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Article

Quantum Space-Time Symmetries: A Principle of Minimum Group Representation

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Abstract: We show that, as in the case of the principle of minimum action in classical and quantum mechanics, there exists an even more general principle in the very fundamental structure of *quantum space-time*: This is the principle of *minimal group representation* that allows to consistently and simultaneously obtain a natural description of the spacetime dynamics and the physical states admissible in it. The theoretical construction is based on the physical states that are average values of the generators of the Metaplectic group $Mp(n)$, the double covering of $SL(2C)$ in a vector representation, with respect to the *coherent states* carrying the spin weight. Our main results here are: (i) There exists a connection between the dynamics given by the Metaplectic group symmetry generators and the physical states (mappings of the generators through bilinear combinations of the basic states). (ii) The ground states are coherent states of the Perelomov-Klauder type defined by the action of the Metaplectic group which divide the Hilbert space into *even* and *odd* states mutually orthogonal. They carry a spin weight 1/4 and 3/4 respectively from which, two other basic states can be formed. (iii) The physical states, mapped bilinearly with the basic 1/4 and 3/4 spin weight states, plus their symmetric and antisymmetric combinations, have spin contents $s = 0,\ 1/2,\ 1,\ 3/2$ and 2. (iv) The generators realized with the bosonic variables of the harmonic oscillator introduce a natural supersymmetry and a superspace whose line element is the geometrical Lagrangian of our model. (v) From the line element as operator level, a coherent physical state of spin 2 can be obtained and naturally related to the metric tensor. (vi) The metric tensor is *naturally discretized* by taking the discrete series given by the basic states (coherent states) in the n number representation, reaching the classical (continuous) space-time for $n \to \infty$. (vii) There emerges a relation between the eigenvalue α of our coherent state metric solution and the black hole area (entropy) as $A_{bh}/4l_p^2 = |\alpha|$, relating the phase space of the metric found g_{ab} and the black hole area A_{bh} through the Planck length l_p^2 and the eigenvalue $|\alpha|$ of the coherent states. As a consequence of the lowest level of the quantum discrete space-time spectrum, eg the ground state associated to $n = 0$ and its characteristic length, there exists a minimum entropy related to the black hole history.

Keywords: quantum space-time; fundamental principle; minimum group representation; symmetry; metaplectic group; phase space; quantum coherent states

1. Introduction

Quantum space-time is a key concept both for quantum theory in its own and for a full quantum theory of gravity. The basic motivation of this paper is to demonstrate that, as in the case of classical and quantum mechanics in which the minimum action is the regulating and determining principle, there is an even more general principle that intervenes in the very and fundamental structure of quantum space-time: this is the interplay between dynamics and symmetry or alternatively matter/energy and

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space-time. The maximum simplicity to achieve this goal is based on the Metaplectic group $Mp\,(n)$ which is the double covering of the $S_p(2C)$ group, and which for the illustrative case that we intend to establish here we fix to $Mp\,(2)$.

We describe quantum space-time as arising from a mapping $P\,(G, M)$ between the quantum phase space manifold of a group G and the real space-time manifold M. The metric $g_{a\,b}$ on the phase space group manifold determines the space-time metric of M after identification of (one component of the momentum P operator with the time T.

The signature of the metric depends on the compact or non compact nature of the group, but the most cases of physical interest, the real space-time signature and its hyperbolic structure require non compact groups.

One of the clear simplest examples of this construction is the quantum space-time derived from the phase space of the harmonic oscillator Refs. [1–3], and the mapping $(X,\, P) \rightarrow (X,\, iT)$ in the case of the normal (real frequency) oscillator, or alternatively $(X,\, P) \rightarrow (X,\, T)$ in the case of the inverted (imaginary frequency) oscillator. The quantum space-time algebra of non commutative operator coordinates is the quantum oscillator algebra. The line element arises from the Hamiltonian (Casimir operator) and its discretization yields the quantum space levels. The zero point energy yields the new quantum region splitting the light cone origin because the classical generating lines $X = \pm T$ are replaced by the quantum hyperbolae $X^2 - T^2 = [X, T]$ due to the non-zero space and time commutators, [4,5].

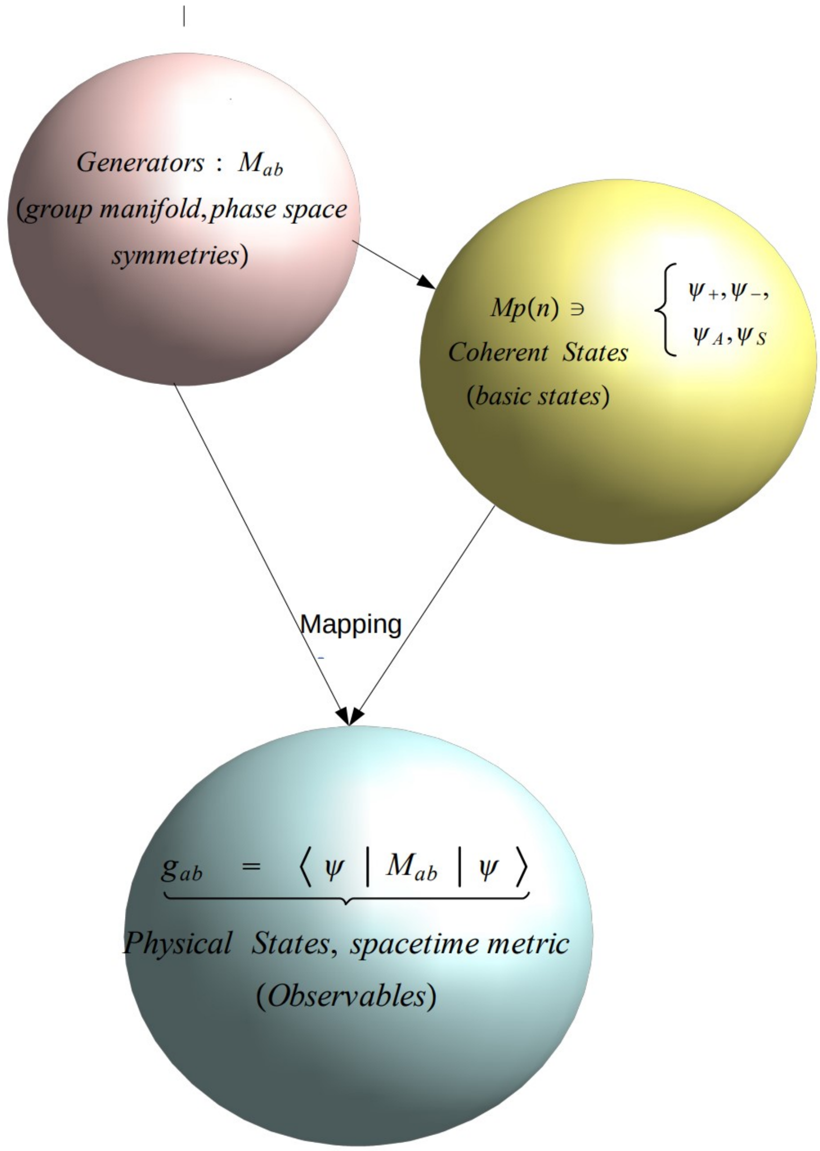
The inverted oscillator is associated to the hyperbolic space-time structure, while the normal oscillator yields euclidean (imaginary time) signature (quantum gravitational instantons).

