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Article

Improved Stability and Instability Results for Neutral Integro-Differential Equations Including Infinite Delay

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Abstract: In this study, we deal with a nonlinear system of neutral Volterra integro-differential equations (NVIDEs) with infinite delay. We study stability, uniform stability and instability of zero solution of the system of NVIDEs with infinite delay. We obtain three new results with regard to these concepts via two new Lyapunov– Krasovskii functionals (LKFs), respectively. As the main contributions of this work, we improve and generalize some earlier results in the relevant database from the linear case to more general nonlinear cases.

Keywords: NVIDE; stability; uniform stability; instability; LKF method

MSC: 34K05; 34K20; 45J05

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1. Introduction

Functional differential equations (FDEs) play important and effective roles not only in mathematics but also in many fields of applied sciences. In recent five decades FDEs have been introduced in many mathematical models of real world problems. Hence, important progresses have been made on linear and nonlinear FDEs and qualitative studies of their solutions. Indeed, FDEs vary from problems in mechanics, technology, population dynamics and economics to problems in biology and medicine, see the books of Hale and Verduyn Lunel [14], Kolmanovskii and Myshkis [15], Krasovskii [16], Kuang [17] and Kuang et al. [18].

The main dissimilarity between ordinary differential equations (ODEs) and FDEs is that in FDEs the progression of a system depends not only upon its current state but also on its history. One can divide FDEs into three classes; retarded, advanced and neutral equations. One easily finds more complicated equations and systems of FDEs, e.g. higher order, more delays, delays depending on $x(t)$, infinite delays, etc. The problems one is interested in FDEs are the same as for ODEs: existence of solutions, periodic solutions, stability and instability of solutions, etc. This is clear from the view of the applied scientist: only stable solutions can be seen in experiments.

In the relevant literature, retarded and neutral equations are treated separately but for both types of these equations the LKF method and the fixed point method are applied, and the linear autonomous case is given extra attention. According to the application of LKF method, it is essential the existence of an LKF which has to be constructed from the given equation. However, this is a nontrivial task, and it is also very hard for systems of NVIDEs. As for the fixed point method, it is needed to find a suitable set, a suitable mapping and a suitable fixed point theorem, respectively. The set has to be consists of points which would be acceptable solutions. Next, the map has fixed points such that they solve the problem under study. Finally, the fixed point theorem make enable that the mapping on this set will have a fixed point.

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We now outline some interesting results from the relative international database on the qualitative studies of FDEs called retarded (delay) and neutral equations, NVIDEs, IDEs, etc.

In 2019, Berezansky and Braverman [3] obtained stability results for a linear scalar neutral differential equation with two retardations as well as for its generalizations. In [3], the technique of the proof is based on the Bohl-Perron theorem and a priori estimates of solutions. Next, in [3], it is also benefited from integral inequalities to obtain stability results.

In 2019, Berezansky and Braverman [4] derived exponential stability assumptions for the following linear scalar neutral differential equation with two variable and bounded retardations:

$$\frac{d}{dt}[x(t) - a(t)x(g(t))] + b(t)x(h(t)) = 0.$$

In the paper of Berezansky and Braverman [4], the Bohl-Perron theorem and a reduction of the considered neutral differential equation are applied as basic tools in the proof to investigate the exponential stability.

In 1986, Qian [25] considered a class of linear NIDEs

$$\frac{d}{dt}\left(x(t) - \int_0^t D(t,s)x(s)ds\right) = Ax(t) + \int_0^t C(t,s)x(s)ds$$

and its convolution type

$$\frac{d}{dt}\left(x(t) - \int_0^t D(t,s)x(s)ds\right) = Ax(t) + \int_0^t C(t,s)x(s)ds.$$

Using LKF method and variation of parameters formula, Qian [25] obtained stability criteria, instability criteria for the first NIDE and some equivalent statements with regard to the stability of null solution of the second NIDE, respectively.

In 1987, Gopalsamy [11] derived a set of interesting assumptions for asymptotic stability of trivial solution of a class of scalar linear neutral IDEs . In Gopalsamy [11], the technique of LKFs is employed to derive conditions for asymptotic stability.

