\ \varphi \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$. Then we have

$$\begin{aligned}
\|S_1(k,\xi) - S_1(k,\varphi)\|_{\mathcal{X}} &= \left\| \sum_{j=-\infty}^{k} \Lambda_1(k, j, \xi(j)) - \sum_{j=-\infty}^{k} \Lambda_1(k, j, \varphi(j)) \right\|_{\mathcal{X}} \\
&\le \sup_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} \|\Lambda_1(k, j, \xi(j)) - \Lambda_1(k, j, \varphi(j))\|_{\mathcal{X}} \\
&\le \sup_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} m_1(k, j)\, \|\xi - \varphi\|_{\mathcal{X}} \\
&= M_1 \|\xi - \varphi\|_{\mathcal{X}}.
\end{aligned}$$

Similarly, we easily observe that

$$\|S_2(k,\xi) - S_2(k,\varphi)\|_{\mathcal{X}} \le M_2 \|\xi - \varphi\|_{\mathcal{X}}.$$

By Theorem 1, the proof of the assertion is complete. □

Lemma 3. *In addition to C1, C2, and C3, also assume that the condition*

C4 *For every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} x(t_0 \pm v_n) = x(t_0) = x_0$$

holds. Then, H maps $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ into $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$.

Proof. Suppose that $x \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$. By Lemma 2, the functions $S_1(t, x(t))$ and $S_2(t, x(t))$, which are defined in (7-8), are discrete almost automorphic functions in t for each x. That is, for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that

$$\begin{aligned}
\lim_{n\to\infty} S_1(t + v_n, x(t + v_n)) &=: \overline{S_1}(t, \overline{x}(t)), \\
\lim_{n\to\infty} \overline{S_1}(t - v_n, \overline{x}(t - v_n)) &:= S_1(t, x(t))
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} S_2(t + v_n, x(t + v_n)) &=: \overline{S_2}(t, \overline{x}(t)), \\
\lim_{n\to\infty} \overline{S_2}(t - v_n, \overline{x}(t - v_n)) &:= S_2(t, x(t))
\end{aligned}$$

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for each $t \in \mathbb{Z}$. Let us write

$$(Hx)(t+v_n) = x(t_0+v_n)\prod_{s=t_0+v_n}^{t+v_n-1} a(s) + \sum_{k=t_0+v_n}^{t+v_n-1}\left(\prod_{s=k+1}^{t+v_n-1} a(s)\right)(S_1(k,x(k)) + S_2(k,x(k)))$$
$$= x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n)$$
$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+v_n+1}^{t+v_n-1} a(s)\right)(S_1(k+v_n,x(k+v_n)) + S_2(k+v_n,x(k+v_n)))$$
$$= x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n)$$
$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s+v_n)\right)(S_1(k+v_n,x(k+v_n)) + S_2(k+v_n,x(k+v_n))).$$

If we take the limit of $(Hx)(t+v_n)$ as $n \to \infty$ and utilize the Lebesgue convergence theorem, then we have

$$\left(\overline{Hx}\right)(t) = x_0\prod_{s=t_0}^{t-1}\bar{a}(s) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k)) + \overline{S_2}(k,\bar{x}(k))\right).$$

For the converse part, we can follow a similar procedure. Consider

$$\left(\overline{Hx}\right)(t-v_n) = x(t_0-v_n)\prod_{s=t_0-v_n}^{t-v_n-1}\bar{a}(s) + \sum_{k=t_0-v_n}^{t-v_n-1}\left(\prod_{s=k+1}^{t-v_n-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k)) + \overline{S_2}(k,\bar{x}(k))\right)$$
$$= x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n)$$
$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k-v_n+1}^{t-v_n-1}\bar{a}(s)\right)\left(\overline{S_1}(k-v_n,\bar{x}(k-v_n)) + \overline{S_2}(k-v_n,\bar{x}(k-v_n))\right),$$

which results in

$$\left(\overline{Hx}\right)(t-v_n) = x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n)$$
$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)\right)\left(\overline{S_1}(k-v_n,\bar{x}(k-v_n)) + \overline{S_2}(k-v_n,\bar{x}(k-v_n))\right).$$