The inverted oscillator in its different representations does appear in a variety of interesting physical situations from particle physics, black holes and modern cosmology (inflation and today dark energy), Refs. [4–9].

An important point in this principle of minimum group representation is the description of quantum spacetime symmetries: The "algebro-pseudo-differential correspondence" plays a key role. This correspondence establishes that a radical operator (e.g. a Hamiltonian) is equivalent in the context of the metaplectic description, to a Majorana-Dirac type operator with internal variables in the oscillator representation . This correspondence is exemplified in the expression Equation (13), Section IV in this paper. This algebraic interpretation is very important because it brings the possibility to make a link with pseudodifferential operators and semigroup (Fourier-Integral) representations as follows [10]:

$$\begin{array}{ccc} & \begin{array}{c} algebraic \\ interpretation \end{array} & \\ \nearrow & & \nwarrow \\ \textit{pseudo differential operators} & \longleftrightarrow & \begin{array}{c} semigroup \\ (Fourier - Integral\ representations) \end{array} \end{array}$$

In this theoretical and physical context, the solutions obtained consist in two types of states: the basic (non-observable) ones (carrying the weight of spin), and the observable physical states that are bilinear (e.g. mean values) with respect to the basic states. The basic states are coherent states corresponding to the Metaplectic group which is the double covering of the $SL(2C)$ group, [10–13].



We consider as in Ref. [10] a N = 1 superspace equipped with a nondegenerate and invertible supermetric where the unconstrained quantization is exactly performed with new methods based on coherent states and respecting the form of the Hamiltonian. In this way, a discrete structure of the spacetime naturally emerges without any prescription of discretization, but appears from the discrete spectrum of the states itself.

Due to the Metaplectic representation (double covering of the $SL(2C)$) of the coherent state solution representing the emergent spacetime, the crossover from the quantum microscopic regime to the macroscopical regime (classical or not) is natural and consistent. This important fact allows us to conciliate apparently different pictures as that of a macroscopical quantum gravity regime and that of a dynamical quantum microscopic picture (the complete process of black hole emission in all its stages being a clear example).

Despite its simplicity, the framework introduced here have provided physically and geometrically important answers with respect to a consistent quantum gravity formulation.

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It is convenient to consider that this type of coherent states is based on a Lie group G with a unitary, irreducible representation T acting on a Hilbert space $\mathcal{H}$. For a fixed vector ψ_0, we define the coherent state system $\{T, \psi_0\}$ to be the set of vectors $\psi \in \mathcal{H}$ such that $\psi = T(g)\,\psi_0$ for some $g \in G$. Generalized coherent states are defined as the states $|\psi\rangle$ corresponding to these vectors in $\mathcal{H}$.

2. The Metaplectic group and the principle of minimal representation

Mp(2), SU(1,1) and Sp(2)

Let us briefly describe the relevant symmetry group to perform the realization of the Hamiltonian operator of the problem. Specifically, this group is the metaplectic $Mp\,(2)$ as the groups $SU(1,1)$ and $Sp(2)$ that are topologically covered by it. In terms of the $(q,\ p)$ operators, or alternatively $(a,\ a^+)$ the variables of the standard harmonic oscillator, the generators of the group $Mp\,(2)$ are the following ones:

$$\begin{aligned} T_1 &= \frac{1}{4}\left(q\,p + p\,q\right) = \frac{i}{4}\left(a^{+2} - a^2\right), \\ T_2 &= \frac{1}{4}\left(p^2 - q^2\right) = -\frac{1}{4}\left(a^{+2} + a^2\right), \\ T_3 &= -\frac{1}{4}\left(p^2 + q^2\right) = -\frac{1}{4}\left(a^+ a + a\,a^+\right) \end{aligned} \tag{1}$$

With the following commutation relations,

$$[\,T_3\,,\,T_1\,] = i\,T_2\,; \qquad [\,T_3\,,\,T_2\,] = -\,i\,T_1\,; \qquad [\,T_1, T_2\,] = -\,i\,T_3$$

We rewrite the commutation relations as: $[\,T_3\,,\,T_1 \pm iT_2\,] = \pm\,(T_1 \pm\, i\,T_2)\,;$ $[\,T_1 + i\,T_2\,,\,T_1 - iT_2\,] = -\,2\,T_3$, then it is easy to see that: $T_1 + i\,T_2 = -\,\frac{i}{2}\,a^2$ and $T_1\,-\,i\,T_2 = \frac{i}{2}\,a^{+2}$. Therefore, the oscillator states $|\,n\,\rangle$ of the number operator are eigenstates of the T_3 generator

$$T_3\,\,|\,n\,\rangle = -\frac{1}{2}\left(n + \frac{1}{2}\right)|\,n\,\rangle$$

3. The Mp(2) vector representation and its coverings

The main characteristics of the particular representation introduced in [2] is the following commutation relation that defines the generators L_i :

$$[\,L_i, a^\alpha\,] = \frac{1}{2}\,a^\beta\,(\sigma_i)_\beta^{\ \alpha} \tag{2}$$

The above representation corresponds to a non-compact Lie algebra with the following matrix form:

$$\sigma_i = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

that fulfills evidently:

$$\sigma_i \wedge \sigma_j = -i\sigma_k, \qquad \sigma_k \wedge \sigma_i = i\sigma_j, \qquad \sigma_j \wedge \sigma_k = i\sigma_i \tag{4}$$

The equivalence that we want to remark is the following :

The generators in the representation of Equation (2) fulfil the relation:

$$L_i = \frac{1}{2}\,a^\beta\,\,(\sigma_i)_\beta^{\ \alpha}\,a_\alpha = T_i \tag{5}$$

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where T_i are the Metaplectic generators namely [6,7]

$$T_1 = \frac{i}{4}\left(a^{+2} - a^2\right) \tag{6}$$
$$T_2 = -\frac{1}{4}\left(a^{+2} + a^2\right) \tag{7}$$
$$T_3 = -\frac{1}{4}\left(aa^{+} + a^{+}a\right) \tag{8}$$

Proof: Explicitly, in matrix form we can write the generators L_i as

$$L_i = \overline{u}\,\mathbb{M}_i\,v \tag{9}$$

$$\overline{u} \equiv \begin{pmatrix} a^{+} & a \end{pmatrix}, \qquad v \equiv \begin{pmatrix} a \\ a^{+} \end{pmatrix}$$

In the representation Equation (2) that is faithful, we take into account that σ_k enter as a "metric" in the sense given by Sannikov Ref. [14], that is, it introduces the signature in the quadratic terms in a and a^{+} Equation (9) explicitly giving rise to the expression Equation (5). Therefore, we have:

$$M_1 = \frac{i}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4}\,\sigma_k\,\sigma_i \tag{10}$$
$$M_2 = -\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{4}\,\sigma_k\,\sigma_j \tag{11}$$
$$M_3 = -\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{4}\,\sigma_k^2 \tag{12}$$

Consequently, and by inspection, Equation (2) coincides with Equation (9): Thus, the equivalence Equation (5) is proved.