In 1995, Ma [21] studied stability of IDEs of neutral type,

$$\frac{d}{dt}\left(Z(t) - \int_a^t C(t,s)Z(s)ds\right) = A(t)x(t) + \int_a^t B(t,s)Z(s)ds.$$

In 1996, Yi [41] considered specify neutral Lotka-Volterra systems including bounded and unbounded delays, respectively. In the cases of bounded and unbounded delays, asymptotic and uniform stability results were derived according to the parameters of these systems, respectively.

In 2003, Chen and Sun [8] dealt with existence of periodic solutions of NVIDE with infinite delay,

$$x'(t) = a(t)x(t) + \int_{-\infty}^t C(t,s)x(s)ds + \int_{-\infty}^t C(t,s)x'(s)ds.$$

In Chen and Sun [8], sufficient conditions with regard to this concept and its uniqueness for NVIDE above are obtained by the contraction mapping theorem.

In 2013, Ardjouni and Djoudi [2] obtained asymptotic stability results for trivial solution of a scalar nonlinear NVIDE with variable retardations via a contraction mapping theorem.

In 2017, Yankson [40] considered a scalar neutral IDE with multiple functional delays. The work of Yankson [40] is devoted to get necessary and sufficient conditions for the considered scalar neutral

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IDE to be asymptotically stable. In Yankson [40], it is benefited from the contraction mapping principle to acquire the proof.

In 2021, Mansouri et al. [22] considered a scalar system of coupled nonlinear NVIDEs with two delays. Mansouri et al. [22] studied the existence and asymptotic stability of periodic solutions of that NVIDEs with two delays. Here, Krasnosel'skii's fixed point theorem is adopted to derive the main results of [22].

In 2021, Andreev and Peregudova [1] considered the stability problem for the non-autonomous and nonlinear IDE of Volterra type with infinite delay of the form:

$$x'(t) = f\left(t, x(t), \int\limits_{-\infty}^{t} g(t,s,x(s))ds\right).$$

In Andreev and Peregudova [1], the development of the LKF method is carried out in both the limiting behavior study of a bounded solution as well as the asymptotic stability of trivial solution. Additionally, in [1], the problems on the study of the motion limiting properties for a mechanical system with linear heredity as well as the stationary motion stabilization of a manipulator with viscoelastic cylindrical and spherical joints are also solved.

In 2023, Nowak et al. [23] considered nonlinear neutral IDEs with variable delays of the form

$$\frac{d}{dt}\left[x(t) - Q(t, x(t-\tau(t)))\right] = -\int\limits_{t-\tau(t)}^{t} a(t,s)x(s)ds.$$

Nowak et al. [23] obtained asymptotic stability results for those nonlinear neutral IDEs with a variable delay. In [23], the authors utilized using Sadovskii's fixed point theorem to fulfill the proofs.

Li and Jiang [20] investigated the stability, uniform stability and instability of trivial solution of the system of linear NVIDEs of first order with infinite delay via the LKF method:

$$\frac{d}{dt}\left(x(t) - \int\limits_{-\infty}^{t} B(t,s)x(s)ds\right) = A(t)x(t) + \int\limits_{-\infty}^{t} C(t,s)x(s)ds. \tag{1}$$

Li and Jiang [20] obtained four new qualitative results with regard to zero solution of the system of linear NVIDEs (1) of first order with infinite delay.

On the other hand, as for some other qualitative results with regard to IDEs with and without delay(s), however not with regard to NVIDEs, one can find numerous results in papers of Bohner et al. [5], Bohner and Tunç [24], Burton and Mahfoud [6], Chang and Wang [7], Eloe et al. [9], Funakubo et al. [10], Gözen and Tunç [12], Graef and Tunç [13], Qian [26], Staffans [27], Taie and Bakhit [28], Tunç [29–31], Tunç and Tunç [32,33], Tunç et al. [34,35], Vanualailai and Nakagiri [36], Wang [37], Wang et al. [38], Xu [39], Zhang [43,44] , the references of these papers and the books of Lakshmikantham and Rama Mohana Rao [19], Volterra [42]. In these papers, the main results have been proved via the LKF method or Lyapunov's function method. We should express that the IDEs in these papers or the related ones in the references of these papers are not in the neutral forms. They include constant delays or variable delays or they are without delays and infinite delays. From these information, we see that in despite of the existence of numerous results with regard to IDEs with and without delays, there exists a number of limited results on the qualitative results of IDEs in neutral forms such that in which the LKF method used as basic tool in the proofs. In addition, as for qualitative results with regard to systems of NVIDEs, in which the LKF method used as basic tool in the proofs, we found only the above results, see Li and Jiang [20]. The proper reason for this case is that because of the difficulty to find a suitable LK F which allows meaning results for nonlinear systems of NVIDEs. This is an open problem in the relevant literature by this time. From the above data, we may ask that the investigation

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of stability, uniform stability, instability, etc., of solutions of nonlinear systems of NVIDEs with infinite delay via LKF method has novelty and new contributions to qualitative theory of NVIDEs with and without delay.