By taking the limit of $\left(\overline{Hx}\right)(t-v_n)$ as $n \to \infty$, and using the Lebesgue convergence theorem, we obtain $\lim_{n\to\infty}\left(\overline{Hx}\right)(t-v_n) = (Hx)(t)$. This completes the proof. □

Remark 1. *It should be highlighted that the condition C4 is a compulsory technical condition for the construction of existence results. A similar condition can be found in the pioneering work of Bohner and Mesquita (see [20], Theorem 3.10]). On the other hand, the main results of [21] do not require such an abstract condition since the authors concentrate on the solutions of integral equations rather than the solutions of integro-differential equations.*

3.1. Existence Results

Now, we are ready to present our first existence result.

Theorem 2. *Assume that C1-C4 hold, and the condition*

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C5
$$\sup_{t\in\mathbb{Z}}\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}(M_1+M_2)=\kappa<1$$
is satisfied. Then, the abstract difference equation (4) has a unique discrete almost automorphic solution.

Proof. In addition to **C1-C4**, also suppose that **C5** holds. By taking Lemma 2 and Lemma 3 into consideration, it remains to show that the mapping $H(\cdot)$ given in (6) is a contraction. Let ξ, $\varphi\in\mathcal{AA}(\mathbb{Z},\mathcal{X})$; then we have the following:

$$\begin{aligned}
&\|H\xi-H\varphi\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&=\sup_{t\in\mathbb{Z}}\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,\xi(k))-S_1(k,\varphi(k))+S_2(k,\xi(k))-S_2(k,\varphi(k)))\right\|_{\mathcal{X}}\\
&\le\sup_{t\in\mathbb{Z}}\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}(M_1+M_2)\|\xi-\varphi\|_{\mathcal{X}}\\
&\le\kappa\|\xi-\varphi\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}.
\end{aligned}$$

This indicates H is a contraction; by the Banach fixed point theorem, it has a unique fixed point. Thus, the nonlinear difference equation (4) has a unique discrete almost automorphic solution. □

Theorem 3. *Assume that the conditions* **C1-C5** *hold. For a positive constant γ, we define the set*
$$W_\gamma=\left\{x\in\mathcal{AA}(\mathbb{Z},\mathcal{X}):\left\|x-x^0\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\gamma\right\},\tag{9}$$
where
$$x^0(t)=\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,0)+S_2(k,0)).\tag{10}$$
Let $\|x\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\gamma$ and

C6 $\left\|\prod_{s=t_0}^{t-1}a(s)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\psi$ *for all t.*

If
$$\|x_0\|_{\mathcal{X}}\psi+\kappa\gamma\le\gamma,\tag{11}$$
then the nonlinear difference equation (4) has a unique discrete almost automorphic solution in W_γ.

Proof. Consider the operator H which is defined in (6). In the proof of Theorem 2, it is already showed that H is a contraction when the condition **C5** holds. Thus, we have to prove that H maps W_γ into W_γ to conclude the proof. We suppose that $x\in W_\gamma$, and the condition (11) holds. Then, we obtain

$$\begin{aligned}
&\left\|(Hx)(t)-x^0(t)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&\le\|x_0\|_{\mathcal{X}}\left\|\prod_{s=t_0}^{t-1}a(s)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,x(k))-S_1(k,0)+S_2(k,x(k))-S_2(k,0))\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}
\end{aligned}$$

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$$
\begin{aligned}
&\leq \|x_0\|_{\mathcal{X}}\,\psi + \sup_{t\in\mathbb{Z}} \sum_{k=t_0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1+M_2)\,\|x\|_{\mathcal{X}} \\
&\leq \|x_0\|_{\mathcal{X}}\,\psi + \kappa\gamma \leq \gamma.
\end{aligned}
$$

Thus $H(W_\gamma) \subset W_\gamma$. This implies that H has a unique fixed point due to contraction mapping principle, and consequentially, (4) has a unique almost automorphic solution in W_γ. □

Theorem 4. *Suppose that the conditions **C1-C6** hold, and x^0 is as in (10). Consider the closed ball*

$$
W_\phi = W_\phi(x_0,\phi) = \left\{ x \in \mathcal{AA}(\mathbb{Z},\mathcal{X}) : \left\| x - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi \right\}.
$$

If

$$
\|x_0\|_{\mathcal{X}}\,\psi + \kappa\phi + \left\| Hx^0 - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi, \tag{12}
$$

then (4) has a unique discrete almost automorphic solution in W_ϕ.