4. Symmetry and Dynamics Principle: Steps to follow

One of the basis of the dynamical description is the Hamiltonian or Lagrangian of the square root type, that is, a non-local and non-linear operator in principle. This is because the *invariance under reparametrizations* as a Lagrangian and as an associated Hamiltonian, generates the correct physical spectrum. The essential guidelines of our approach here are based on the items specifically described in the sequel:

4.1. The invariant action

(i) The elementary distance function (positive square root of the line element) is taken as the fundamental geometric object of the space-time-matter structure, the geometric Lagrangian (functional action) of the theory.

The symmetry of the line element corresponds at least to that given by the Cartan-Killing form of Osp(1,2) or super-Poincaré, which allows having a bosonic realization as a function of the a and a^{+} operators of the standard harmonic oscillator. The metric is consequently non-degenerate, and contains additional odd (fermionic) coordinates.

4.2. Extended Hamiltonian of the system

From (i) the geometric Hamiltonian is obtained in the usual way : this will be the fundamental classical-quantum operator. This universal Hamiltonian (square root Hamiltonian) has an extended phase space: it contains a zero moment P_0 characteristic of the complete phase space at the maximum level from the point of view of the physical states.

The inclusion of a zero momentum P_0 prevents the arbitrary nullification of the Hamiltonian, a fact that occurs in the proper time system in which the evolution coincides with the time coordinate: in this case time "disappears" from the dynamic equations.

Another important point is that the extension of the operator including P_0 allows preserving its square root form.

4.3. Relativistic wave equation and the algebraic interpretation

(iv) The Hamiltonian $\mathcal{H}_s$, when rewritten in differential form, defines a new relativistic wave equation of second order and degree 1/2 (square root form). This can be reinterpreted as a Dirac-Sudarshan type equation of positive energies and internal variables (e.g. oscillator type variables) contained as components of the auxiliary or internal vector L_α:

$$\mathcal{H}_s \Psi \equiv \sqrt{\mathcal{F}} \mid \Psi \rangle \leftrightarrow \left\{ [\mathcal{F}]^\alpha_\beta \, L_\alpha \right\} \Psi^\beta \tag{13}$$

having a para-Bose or para-Fermi interpretation of the basic solution-states of the system $\mid \Psi \rangle$. This gives rise to the main justification for an algebraic interpretation of the radical operator: we have operability and a consistent number of states of the system (the Lagrange multiplier method eliminates the square root, consequently doubling the spectrum of physical states).

4.4. Basic states of representation and the spectrum of physical states

The basic states $\mid \Psi_s \rangle$ belong to the group Mp(n) and have a spin weight s = 1/4, 3/4 in the simplest case Mp(2): They contain *even* and *odd* sectors (s = 1/4, 3/4) in the number of levels of the Hilbert space respectively and therefore, they span non-dense irreducible spaces.

The physical spectrum will be formed by states that are bilinear in fundamental functions (corresponding to $\mid \Psi_{s=1/4,\,3/4} \rangle$, which in the case of $Mp\,(2)$ are $f_{1/4}$ and $f_{3/4}$, having a spin weight $s = 1/4$ and $3/4$ respectively. They are supported and connected by a vector representation of the generators of $Mp\,(n)$ namely L_α, or those covered by $Mp\,(n)$, e.g. $SU\,(p, n-p)$, $SL(nR)$, etc. A characteristic physical state of $Mp\,(2)$ is of the form $\Phi_\mu = \langle s \mid L_\mu \mid s' \rangle$ with $(s, s' = 1/4, 3/4)$, and L_μ being the vector representation of one of the generators of $Mp\,(2)$.

With the Mp(2) interpretation is also possible to describe a *complete* multiplet spanning *spins* from $(0, 1/2, 1, 3/2, 2)$. This is so because with the fundamental states and the allowed vectorial generators, the tower of states is finite and the states involved are *all physical*, as it must be from the physical viewpoint.

It is notable that in the general case, $Sp(2m)$ can be embedded somehow in a larger algebra as $Sp(2m) + R^{2m}$, admitting an Hermitian structure with respect to which it becomes the orthosymplectic superalgebra $Osp(2m, 1)$. Consequently the metaplectic representation of $Sp(2m)$ extends to an irreducible representation (IR) of $Osp(2m, 1)$ which can be realized in terms of the space H of all holomorphic functions $h : C^m \to C / \int |h\,(z)|^2 e^{-|z|^2} d\lambda\,(z) < \infty$ with $\lambda\,(z)$ the Lebesgue measure on C^m.

The restriction of the $Mp\,(n)$ representation to $Sp(2m)$, implies that the two irreducible sectors are supported by the subspaces $H^\pm$ of H, where H^+ and H^- are the spans (closed) of the set of functions $z^n \equiv \left(z_1^{n_1}, ..., z_m^{n_m}\right)$ with $n_\theta \in Z$, $|n| = \sum n_\theta$, *even* and *odd* respectively.

5. Statement of the problem

Geometrically, we take as the starting point the functional action that will describe the world-line (measure on a superspace) of the superparticle as follows:

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$$S = \int_{\tau_1}^{\tau_2} d\tau \, L\left(x, \theta, \bar{\theta}\right) = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\overset{\circ}{\omega}_{\mu} \overset{\circ}{\omega}^{\mu} + \mathbf{a}\dot{\theta}^{\alpha} \dot{\theta}_{\alpha} - \mathbf{a}^{*}\dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}} \tag{14}$$

where $\overset{\circ}{\omega}_{\mu} = \dot{x}_{\mu} - i\left(\dot{\theta}\, \sigma_{\mu}\bar{\theta} - \theta\, \sigma_{\mu}\dot{\bar{\theta}}\right)$, and the dot indicates derivative with respect to the parameter τ, as usual. The above Lagrangian was constructed considering the line element (e.g. the measure, positive square root of the interval) of the non-degenerated supermetric

$$ds^2 = \omega^{\mu}\, \omega_{\mu} + \mathbf{a}\, \omega^{\alpha}\omega_{\alpha} - \mathbf{a}^{*}\, \omega^{\dot{\alpha}}\omega_{\dot{\alpha}},$$

where the bosonic term and the Majorana bispinor compose a superspace $(1, 3\,|1)$, with coordinates $(t, x^i, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$, and where the Cartan forms of the supersymmetry group are described by: $\omega_{\mu} = dx_{\mu} - i\left(d\theta\sigma_{\mu}\bar{\theta} - \theta\sigma_{\mu}d\bar{\theta}\right)$, $\quad \omega^{\alpha} = d\theta^{\alpha}$, $\quad \omega^{\dot{\alpha}} = d\theta^{\dot{\alpha}}$ (obeying evident supertranslational invariance).