This article is devoted to the following system of NVIDEs of first order containing infinite delay

$$\frac{d}{dt}\left(x(t)-\int_{-\infty}^{t} C(t,s)x(s)ds\right)=A(t)x(t)+\int_{-\infty}^{t} F(t,s,x(s))ds \tag{2}$$

where $x\in\mathbb{R}^n$, $A(t)$ and $C(t,s)$ are $n\times n$ matrices of continuous functions for $-\infty<s\leq t<\infty$ and $F\in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ with $F(t,s,0)=0$. The main aim of this article is to study stability, uniform stability and instability of zero solution of nonlinear system of NVIDEs (2) with infinite delay. We establish three new outcomes as new theorems with regard to these concepts defining and using two new LKFs. The motivation of this study has been inspired from the results of the papers, the books above and in particular from the results of Li and Jiang [20]. We aim to obtain new outcomes for the concepts mentioned and do new contributions to qualitative theory of NVIDEs via the LKF method.

Let $BC(-\infty,t_0]$ denote the class of bounded and continuous functions ϕ from the interval $(-\infty,t_0]$ to $\mathbb{R}^n$. Next, let $\phi\in BC(-\infty,t_0]$. Then, we describe the norm of the initial function ϕ by

$$\|\phi\|=\sup\left\{\|\phi(t)\|:\ t\in(-\infty,t_0]\right\}.$$

We will denote a solution of system of NVIDE (2) of first order with infinite delay by $x(t)$ either $x(t,t_0,\phi)$, which fulfill the initial condition $x(t)=\phi(t)$ on the interval $(-\infty,t_0]$, in which $\phi\in BC(-\infty,t_0]$.

Let $\|x\|=\sum_{i=1}^{n}|x_i|$, $x\in\mathbb{R}^n$, $\|A\|=\max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}|$ and the matrix measure

$$\mu(A)=\max_{1\leq j\leq n}\left\{a_{jj}+\sum_{i=1,i\neq j}^{n}|a_{ij}|\right\}.$$

Letting

$$Dx_t=\left(x(t)-\int_{-\infty}^{t} C(t,s)x(s)ds\right),$$

then from system of NVIDEs (2) we have

$$\frac{d}{dt}Dx_t=A(t)Dx_t+\int_{-\infty}^{t}\aleph(t,s,x(s))ds,$$

where

$$\aleph(t,s,x(s))=A(t)C(t,x(s))x(s)+F(t,s,x(s)).$$

The paper is formed as in the bellow lines. In Section 2, outcomes of this study such as stability and uniformly stability are stated and proved. In Section 3, instability outcome of this study is hold forth and proved. Section 4 explains the contributions of this study to the relevant literature briefly. At the end, in Section 5, the conclusion of this study is designed.

2. Stability

In this part, for the stability and uniformly stability results, we arrange the following sufficient conditions:

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$(As1^0)$

$$\mu(A(t)) = \max_{1\le j\le n}\left\{a_{jj}(t) + \sum_{i=1,i\neq j}^{n}\left|a_{ij}(t)\right|\right\} \le 0;$$

$(As2^0)$

$$\left\|\aleph(u,t,x(t))\right\| \le \aleph_0 \left\|\Omega(u,t)\right\| \;\; \left\|x(t)\right\|,$$

where $\Omega(u,t)$ is an $n\times n$ matrix of continuous functions for $-\infty < s \le t < \infty$, $\aleph_0 \in \mathbb{R}$, $\aleph_0 > 0$;
$(As3^0)$

$$\mu(A(t)) + \int_t^{\infty} \|L(u,t)\| du \le 0,$$

where

$$\|L(u,t)\| = \|A(u)\| \, \|C(u,t)\| + \aleph_0 \left\|\Omega(u,t)\right\|;$$

$(As4^0)$

$$\int_{-\infty}^{t} \|C(t,s)\| ds \le K_1 < 1, K_1 \in \mathbb{R}, K_1 > 0,$$

$$\int_{-\infty}^{t_0}\int_{t_0}^{\infty} \left\|\Omega(u,s)\right\| duds \le K_2 < \infty, K_2 \in \mathbb{R}, K_2 > 0,$$

$$\int_{-\infty}^{t_0}\int_{t_0}^{\infty} \|C(u,s)\| \;\; \|A(u)\| duds \le K_3 < \infty, K_3 \in \mathbb{R}, K_3 > 0.$$