Proof. Pick $x \in W_\phi$, and assume that (12) is satisfied. Then,

$$
\begin{aligned}
&\left\| (Hx)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \left\| (Hx)(t) - \left(Hx^0\right)(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} + \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \|x_0\|_{\mathcal{X}} \left\| \prod_{s=t_0}^{t-1} a(s) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(S_1(k,x(k)) - S_1(k,x^0(k)) + S_2(k,x(k)) - S_2(k,x^0(k)) \right) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&+ \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \|x_0\|_{\mathcal{X}}\,\psi + \sup_{t\in\mathbb{Z}} \sum_{k=t_0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1+M_2) \left\| x - x^0 \right\|_{\mathcal{X}} + \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}.
\end{aligned}
$$

This implies

$$
\left\| (Hx)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \|x_0\|_{\mathcal{X}}\,\psi + \kappa\phi + \left\| Hx^0 - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi,
$$

and consequentially, $H(W_\phi) \subset W_\phi$. Since the mapping H is a contraction, we deduce that (4) has a unique discrete almost automorphic solution in W_ϕ. □

Example 1. *Consider the nonlinear difference equation given by*

$$
\begin{aligned}
x(t+1) &= \frac{1}{2} sgn\left(\cos 2\pi t\Omega\right) x(t) \\
&+ \sum_{j=-\infty}^{t-1} \frac{1}{20} \left(\frac{1}{4} \left(\sin\left(\frac{\pi}{2} j\right) + \sin\left(\frac{\pi}{2} j\sqrt{2}\right) \right) \right)^{t-j} x(j) + \sum_{j=t}^{\infty} \frac{1}{20} \arctan\left(3^{t-j} x(j)\right),
\end{aligned} \tag{13}
$$

where Ω is an irrational number, and $x(0) = x_0$. A comparison between (4) and (13) results in

$$
a(t) = \frac{1}{2} sgn\left(\cos 2\pi t\Omega\right),
$$

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$$\Lambda_1(t,s,x) = \frac{1}{20}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\sqrt{2}\right)\right)\right)^{t-s} x,$$

and

$$\Lambda_2(t,s,x) = \frac{1}{20}\arctan\left(3^{t-s}x\right).$$

The function $a(\cdot)$ is discrete almost automorphic for any irrational number Ω (see [41]). Besides that, the function $f(t) = \sin\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\sqrt{2}\right)$ is discrete almost periodic, and consequently discrete almost automorphic. Thus, the function Λ_1 is discrete bi-almost automorphic. Despite the fact that the function Λ_2 does not contain any almost automorphic arguments, it can be considered as a discrete bi-almost automorphic function since it is a convolution term. Next, we analyze Λ_1 and Λ_2 in details. We focus on

$$\|\Lambda_1(t,s,x_1) - \Lambda_1(t,s,x_2)\|_{\mathcal{X}} \le \left|\frac{1}{20}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\sqrt{2}\right)\right)\right)^{t-s}\right| \|x_1 - x_2\|_{\mathcal{X}},$$

and set

$$m_1(t,s) = \left|\frac{1}{20}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\sqrt{2}\right)\right)\right)^{t-s}\right|.$$

Subsequently, we write

$$\sup_{t\in\mathbb{Z}} \sum_{j=-\infty}^{t-1} m_1(t,j) \le \sup_{t\in\mathbb{Z}} \sum_{j=-\infty}^{t-1} \frac{1}{20}\left(\frac{1}{2}\right)^{t-j},$$

and obtain the constant $M_1 = \frac{1}{20}$. Similarly, we consider

$$\|\Lambda_2(t,s,x_1) - \Lambda_2(t,s,x_2)\|_{\mathcal{X}} \le \frac{1}{20}3^{t-s}\|x_1 - x_2\|_{\mathcal{X}},$$