As we have extended our manifold to include fermionic coordinates, it is natural to extend also to the superspace the concept of trajectory for a point particle. To do this, we take the coordinates $x(\tau)$, $\theta^{\alpha}(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ.

The Hamiltonian in square root form, namely $\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}\,|\Psi\rangle = 0$, is constructed defining the supermomenta as usual and due the nullification of this Hamiltonian the Lanczos method for constrained Hamiltonian systems was used.

Consequently, there exist an algebraic interpretation of the pseudo-differential operator (square root) in the case of an underlying Mp(n) group structure:

$$\sqrt{\mathcal{H}}\,|\Psi\rangle \equiv \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}\,|\Psi\rangle = 0 \tag{15}$$

$$\left\{[\mathcal{H}]_{\beta}^{\alpha}\left(\Psi L_{\alpha}\right)\right\}\Psi^{\beta} \equiv \left\{\left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right]_{\beta}^{\alpha}\left(\Psi L_{\alpha}\right)\right\}\Psi^{\beta} = 0 \tag{16}$$

Therefore, both structures can be identified: e.g. $\sqrt{\mathcal{H}} \longleftrightarrow [\mathcal{H}]_{\beta}^{\alpha}\left(\Psi L_{\alpha}\right)$, being the state Ψ the square root of a spinor Φ (on which the "square root" Hamiltonian acts) such that it can be bilinearly defined as $\Phi = \Psi L_{\alpha}\Psi$.

The key observation here is that the operability of the pseudo-differential "square root" Hamiltonian can be clearly interpreted if it acts on the square root of the physical states. In the case of the Metaplectic group, the square root of a spinor certainly exist [14–17] making this interpretation Equation (15) and Equation (16) fully consistent from the relativistic and group theoretical viewpoint.

We must emphasize that in the literature there is a history of attempts to establish some type of operator procedure:

As we have remarked before [10], for the typical example in Ref. [18] the Dirac factorization of the one dimensional relativistic Schrodinger equation was treated introducing the so called quantum simulation of the Dirac equation: In Ref. [18], the vector $\underline{\Psi}$ is not an spinor and Equation (37) in Ref. [18] is not the relatvistic counterpart of the Pauli equation: As we have pointed out before there are not spin degrees of freedom and relativistic invariance in Ref. [18].

Consequently, the construction given in Ref. [18] as in other attempts , it was a mathematical artifact in order to mimify the relativistic effects in a sharp contrast with Equation (16) here that is fully relativistic and capable of including a complete (super) multiplet spanning spins from 0, 1/2, 1, 3/2, 2 of physical states.

In the next paragraph, we will describe these states (truly spinorial and relativistic ones) coming from the algebraic correspondence.

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6. Physical states from Symmetries

Generators (dynamical symmetries) being into a oscillator-like vector representation (spinorial) are mapped through their mean values with respect to the basic states (the Mp(n) coherent states) giving rise to the observable physical states. That is to say, there is an interrelation between symmetries and physical states. This gives rise to the first important consequence that, taking into account the unobservable basic states, the bilinear states that are observable can only contain spins (0, 1/2, 1, 3/2, 2).

Next, we will provide a brief theoretical justification to the above construction and then, in the following Section, we describe the emergent space-time discretization mechanism.

There exist some type of operators, L_{ab} for example, where the integral involves *non-diagonal* projectors. This means that is necessary an extension of the set of "acceptable classical observables" to those coherent state distributions $T \in \mathcal{D}'\left(\mathbb{R}^2\right)$ (space of distributions) such that the product $e^{-\eta\,|z|^2}T \in \mathcal{S}'\left(\mathbb{R}^2\right)$, e.g. the tempered distributions in $\mathcal{D}'\left(\mathbb{R}^2\right)$, belong to the Schwartz space $\mathcal{S}'\left(\mathbb{R}^2\right)$.

As pointed out in Ref. [35], an increase in the family of representations of various systems offers new ways to study such systems: Representations of Hilbert space operators in the manner of the Weyl representation may be carried out for a great variety of groups, and asymmetric representations of various forms. (Analogous representations to those presented in [35] for the Weyl group can be introduced for other groups).

In our case here, the big group involved is the *Metaplectic group* $Mp\,(2)$ (the covering group of $SL(2C)$). This important group $Mp\,(2)$ have been studied with some detail in several references [36] and is closely related with the para-Bose coherent states and squeezed states (CS and SS).

In what follows, it is convenient to consider generalized coherent states (CS) based in a Lie group G with a unitary, irreducible representation T acting on some Hilbert space $\mathcal{H}$. If we take a fixed vector ψ_0, we define the coherent state system $\{T, \psi_0\}$ to be the set of vectors $\psi \in \mathcal{H}$ such that $\psi = T(g)\,\psi_0$ for some $g \in G$. Generalized coherent states are defined as the states $|\psi\rangle$ corresponding to these vectors in $\mathcal{H}$.

The basic point for our analysis is the following coherent state reconstructing Kernel for any operator A (not necessarily bounded :

$$K_{\widehat{A}}\left(\alpha,\alpha';g\right) \;=\; e^{\left[\,|\alpha|^2-|\alpha'|^2\,\right]}\left\langle\,\alpha\mid A\mid\alpha'\,\right\rangle \tag{17}$$

where α and α' are complex variables that characterize a respective coherent state, and g is an element of $Mp\,(\,2\,)$. The possible *basic CS states* are classified as:

$$\begin{aligned} |\,\Psi_{1/4}\,(t,\xi,q)\,\rangle \;&=\; f\,(\xi)\;|\,\alpha_+\,(t)\,\rangle \\ |\,\Psi_{3/4}\,(t,\xi,q)\,\rangle \;&=\; f\,(\xi)\;|\,\alpha_-\,(t)\,\rangle \end{aligned} \tag{18}$$

with the following independent, non-equivalent, *symmetric* and *anti-symmetric* combinations

$$\begin{aligned} \left|\,\Psi^S\,\right\rangle \;&=\; \frac{f\,(\xi)}{\sqrt{2}}\,\left(\,|\,\alpha_+\,\rangle\,+\,|\,\alpha_-\rangle\,\right) \;=\; f\,(\xi)\,\left|\alpha^S\,(t)\right\rangle \\ \left|\,\Psi^A\,\right\rangle \;&=\; \frac{f\,(\xi)}{\sqrt{2}}\,\left(\,|\,\alpha_+\,\rangle\,-\,|\,\alpha_-\rangle\,\right) \;=\; f\,(\xi)\,\left|\alpha^A\,(t)\right\rangle \end{aligned} \tag{19}$$