The first result of this article as the stability of trivial solution of system of NVIDEs (2) is introduced in the subsequent theorem.

Theorem 1. *If $(As1^0) - (As4^0)$ hold, then the solution $x = 0$ of system of NVIDEs (2) is stable.*

Proof. As similar to Li and Jiang [20], from the definition of Dx_t and system of NVIDEs (2), we obtain $D^+ \|Dx_t\|$, which denotes the upper right derivative , as the below:

$$\begin{aligned} D^+ \|Dx_t\| =& \lim_{h\to 0^+} \sup \frac{\|Dx_{t+h}\| - \|Dx_t\|}{h} \\ \le& \lim_{h\to 0^+} \sup \left(\frac{\|I + hA(t)\|}{h} \text{-1}\right) \|Dx_t\| + \left\|\int_{-\infty}^{t} \aleph(t,s,x(s)) ds\right\| \\ \le& \lim_{h\to 0^+} \sup \left(\frac{\|I + hA(t)\|}{h} \text{-1}\right) \|Dx_t\| + \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds \\ \le& \mu(A(t)) \|Dx_t\| + \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds. \end{aligned} \tag{3}$$

We will now introduce an LKF $W = W(t,x_t)$ by

$$W = \|Dx_t\| + \int_{-\infty}^{t}\int_{t}^{\infty} \|\aleph(u,s,x(s))\| duds + \int_{-\infty}^{t} \|x(s)\| ds \int_{t}^{\infty} \|C(u,s)\| \;\; \|A(u)\| du.$$

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As coming calculations, the upper right derivative of this LKF along the solutions of system of NVIDEs (2) results in

$$\begin{aligned} D^{+}W(t,x_t) =& D^{+}\left\|Dx_t\right\| + \int_t^{\infty} \|\aleph(u,t,x(t))\|du - \int_{-\infty}^{t} \|\aleph(t,s,x(s))\|ds \\ & + \|x(t)\| \int_t^{\infty} \|A(u)\| \ \|C(u,t)\|\, du - \int_{-\infty}^{t} \|A(t)\| \ \|C(t,s)\| \ \|x(s)\|\, ds. \end{aligned} \tag{4}$$

From system of NVIDEs (2), inequality (3), $(As2^0)$ and equality (4), we notify that

$$\begin{aligned} D^{+}W(t,x_t) \leq& \mu(A(t))\left\|Dx_t\right\| + \int_{-\infty}^{t} \|\aleph(t,s,x(s))\|\, ds \\ & + \int_t^{\infty} \|\aleph(u,t,x(t))\|du - \int_{-\infty}^{t} \|\aleph(t,s,x(s))\|ds \\ & + \|x(t)\| \int_t^{\infty} \|A(u)\| \ \|C(u,t)\|\, du - \int_{-\infty}^{t} \|A(t)\| \ \|C(t,s)\| \ \|x(s)\|\, ds \\ \leq& \mu(A(t))\left\| x(t) - \int_{-\infty}^{t} C(t,s)x(s)ds \right\| + \aleph_0 \|x(t)\| \int_t^{\infty} \|\Omega(u,t)\|du \\ & + \|x(t)\| \int_t^{\infty} \|A(u)\| \ \|C(u,t)\|\, du - \int_{-\infty}^{t} \|A(t)\| \ \|C(t,s)\| \ \|x(s)\|\, ds. \end{aligned}$$

According to $(As1^0)$, we know that

$$\mu(A(t)) = \max_{1\leq j\leq n}\left\{ a_{jj}(t) + \sum_{i=1, i\neq j}^{n} \left|a_{ij}(t)\right| \right\} \leq 0.$$