*and get $m_2(t,s) = \frac{1}{20}3^{t-s}$. Accordingly, we have the constant $M_2 = \frac{3}{40}$. Thus, the conditions **C1-C3** are satisfied. Furthermore, the condition **C5** holds since*

$$\sup_{t\in\mathbb{Z}} \sum_{k=0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1 + M_2) = \sup_{t\in\mathbb{Z}} \sum_{k=0}^{t-1} \frac{1}{8} \left\| \prod_{s=k+1}^{t-1} \frac{1}{2} sgn(\cos 2\pi s\Omega) \right\|_{\mathcal{X}} \le \frac{1}{16}.$$

*Then, Theorem 2 implies that the nonlinear difference equation (13) has a unique discrete almost automorphic solution whenever the technical condition **C4** holds.*

Furthermore, it is obvious that

$$\left\| \prod_{s=0}^{t-1} \frac{1}{2} sgn(\cos 2\pi s\Omega) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \le 1.$$

If we concentrate on the Theorem 3, then we obtain the existence of unique discrete almost automorphic solution of (13) in the set

$$W_\gamma = \left\{ x \in \mathcal{AA}(\mathbb{Z},\mathcal{X}) : \left\| x - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \le \gamma \right\}$$

*for $\frac{16}{15}\|x_0\|_{\mathcal{X}} \le \gamma$ by tacitly assuming that the condition **C4** holds.*

3.2. Bohr-Neugebauer Criterion

In this part of the manuscript, we focus on the connection between the existence of discrete almost automorphic solutions and bounded solutions of nonlinear difference equations with almost automorphic arguments. Since this result is originated as the Bohr-Neugebauer theorem, the next

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result can be regarded as a discrete variant of Bohr-Neugebauer theorem for a particular class of nonlinear difference equations.

Theorem 5. *Suppose that the conditions **C1-C5** are satisfied. Then, a bounded solution of nonlinear abstract difference equation is discrete almost automorphic if and only if it has a relatively compact range.*

Proof. Necessity: Suppose that $x(\cdot)$ is an almost automorphic solution of (4). This directly implies that its range $\mathcal{R}$ is relatively compact.

Sufficiency: Assume that **C1-C5** hold, and $x(\cdot)$ is a bounded solution of (4) with a relatively compact range $\mathcal{R}$, that is $\bar{\mathcal{R}}$ is compact. By **C1** and **C2**, for any arbitrary integer sequence $\{v_n''\}$, there exists a subsequence $\{v_n'\}$ of $\{v_n''\}$ such that the following limits hold:

$$\lim_{n\to\infty} a(t+v_n') = \bar{a}(t), \ \lim_{n\to\infty} \bar{a}(t-v_n') = a\left(t\right),$$

and

$$\lim_{n\to\infty} \Lambda_{1,2}\left(t+v_n', s+v_n', x\right) = \bar{\Lambda}_{1,2}\left(t,s,x\right), \ \lim_{n\to\infty} \bar{\Lambda}_{1,2}\left(t-v_n', s-v_n', x\right) = \Lambda_{1,2}\left(t,s,x\right).$$

Next, it is clear that $x(t+v_n')$ is a sequence in $\bar{\mathcal{R}}$, and by sequential compactness there exists a subsequence $\{v_n\}$ of $\{v_n'\}$ so that $x(t+v_n) \to \bar{x}\left(t\right)$ as $n \to \infty$. For the sequel, define

$$\varsigma\left(t\right) := x\left(t_0\right)\left(\prod_{s=t_0}^{t-1} \bar{a}\left(s\right)\right) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} \bar{a}\left(s\right)\right)\left(\overline{S_1}\left(k,\overline{x}\left(k\right)\right) + \overline{S_2}\left(k,\overline{x}\left(k\right)\right)\right), \tag{14}$$

where

$$\overline{S_1}\left(k,\overline{x}\left(k\right)\right) = \sum_{j=-\infty}^{k} \bar{\Lambda}_1\left(k,j,\bar{x}\left(j\right)\right),$$

and

$$\overline{S_2}\left(k,\overline{x}\left(k\right)\right) = \sum_{j=k+1}^{\infty} \bar{\Lambda}_2\left(k,j,\bar{x}\left(j\right)\right).$$