The important fact in order to evaluate the kernels Equation (17) is the action of a and a^2 over the states previously defined

$$a\,|\Psi_{1/4}\rangle = \alpha\,|\Psi_{3/4}\rangle\,;\;\; a\,|\Psi_{3/4}\rangle = \alpha\,|\Psi_{1/4}\rangle\,;\;\; a\left|\Psi^S\right\rangle = \alpha\left|\Psi^S\right\rangle;\;\; a\left|\Psi^A\right\rangle = -\alpha\left|\Psi^A\right\rangle$$

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$$a^2 \left|\Psi_{1/4}\right\rangle = \alpha^2 \left|\Psi_{1/4}\right\rangle ; \; a^2 \left|\Psi_{3/4}\right\rangle = \alpha^2 \left|\Psi_{3/4}\right\rangle ; \; a^2 \left|\Psi^S\right\rangle = \alpha^2 \left|\Psi^S\right\rangle ; \; a^2 \left|\Psi^A\right\rangle = \alpha^2 \left|\Psi^A\right\rangle$$

and similarly for the states $\overline{\Psi}$.

We have that the *physical states* are particular representations of the operators L_{ab} and $\mathbb{L}_{ab} \in Mp\,(2)$ in spinorial form in the sense of quasi-probabilities (tomograms in the Ψ_s plane) or as mean values with respect to the basic coherent states Equations (18) and (19): $|\Psi_\lambda\rangle,\; \lambda \;=\; (1/4,\; 1/2,\; 3/4,\; 1\,)$. There are six possible generalized kernels Equation (17): Two $g\,(\,t,\; s, \pm\,\alpha\,),\; s = 1, 2$ in the Heinsenberg Weil (HW) oscillator representation corresponding to the symmetric and anti-symmetric states respectively:

$$g_{ab}\;(\,t,\,2,\,\alpha\,)\,|_{HW} \;=\; \left\langle\, \Psi^S\,(t) \,\middle|\, L_{ab} \,\middle|\, \Psi^S\;(\,t\,) \,\right\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{(2)\,ab} \tag{20}$$

$$g_{ab}\;(\,t,\,1,\,-\,\alpha\,)\,|_{\,HW} \;=\; \left\langle\, \Psi^A\,(t) \,\middle|\, L_{ab} \,\middle|\, \Psi^A\,(t) \,\right\rangle \;=\; \mathcal{F}\, \begin{pmatrix} -\alpha \\ -\alpha^* \end{pmatrix}_{(1)\,ab} \tag{21}$$

where :

$$\mathcal{F} \;=\; e^{\left[-\left(\frac{m}{\sqrt{2}\,|\,\mathbf{a}\,|}\right)^2 [\,(\,\alpha+\alpha^*\,)-B\,]^2 + D\,\right]}\, e^{[\,\xi\,\varrho\,(\,\alpha+\alpha^*\,)\,]}\;|\,f\,(\,\xi\,)\,|^2$$

and four $g_{ab}\,(\,t,\; s,\; \alpha^2\,)\,,\, s = (\,1,\; 2,\; 1/2,\; 3/2\,)$ for $SU(1,1)$, with the symmetric Ψ^S, anti-symmetric Ψ^A, and $\Psi_{1/4},\,\Psi_{3/4}$ states:

$$g_{ab}\left(t,\,2,\,\alpha^2\right)_{SU(1,1)} \;=\; \left\langle \Psi^S\,(t) \right|\, \mathbb{L}_{ab} \left|\Psi^S\,(t) \right\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha^2 \\ \alpha^{*2} \end{pmatrix}_{(2)\,ab} \tag{22}$$

$$g_{ab}\left(t,\,1,\,\alpha^2\right)_{SU(1,1)} \;=\; \left\langle \Psi^A\,(t) \right|\, \mathbb{L}_{ab} \left|\Psi^A\,(t) \right\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha^2 \\ \alpha^{*2} \end{pmatrix}_{(1)\,ab} \tag{23}$$

$$g_{ab}\left(t,\,3/2,\,\alpha^2\right)_{SU(1,1)} \;=\; \langle \Psi_{3/4}\,(t)|\, \mathbb{L}_{ab}\, |\Psi_{3/4}\,(t)\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha^2 \\ \alpha^{*2} \end{pmatrix}_{(3/2)ab} \tag{24}$$

$$g_{ab}\left(t,\,1/2,\,\alpha^2\right)_{SU(1,1)} \;=\; \langle \Psi_{1/4}\,(t)|\, \mathbb{L}_{ab}\, |\Psi_{1/4}\,(t)\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha^2 \\ \alpha^{*2} \end{pmatrix}_{(1/2)\,ab} \tag{25}$$

where B and D are given by:

$$B \;=\; \left(\frac{|\,\mathbf{a}|}{m}\right)^2 c_1, \quad , \quad D \;=\; \left(\frac{|\,\mathbf{a}\,|\; c_1}{\sqrt{2}\,m}\right)^2 + \,c_2 \tag{26}$$

c_1 and c_2 being constants characterizing the solution or its initial conditions.

Equation (25) are expressed in the so called Sudarshan's diagonal-representation that lead, as an important consequence, the *physical states* with spin content $\lambda \;=\; (\,1/2,\; 1,\; 3/2,\; 2\,)$. Precisely, the generalized coherent states here generate a map that relates the metric, solution of the wave equation g_{ab} to the specific subspace of the full Hilbert space where these coherent states live. Moreover, there exists for operators $\in Mp\,(2)$an asymmetric - kernel leading for our case the following $\lambda = 1$ state :

$$g_{ab}\,(\,t,\,1,\,\alpha)\,|_{HW} \;=\; \langle \Psi_{3/4}\,(t)|\, L_{ab}\, |\Psi_{1/4}\,(t)\rangle = \langle \Psi_{1/4}\,(t)|\, L_{ab}\, |\Psi_{3/4}\,(t)\rangle \;=\; \mathcal{F}\, \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{(1)\,ab}$$

This is so because the non-diagonal projector involved in the reconstruction formula of L_{ab} is formed with the $\Psi_{1/4}$ and $\Psi_{3/4}$ states which span completely the *full* Hilbert space.

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Observation 1: Due to the non observability of isolated basic states, the spin zero physical states appear as bounded states $(g\,\overline{g})$, where $g_{ab}\,(t,s,w)$ and $\overline{g}_{ab}\,(t,s,w)$ are given by the bilinear expressions Equation (25).