Next, we also note that

$$\|A(t)\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} \left|a_{ij}(t)\right|$$

and

$$|\mu(A(t))| \leq \|A(t)\|.$$

Hence, it is explicit that

$$-\|A(t)\| \leq \mu(A(t)) \leq \|A(t)\|.$$

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Using the inequality $-\mu(A(t)) - \|A(t)\| \leq 0$ and an elementary one, it is now remarkable that

$$\begin{aligned} D^{+}W(t,x_t) \leq & \mu(A(t)) \|x(t)\| - \mu(A(t)) \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds \\ & + \|x(t)\| \left[\aleph_0 \int_{t}^{\infty} \|\Omega(u,t)\| + \|A(u)\| \ \|C(u,t)\| \right] du \\ & - \int_{-\infty}^{t} \|A(t)\| \ \|C(t,s)\| \ \|x(s)\| \, ds \\ & = \left[\mu(A(t)) + \int_{t}^{\infty} (\|A(u)\| \ \|C(u,t)\| + \aleph_0 \|\Omega(u,t)\|) du \right] \|x(t)\|. \end{aligned}$$

Letting

$$\|L(u,t)\| = \|A(u)\| \ \|C(u,t)\| + \aleph_0 \|\Omega(u,t)\|,$$

we acquire

$$D^{+}W(t,x_t) \leq \left[\mu(A(t)) + \int_{t}^{\infty} \|L(u,t)\| du \right] \|x(t)\| \leq 0.$$

Next, for any $\varepsilon > 0$, $t_0 \geq 0$ and $\phi \in BC(-\infty, t_0]$, we will indicate that there exists a $\delta = \delta(t_0, \varepsilon)$ such that if $\|\phi(t)\| < \delta$, $t \in (-\infty, t_0]$, then $\|x(t, t_0, \phi)\| < \varepsilon$ for all $t \geq t_0$.

Since $D^{+}W(t,x_t) \leq 0$, $t \geq t_0$, then it is clear that the LKF $W(t,x_t)$ is decreasing. Thus, we have

$$W(t,x_t) \leq W(t_0,\phi).$$

Hence, considering this decreasing property, the LKF $W(t,x_t)$ and using some basic elementary inequalities, it follows that

$$\begin{aligned} W(t,x_t) \leq & W(t_0,\phi) = \|D\phi\| + \int_{-\infty}^{t_0} \int_{t_0}^{\infty} \|\aleph(u,s,\phi(s))\| du ds \\ & + \int_{-\infty}^{t_0} \|\phi(s)\| ds \int_{t_0}^{\infty} \|C(u,s)\| \ \|A(u)\| \, du \\ \leq & \left\| \phi(t_0) - \int_{-\infty}^{t_0} C(t_0,s)\phi(s) ds \right\| + \int_{-\infty}^{t_0} \int_{t_0}^{\infty} \|\aleph(u,s,\phi(s))\| du ds \\ & + \int_{-\infty}^{t_0} \|\phi(s)\| ds \int_{t_0}^{\infty} \|A(u)\| \ \|C(u,s)\| \, du \\ \leq & \delta + K_1\delta + \delta \int_{-\infty}^{t_0} \int_{t_0}^{\infty} [\aleph_0 \|\Omega(u,s)\| + \|A(u)\| \ \|C(u,s)\|] du ds \\ \leq & \delta \left(1 + K_1 + \aleph_0 K_2 + K_3\right) = \delta M, \end{aligned}$$

where

$$M = 1 + K_1 + \aleph_0 K_2 + K_3.$$

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On the other hand, in addition to the above outcomes, from the LKF $W(t, x_t)$, one can derive that

$$W(t,x_t) = \left\| x(t) - \int_{-\infty}^{t} C(t,s)x(s)ds \right\| + \int_{-\infty}^{t}\int_{t}^{\infty} \|\aleph(u,s,x(s))\| duds$$
$$\geq \|x(t)\| - \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds.$$

Hence, we have

$$\|x(t)\| \leq \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds + W(t,x_t)$$
$$\leq \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds + \delta M, t \geq t_0.$$

Keeping in the mind a similar procedure, for any $\varepsilon > 0$ and $t_0 \geq 0$, we can select a suited value of δ such that

$$0 < \delta < \min\left\{\varepsilon, \varepsilon(1 - K_1)M^{-1}\right\}.$$

We will now show that when $\|\phi(t)\| < \delta$, $t \in (-\infty, t_0]$, then $\|x(t, t_0, \phi)\| < \varepsilon$ for all $t \geq t_0$. On the contrary, we assume that there exists $t_1 > t_0$ such that $\|x(t)\| < \varepsilon$ for $t \in [t_0, t_1)$ and $\|x(t_1)\| = \varepsilon$.