We have

$$
\begin{aligned}
&\left\|x(t+v_n)-\varsigma(t)\right\|_{\mathcal{X}}\\
&=\left\|x(t_0+v_n)\prod_{s=t_0+v_n}^{t+v_n-1}a(s)+\sum_{k=t_0+v_n}^{t+v_n-1}\left(\prod_{s=k+1}^{t+v_n-1}a(s)\right)(S_1(k,x(k))+S_2(k,x(k)))\right.\\
&\left.-x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)+\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k))+\overline{S_2}(k,\bar{x}(k))\right)\right\|_{\mathcal{X}}\\
&\leq\left\|x(t_0+v_n)\prod_{s=t_0}^{t-1}a(s+v_n)-x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}}\\
&+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s+v_n)\right)(S_1(k+v_n,x(k+v_n))+S_2(k+v_n,x(k+v_n)))\right.\\
&\left.-x(t_0)\left(\prod_{s=t_0}^{t-1}\bar{a}(s)\right)+\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k))+\overline{S_2}(k,\bar{x}(k))\right)\right\|_{\mathcal{X}}\\
&\leq\left\|x(t_0+v_n)\prod_{s=t_0}^{t-1}a(s+v_n)-x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}}\\
&+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s+v_n)-\prod_{s=t_0}^{t-1}\bar{a}(s)\right)(S_1(k+v_n,x(k+v_n))+S_2(k+v_n,x(k+v_n)))\right\|_{\mathcal{X}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{t-1}\bar{a}(s)\right)(S_1(k+v_n,x(k+v_n))+S_2(k+v_n,x(k+v_n))\right.\\
&\left.-\overline{S_1}(k,\bar{x}(k))-\overline{S_2}(k,\bar{x}(k)))\right\|_{\mathcal{X}}\\
&\leq\left\|x(t_0+v_n)\prod_{s=t_0}^{t-1}a(s+v_n)-x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}}\\
&+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s+v_n)-\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}}\left\|S_1(k+v_n,x(k+v_n))+S_2(k+v_n,x(k+v_n))\right\|_{\mathcal{X}}\\
&+\sum_{k=t_0}^{t-1}\left\|\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|\left(\left\|S_1(k+v_n,x(k+v_n))-\overline{S_1}(k,\bar{x}(k))\right\|_{\mathcal{X}}\right.\\
&\left.+\left\|S_2(k+v_n,x(k+v_n))-\overline{S_2}(k,\bar{x}(k))\right\|_{\mathcal{X}}\right).
\end{aligned}
$$

In the light of Lebesgue convergence theorem, we get $\left\|x(t+v_n)-\varsigma(t)\right\|_{\mathcal{X}}\to 0$ as $n\to\infty$. So, $\bar{x}(t)=\varsigma(t)$, and $\bar{x}$ satisfies (14).

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Now, it remains to show that $\lim_{n\to\infty} \bar{x}(t - v_n) = x(t)$ for each $t \in \mathbb{Z}$. We focus on