Observation 2: Each kernel represents a global *physical* state composed by fundamental states that separately are *basic* and *unobservable*.

Notice that the spectrum of the physical states are labeled not only by their spin content λ, but also by the "eigenspinors" $\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{(\lambda)\,ab}$ and $\begin{pmatrix} \alpha^2 \\ \alpha^{*2} \end{pmatrix}_{(\lambda)\,ab}$ corresponding to the vector representations of L_{ab} and $\mathbb{L}_{ab}$ respectively, (maps over a region of $\mathcal{H}$).

7. Supermetric and emergent spacetime

The Lagrangian density from the action Equation (14) represents a free particle in a superspace with coordinates $z_A \equiv \left(x_\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}\right)$. In these coordinates, the line element of the superspace reads,

$$ds^2 \longrightarrow \dot{z}^A \dot{z}_A \,=\, \dot{x}^\mu \dot{x}_\mu - 2\,i\,\dot{x}^\mu\,(\,\dot{\theta}\,\sigma_\mu\,\bar{\theta} - \theta\,\sigma_\mu \dot{\bar{\theta}}\,) + (\mathbf{a} - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\,\dot{\theta}^\alpha\dot{\theta}_\alpha - (\mathbf{a}^* + \theta^\alpha\theta_\alpha\,)\,\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}$$

It is important to notice that following the steps detailed in Section IV, the quantization is exactly performed providing the correct physical and mathematical interpretation to the square root Hamiltonian, and the correct spectrum of physical states.

Without lose of generality, and for simplicity, we take the solution Equation (20) to represent the metric and with three compactified dimensions ($s = 2$ spin fixed), we have :

$$g_{AB}\,(t) \,=\, e^{\,A(t)\,+\,\xi\,\varrho\,(t)}\,g_{AB}\,(0), \tag{27}$$

where the initial values of the metric components are given by

$$g_{ab}\,(0) \,=\, \langle\psi(0)|\begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab}|\psi(0)\rangle, \tag{28}$$

or, explicitly,

$$g_{\mu\nu}(0) = \eta_{\mu\nu}\,, \qquad g_{\mu\alpha}(0) = -\,i\,\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,, \qquad g_{\mu\dot{\alpha}}(0) = -\,i\,\theta^\alpha\sigma_{\mu\alpha\dot{\alpha}}\,, \tag{29}$$

$$g_{\alpha\beta}(0) = (a - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\,\epsilon_{\alpha\beta}\,, \qquad g_{\dot{\alpha}\dot{\beta}}(0) = -\,(a^* + \theta^\alpha\theta_\alpha)\,\epsilon_{\dot{\alpha}\dot{\beta}}\,. \tag{30}$$

The bosonic and spinorial parts of the exponent in the superfield solution Equation (27) are, respectively,

$$\begin{aligned} A(t) \;&=\; -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 \,+\, c_1\,t \,+\, c_2, \\ \xi\,\varrho\,(t) \;&=\; \xi\,(\,\phi_\alpha\,(t) \,+\, \bar{\chi}_{\dot{\alpha}}\,(t)\,) \\ &=\; \theta^\alpha\left(\overset{\circ}{\phi}_\alpha \cos\,(\,\omega\,t/\,2\,) \,+\, \frac{2}{\omega}\,Z_\alpha\right) - \bar{\theta}^{\dot{\alpha}}\left(-\overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \sin\,(\,\omega t/2\,) - \frac{2}{\omega}\overline{Z}_{\dot{\alpha}}\right) \\ &=\; \theta^\alpha\overset{\circ}{\phi}_\alpha \cos\,(\,\omega t/2\,) \,+\, \bar{\theta}^{\dot{\alpha}}\,\overset{\circ}{\bar{\phi}}_{\dot{\alpha}}\,\sin\,(\,\omega t/\,2\,) \,+\, 4\,|\,\mathbf{a}\,|\,Re(\,\theta Z\,), \end{aligned} \tag{31}$$

where $\overset{\circ}{\phi}_\alpha$, Z_α, $\overline{Z}_{\dot{\beta}}$ are constant spinors, $\omega \,=\, 1/|\,\mathbf{a}\,|$ and the constant $c_1 \in \mathbb{C}$, due to the obvious physical reasons and the chiral restoration limit of the superfield solution. We see in the next Section the associated emerging discrete space-time structure.

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8. Superspace and discrete spacetime structure

Now we will see how the discrete spacetime structure naturally arise from the model. Expanding on a basis of eigenstates of the number operator:

$$\sum_m |m\rangle \langle m| \;=\; 1, \tag{32}$$

we have

$$g_{ab}\,(0) \;=\; \sum_{n,m} \langle\, \psi(0)\,|m\rangle\; \langle\, m\,|\; L_{ab}\;|\, n\,\rangle\; \langle\, n\,|\, \psi(0)\,\rangle \tag{33}$$

Then,

$$g_{ab}(t) \;=\; \underbrace{e^{A(t)\,+\,\xi\,\rho(t)}}_{f(t)} \sum_{n,m} \langle \psi(0)\,|\,m\rangle\; \langle n\,|\,\psi(0)\,\rangle\; \langle m|\; L_{ab}\,|\,n\rangle$$

$$\langle m\,|\; L_{ab}\;|\,n\rangle = \langle m| \begin{pmatrix} a \\ a^{\dagger} \end{pmatrix}_{ab} |n\rangle = \begin{pmatrix} \langle m|n-1\rangle\sqrt{n} \\ \langle m|n+1\rangle\sqrt{n+1} \end{pmatrix}_{ab} = \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab} \tag{34}$$

It follows

$$g_{ab}(0) = \sum_{n,m} \langle\, \psi(0)\,|m\rangle \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab} \langle\, n\,|\, \psi(0)\,\rangle$$

$$g_{ab}(0) = \sum_n \sqrt{n}\; \langle\, \psi(0)\,|\, n-1\,\rangle\; \langle\, n|\, \psi(0)\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \sum_m \sqrt{n+1}\; \langle\, \psi(0)\,|n+1\rangle\; \langle n|\, \psi(0)\,\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab}$$

From the equation above we see that the only clear sense for it is due the decomposition of ψ into the basic states of the metapletic representation

$$|\;\psi(0)\;\rangle \;=\; A\;|\alpha_+\rangle + B\;|\alpha_-\rangle \tag{35}$$

where the constants A and B are arbitrary and they control the classical behavior of the spectrum at the macroscopic level. We, without lose generality in this part of the discussion, take $A = B$ such that $|\psi(0)\rangle = |\alpha_+\rangle + |\alpha_-\rangle$, but we will return to this important point later.