$$
\begin{aligned}
&\|\bar{x}(t - v_n) - x(t)\|_{\mathcal{X}} \\
&\le \left\| x(t_0 - v_n) \prod_{s=t_0-v_n}^{t-v_n-1} \bar{a}(s) - x(t_0) \prod_{s=t_0}^{t-1} a(s) \right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0-v_n}^{t-v_n-1} \left(\prod_{s=k+1}^{t-v_n-1} \bar{a}(s) \right) \left(\overline{S}_1(k, \overline{x}(k)) + \overline{S}_2(k, \overline{x}(k)) \right) \right. \\
&\left. - \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(S_1(k, x(k)) + S_2(k, x(k)) \right) \right\|_{\mathcal{X}} \\
&= \left\| x(t_0 - v_n) \prod_{s=t_0-v_n}^{t-v_n-1} \bar{a}(s) - x(t_0) \prod_{s=t_0}^{t-1} a(s) \right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} \bar{a}(s - v_n) \right) \left(\sum_{j=-\infty}^{k} \bar{\Lambda}_1(k - v_n, j - v_n, \bar{x}(j - v_n)) \right. \right. \\
&\left. + \sum_{j=k+1}^{\infty} \bar{\Lambda}_2(k - v_n, j - v_n, \bar{x}(j - v_n)) \right) - \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(\sum_{j=-\infty}^{k} \Lambda_1(k, j, x(j)) \right. \\
&\left. \left. + \sum_{j=k+1}^{\infty} \Lambda_2(k, j, x(j)) \right) \right\|_{\mathcal{X}} \\
&\le \left\| x(t_0 - v_n) \prod_{s=t_0}^{t-1} \bar{a}(s - v_n) - x(t_0) \prod_{s=t_0}^{t-1} a(s) \right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} \bar{a}(s - v_n) \right) \left(\sum_{j=-\infty}^{k} \bar{\Lambda}_1(k - v_n, j - v_n, \bar{x}(j - v_n)) \right. \right. \\
&\left. \left. + \sum_{j=k+1}^{\infty} \bar{\Lambda}_2(k - v_n, j - v_n, \bar{x}(j - v_n)) - \sum_{j=-\infty}^{k} \Lambda_1(k, j, \bar{x}(j - v_n)) - \sum_{j=k+1}^{\infty} \Lambda_2(k, j, \bar{x}(j - v_n)) \right) \right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} \bar{a}(s - v_n) - \prod_{s=k+1}^{t-1} a(s) \right) \left(\sum_{j=-\infty}^{k} \Lambda_1(k, j, \bar{x}(j - v_n)) + \sum_{j=k+1}^{\infty} \Lambda_2(k, j, \bar{x}(j - v_n)) \right) \right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(\sum_{j=-\infty}^{k} \left(\Lambda_1(k, j, \bar{x}(j - v_n)) - \Lambda_1(k, j, x(j)) \right) \right. \right.
\end{aligned}
$$

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$$+\sum_{j=k+1}^{\infty}\left(\Lambda_2\left(k,j,\bar{x}\left(j-v_n\right)\right)-\Lambda_2\left(k,j,x\left(j\right)\right)\right)\Bigg\|_{\mathcal{X}}$$

$$\leq\left\|x\left(t_0-v_n\right)\prod_{s=t_0}^{t-1}\bar{a}\left(s-v_n\right)-x\left(t_0\right)\prod_{s=t_0}^{t-1}a\left(s\right)\right\|_{\mathcal{X}} \tag{15}$$

$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}\bar{a}\left(s-v_n\right)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}\left\|\bar{\Lambda}_1\left(k-v_n,j-v_n,\bar{x}\left(j-v_n\right)\right)-\Lambda_1\left(k,j,\bar{x}\left(j-v_n\right)\right)\right\|_{\mathcal{X}}\right. \tag{16}$$

$$\left.+\sum_{j=k+1}^{\infty}\left\|\bar{\Lambda}_2\left(k-v_n,j-v_n,\bar{x}\left(j-v_n\right)\right)-\Lambda_2\left(k,j,\bar{x}\left(j-v_n\right)\right)\right\|_{\mathcal{X}}\right) \tag{17}$$

$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}\bar{a}\left(s-v_n\right)-\prod_{s=k+1}^{t-1}a\left(s\right)\right\|_{\mathcal{X}}\left\|\sum_{j=-\infty}^{k}\Lambda_1\left(k,j,\bar{x}\left(j-v_n\right)\right)+\sum_{j=k+1}^{\infty}\Lambda_2\left(k,j,\bar{x}\left(j-v_n\right)\right)\right\|_{\mathcal{X}} \tag{18}$$