This is in fact, the effect of the decomposition of the SO(2,1) group in two irreducible representations of the metaplectic group Mp(2): spanning even and odd n respectively.

The important feature of the state $|\,\psi(0)\,\rangle = |\,\alpha_+\,\rangle \;+\; |\alpha_-\rangle$ is that it is invariant (if $A = B$) to the action of the operators a and $a^{\dagger}$. This fact is because in the metaplectic representation the general behaviour of these states are: $a\,|\alpha_+\rangle = a^{\dagger}\,|\alpha_+\rangle = |\alpha_-\rangle$ and $a\,|\alpha_-\rangle = a^{\dagger}|\alpha_-\rangle = |\alpha_+\rangle$.

Statistical distributions and classical limit

Is easily checked from the Poissonian distribution for the coherent states:

$$P_{\alpha}(n) = |\,\langle\, n\,|\,\alpha\,\rangle\,|^2 = \frac{\alpha^n e^{-\alpha}}{n!}$$

obeying

$$\sum_{n=0}^{\infty} P_{\alpha}(n) = 1, \quad \sum_{n=0}^{\infty} n\, P_{\alpha}(n) \;=\; \alpha$$

that it differs with the individual distributions coming from each one of the two irreducible representations of the metaplectic group Mp(2) (spanning even and odd n respectively):

$$\left. \begin{array}{l} \sum\limits_{n=0}^{\infty} P_{\alpha_+}(2n) \;=\; e^{-\alpha}\cosh(\alpha) \\ \sum\limits_{n=0}^{\infty} P_{\alpha_-}(2n+1) \;=\; e^{-\alpha}\sinh(\alpha) \end{array} \right\} \;\rightarrow\; \sum_{n=0}^{\infty} \left(P_{\alpha_+}(n) \;+\; P_{\alpha_-}(n)\right) \;=\; 1 \tag{36}$$

Despite of the different form between above equations, the limit n$\rightarrow\infty$ is the same for the sum of the two distributions coming from the Mp(2) irreducible representations (IR), and for the $SO(2,1)$ representation as it should be.

Having this in mind, the specific form of $|\,\alpha_+\,\rangle$, $|\alpha_-\rangle$ are given by :

$$|\,\alpha_+\,\rangle \;\equiv\; |\,\Psi_{1/4}\,(0,\xi,q)\,\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k}\,(0,\xi)\,|2k\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k}\,(0,\xi\,)\,\frac{\left(a^{\dagger}\right)^{2k}}{\sqrt{(2k)!}}\;|0\rangle \tag{37}$$

$$|\alpha_-\rangle \equiv |\,\Psi_{3/4}\,(\,0,\xi,q\,)\,\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k+1}\,(0,\xi)\;\;|2k+1\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k+1}\,(0,\xi)\,\frac{\left(a^{\dagger}\right)^{2k+1}}{\sqrt{(2k+1)!}}\,|0\rangle$$

where in the parameter ξ all the possible odd n dependence is stored.

Consequently, $|\alpha_+\rangle$ connects only with *even* vectors of the basis number and $|\alpha_-\rangle$ with the *odd* vectors in the basis number. Therefore, using the decomposition Equation (35) and decomposing the base number $|n\rangle$ into *even* and *odd*, we arrive to following result:

$$\begin{aligned} g_{ab}(t) \;=\; \frac{f\,(t)}{2}\sum_m \Bigg\{ &[\,P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)\,]\begin{pmatrix}1\\0\end{pmatrix}_{ab} + \\ &+ \Big[P_{\alpha_+^*}(2m)\cdot 2m + P_{\alpha_-^*}(2m+1)\cdot(2m+1)\Big]\begin{pmatrix}0\\1\end{pmatrix}_{ab}\Bigg\} \end{aligned} \tag{38}$$

This expression is the core of our discussion: It shows explicitly *the discrete structure of the spacetime* as the fundamental basis for a consistent quantum field theory of gravity.

On the other hand, when we reach the limit $n\rightarrow\infty$ the metric solution goes to the continuum one:

$$\sum_{n=0}^{\infty}\left[P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)\right] = \alpha e^{-|\alpha|}\left(\cosh(\alpha)+\sinh(\alpha)\right) = \alpha$$

and similarly for the lower part (spinor down) of the above equation:

$$\sum_{n=0}^{\infty}\left[P_{\alpha_+}(2m)\cdot 2m + P_{\alpha_-}(2m+1)\cdot(2m+1)\right] = \alpha^*$$

Consequently, when the number of discrete levels increases, the metric solution goes to the continuum general relativistic "manifold" behavoiur :

$$g_{ab}(t)_{n\rightarrow\infty} \;\rightarrow\; \frac{f\,(t)}{2}\left\{\alpha\begin{pmatrix}1\\0\end{pmatrix}_{ab} + \alpha^*\begin{pmatrix}0\\1\end{pmatrix}_{ab}\right\} = f\,(t)\,\langle\,\psi(0)\,\begin{pmatrix}a\\a^{\dagger}\end{pmatrix}_{ab}\,|\,\psi(0)\,\rangle \tag{39}$$

as expected.

9. The Lowest n = 0 Level and its Length

Is not difficult to see that for the number $n=0$ the metric solution takes the value

$$\begin{aligned} g_{ab}(t) \;&=\; \frac{f\,(t)}{2}\left[P_{\alpha}\,(1)\begin{pmatrix}1\\0\end{pmatrix}_{ab} + \;P_{\alpha_-^*}\,(1)\begin{pmatrix}0\\1\end{pmatrix}_{ab}\right] \\ &=\; \frac{f\,(t)}{2}\,e^{-|\,\alpha\,|}\left[\alpha\begin{pmatrix}1\\0\end{pmatrix}_{ab} + \;\alpha^*\begin{pmatrix}0\\1\end{pmatrix}_{ab}\right] \end{aligned} \tag{40}$$

This evidently defines an associated characteristic length for the eigenvalues α, α^* due the metric axioms in a Riemannian manifold. In principle, (due to the existence of discrete Poincare subgroups of this supermetric), fundamental symmetries as the Lorentz symmetry can be preserved at this level of discretization.

10. Implications for the Black hole entropy: A superspace solution

As is well known, the black hole entropy $S = k_B\, A_{bh}\, / 4\, l_P^2$ where A is the horizon area and $l_P \equiv \sqrt{\hbar G / c^3}$ is the Planck length, was first found by Bekenstein and Hawking [19] using thermodynamic arguments of preservation of the first and second laws of thermodynamics.

An information theory proof was also found by Bekenstein in which black hole entropy can be treated as the measure of "inaccessibility" of the information of an external observer on an actual internal configuration of the black hole realized in a given state (described by values of mass, charge, and angular momentum).