$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a\left(s\right)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}\left\|\Lambda_1\left(k,j,\bar{x}\left(j-v_n\right)\right)-\Lambda_1\left(k,j,x\left(j\right)\right)\right\|_{\mathcal{X}}\right. \tag{19}$$

$$\left.+\sum_{j=k+1}^{\infty}\left\|\Lambda_2\left(k,j,\bar{x}\left(j-v_n\right)\right)-\Lambda_2\left(k,j,x\left(j\right)\right)\right\|_{\mathcal{X}}\right). \tag{20}$$

The expressions in (15-18) converge to 0 as $n\to\infty$. On the other hand, from (19-20) we get

$$\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a\left(s\right)\right\|_{\mathcal{X}}$$
$$\times\left(\sum_{j=-\infty}^{k}\left\|\Lambda_1\left(k,j,\bar{x}\left(j-v_n\right)\right)-\Lambda_1\left(k,j,x\left(j\right)\right)\right\|_{\mathcal{X}}+\sum_{j=k+1}^{\infty}\left\|\Lambda_2\left(k,j,\bar{x}\left(j-v_n\right)\right)-\Lambda_2\left(k,j,x\left(j\right)\right)\right\|_{\mathcal{X}}\right)$$
$$\leq\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a\left(s\right)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}m_1\left(k,j\right)\left\|\bar{x}\left(j-v_n\right)-x\left(j\right)\right\|_{\mathcal{X}}+\sum_{j=k+1}^{\infty}m_2\left(k,j\right)\left\|\bar{x}\left(j-v_n\right)-x\left(j\right)\right\|_{\mathcal{X}}\right),$$

where we employed C3. Since x is bounded, $\left\|\bar{x}\left(j-v_n\right)-x\left(j\right)\right\|_{\mathcal{X}}$ forms a bounded sequence, and consequently, there exists a subsequence $\{v_p\}$ of $\{v_n\}$ so that

$$\left\|\bar{x}\left(t-v_p\right)-x\left(t\right)\right\|_{\mathcal{X}}\to\theta\left(t\right)$$

as $p\to\infty$. This implies the inequality

$$\theta\left(t\right)\leq\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a\left(s\right)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}m_1\left(k,j\right)\theta\left(j\right)+\sum_{j=k+1}^{\infty}m_2\left(k,j\right)\theta\left(j\right)\right),$$

and this results in $\theta\left(t\right)=0$ due to **C5**. Therefore, $x(\cdot)$ is a discrete almost automorphic solution of (4). The proof is complete. □

Remark 2. *As a direct consequence of Theorem 5, one may easily conclude that any bounded solution of the nonlinear difference equation* (13) *given in Example 1 is discrete almost automorphic.*

4. Conclusions

This study focuses on certain kind of nonlinear difference equations, and provides an elaborative analysis on the existence of discrete almost automorphic solutions under sufficient conditions by fixed point theory. Utilization of the contraction mapping principle in the construction of the main results enables us to get the sufficient conditions regarding the existence and uniqueness of the solutions swiftly and elementarily. In addition to main outcomes regarding existence and uniqueness of almost automorphic solutions, the present work provides a discrete Bohr-Neugebauer type theorem, and polishes the relationship between the existence of bounded and discrete almost automorphic solutions. To the best of our knowledge, our paper is the first one which proposes a Bohr-Neugebauer type result for difference equations. As a continuation of this study, it might be an interesting task to obtain a Bohr-Neugebauer type theorem for q-difference equations by inspiring from the manuscripts [20] and [42].

Author Contributions: Conceptualization, M. Kostić, H. C. Koyuncuoğlu; methodology, M. Kostić, H. C. Koyuncuoğlu and V. Federov; formal analysis, M. Kostić, H. C. Koyuncuoğlu and V. Federov; investigation, M. Kostić, H. C. Koyuncuoğlu and V. Federov; writing—original draft preparation, H. C. Koyuncuoğlu; supervision, M. Kostić, V. Federov; project administration, M. Kostić

Funding: The work of Marko Kostić is partially funded by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. The work of Vladimir E. Fedorov is funded by the grant of President of the Russian Federation to support leading scientific schools, project number NSh-2708.2022.1.1.

Conflicts of Interest: The authors declare no conflict of interest.

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