From the point of view of statistical mechanics, the entropy is the mean logarithm of the density matrix. About this issue, Bekenstein proposed a model of quantization of the horizon area with the title "Demystifying black hole's entropy proportionality to area", Ref. [20]:

The horizon is formed by patches of equal area δl_P^2. Their standard size is important and makes them all equivalent. The horizon can be regarded as having many degrees of freedom, one per each patch, made from equivalent patches all with the same number χ of quantum states.

Consequently, the total number of quantum states of the horizon is $\Omega_H = \chi^{A_{bh}\, /\, \delta\, l_P^2}$ and the statistical (Boltzmann) entropy associated with the horizon is $S = k_B \ln \Omega_H = k_B \left(A_{bh}\, /\, \delta\, l_P^2 \right) \ln \chi$:

Choosing $\delta = 4 \ln \chi$ Bekenstein arrived to the expected thermodynamical black hole formula.

However, Bekenstein do not gives account that introducing δ into the original black hole entropy formula one obtains the Poisson expression for the total number of states:

$$\Omega_H = e^{A_{bh}\, /\, 4\, l_P^2} \tag{41}$$

This expression is precisely *the link* with the structure of the emergent coherent state metric of our approach here. Considering the similar Poissonian expression for the number of states from g_{ab}, namely $e^{|\alpha|}$, the relation between the coherent state eigenvalue α corresponding to our coherent state metric solution and the above equation is clear:

$$A_{bh}\, /\, 4\, l_P^2 = |\, \alpha\, | \tag{42}$$

This expression relates the phase space of the coherent state solution metric g_{ab} and the black hole area A_{bh} through the Planck length l_P^2 and the eigenvalue $|\alpha|$ characterizing the coherent states.

11. Implications for Hawking Radiation

- Due to the interplay between the area of the black hole surface and the black hole mass, it is quantized as well. The mass of the black hole decreases when radiation is emitted due the quantum jump from one quantized value of the mass (energy) to a lower quantized value.
- As a consequence, (because radiation is emitted at quantized frequencies corresponding to the differences between energy levels), quantum gravity implies a discretized emission spectrum for the black hole radiation.
- The spectral lines can be very dense in macroscopic regimes leading physically no contradiction with Hawking's prediction of a continuous thermal spectrum in the semiclassical regime.
- From the point of view of our approach here:
- If we now suppose simply that the constants A, B in the state solution eg. Eq. (27), Eq. (35), are different, $A \neq B$ we have :

$$|\psi(0)\rangle = A\, |\, \alpha_+\, \rangle + B\, |\, \alpha_-\, \rangle,$$

- Then, we cannot reach the thermal (Hawking) spectrum at the macroscopic level.

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- This fact is clear because we need exact balance between the superposition of the two irreducible representations of the Metaplectic group.

  This will lead as a result, non classical states of radiation in the sense of [21] as can be easily seen putting, for example, the constants B (or A) equal to zero:

$$g_{ab}(t) \;=\; A\,\frac{f(t)}{2}\sum_{m}\left[P_{\alpha_+}(2m)\cdot(2m)\;+\;P_{\alpha-}(2m+1)\cdot(2m+1)\right]\begin{pmatrix}1\\0\end{pmatrix}_{ab} \tag{43}$$

- Notice that only the up spinor part survives and the classical (thermal) limit is not reached, even in the continuous limit where the number of levels increases accordingly to

$$g_{ab}(t)\;_{n\rightarrow\infty}\;\rightarrow\;\frac{f(t)}{2}\,A\,\alpha\begin{pmatrix}1\\0\end{pmatrix}_{ab}\;=\;A\,f(t)\,\langle\,\psi(0)\,|\begin{pmatrix}a\\0\end{pmatrix}_{ab}|\,\psi(0)\,\rangle \tag{44}$$

- In such a case where $A=0$, (or $B=0$), the spectrum will takes only *even (or odd)* levels becoming evidently non thermal.

Therefore,

- if $A=B$ the thermal Hawking spectrum is reached at the continuum classical gravity level eg, the Poissonian behaviour of the distribution is complete.
- Otherwise, with $A\neq B$, the spectrum belongs to a non classical one and the quantum properties of gravity are macroscopically manifest.

12. Concluding remarks

Here we have shown that there is a principle of minimal group representation that allows us to consistently and simultaneously obtain a natural description of the dynamics of spacetime and the physical states admissible in it.

The theoretical construction is based on the fact that the physical states are, roughly speaking, average values of the generators of the metaplectic group Mp(n) in a vector representation, with respect to the coherent states that are not observable (carrying the weight of spin). Schematically, we have the following picture where $M_{ab}=\mathbb{L}_{ab}(L_{ab})$:

$$\underbrace{g_{ab}\;=\;\langle\,\psi\,|\;M_{ab}\;|\,\psi\,\rangle}_{\textit{Physical States, spacetime metric (Observables)}}$$

$$\nearrow \qquad\qquad\qquad\qquad \nwarrow$$

$$\begin{matrix}Mp(n)\ni\\ \textit{Coherent States}\\ (\textit{basic states})\end{matrix}\;\left\{\begin{matrix}\psi_+,\psi_-,\\ \psi_A,\psi_S\end{matrix}\right.\qquad\longleftrightarrow\qquad\begin{matrix}\textit{Generators}:\;M_{ab}\\ (\textit{group manifold,}\\ \textit{phase space}\\ \textit{symmetries})\end{matrix}$$

In summary:

(1) We demonstrate that there is a connection between dynamics, given by the generators of the symmetries, and the physically admissible states.

(2) The physically admissible states are mappings of the generators of the relevant symmetry groups covered by the metaplectic group, in the simplest case according to the chain: $Mp(2)\supset SL(2R)\supset SO(1,2)$) through a bilinear combination of basic states.

(3) The ground states are coherent states defined by the action of metaplectic group (Perelomov-Klauder type), these states divide the Hilbert space into *even* and *odd* states, and are mutually orthogonal. They carry a weight of spin $1/4$ and $3/4$ respectively.

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(4) From the basic states combined symmetrically and antisymmetrically, two other basic states can be formed. These new states manifest a change of sign under the action of the creation operator a^+.

(5) The physically admissible states, mapped bilinearly with the basic states with spin weight 1/4 and 3/4, plus their symmetric and antisymmetric combinations, have spin contents $s = 0,\ 1/2,\ 1,\ 3/2$ and 2.

(6) A symmetry of the superspace is formed by a realization of the generators with bosonic variables of the harmonic oscillator as Lagrangian. Taking a line element corresponding to such superspace a physical state of spin 2 can be obtained and related to the metric tensor.

(7) The metric tensor is discretized simply by taking the discrete series given by the basic states (coherent states) in the number n representation, consequently the metric tends to the classical (continuous) value when $n \to \infty$.

(8) The results of this paper have implications for the lowest level of the discrete spectrum of space-time, the ground state associated to $n = 0$ and its characteristic length, in the black hole history of black hole evaporation

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