Hence,

$$\varepsilon = \|x(t_1)\| \leq \int_{-\infty}^{t_1} \|C(t,s)\| \ \|x(s)\| \, ds + \delta M \leq \delta M + \varepsilon K_1 < \varepsilon.$$

This result is a contraction. Thus, $\|x(t, t_0, \phi)\| < \varepsilon$ for all $t \geq t_0$. This inequality therefore completes the proof of Theorem 1. □

The second result of this article as the uniformly stability of trivial solution of system of NVIDEs (2) is introduced in the following theorem.

Theorem 2. *We assume that* $(As1^0) - (As4^0)$ *of Theorem 1 and the subsequent condition hold:*
$(As5^0)$

$$\int_{-\infty}^{t_0}\int_{t_0}^{\infty} (\aleph_0 \|\Omega(u,s)\| + \|A(u)\| \ \|C(u,s)\|) duds$$

is bounded. Then the solution $x = 0$ *of system of NVIDEs (2) is uniformly stable.*

Proof. From $(As5^0)$, there exists a positive constant K_4 such that

$$\int_{-\infty}^{t_0}\int_{t_0}^{\infty} (\aleph_0 \|\Omega(u,s)\| + \|A(u)\| \ \|C(u,s)\|) duds \leq K_4.$$

According to Theorem 1, it is known that the constant δ is independent of t_0. Hence, we conclude that the trivial solution $x = 0$ of system of NVIDEs (2) is uniformly stable. □

3. Instability

The third result of this article as the instability of trivial solution of system of NVIDEs (2) is introduced in the subsequent theorem. Before giving the instability result, we arrange the following

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sufficient conditions:
$(As6^0)$

$$\int_{-\infty}^{t} \|C(t,s)\| ds \le K_1 < 1 \text{ and } \|C(t,s)\| \le K_1 \text{ for } t \ge s;$$

$(As7^0)$ $\aleph_0$, $K_5 \ge 1$ and $K_6 > K_1$ are positive constants such that

$$\|\aleph(u,t,x(t))\| \le \aleph_0 \|\Omega(u,t)\| \ \|x(t)\|,$$

where $\Omega(u,t)$ is an $n \times n$ matrix of continuous functions for $-\infty < s \le t < \infty$,

$$\mu(A(t)) - \aleph_0 K_5 \int_{t}^{\infty} \|\Omega(u,t)\| du - K_5 \int_{t}^{\infty} \|A(u)\| \ \|C(u,t)\| du \ge K_6.$$

Theorem 3. *If $(As6^0)$ and $(As7^0)$ hold, then the solution $x = 0$ of system of NVIDEs (2) is unstable.*

Proof. We will now describe an LKF $W_1 = W_1(t, x_t)$ by

$$W_1 = \|Dx_t\| - K_5 \int_{-\infty}^{t} \int_{t}^{\infty} \|\aleph(u,s,x(s))\| du ds - K_5 \int_{-\infty}^{t} \|x(s)\| ds \int_{t}^{\infty} \|C(u,s)\| \ \|A(u)\| du.$$

In the similar way as that in Theorem 1, we can obtain

$$D^+ \|Dx_t\| \ge \mu(A(t)) \|Dx_t\| - \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds.$$

Differentiating the LKF W_1 along the solutions of system of NVIDEs (2) results in

$$\begin{aligned} D^+ W_1(t,x_t) =& D^+ \|Dx_t\| - K_5 \int_{t}^{\infty} \|\aleph(u,t,x(t))\| du + K_5 \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds \\ & - K_5 \|x(t)\| \int_{t}^{\infty} \|A(u)\| \ \|C(u,t)\| du + K_5 \|A(t)\| \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| ds \\ \ge& \mu(A(t)) \|Dx_t\| - \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds \\ & - K_5 \int_{t}^{\infty} \|\aleph(u,t,x(t))\| du + K_5 \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds \\ & - K_5 \|x(t)\| \int_{t}^{\infty} \|A(u)\| \ \|C(u,t)\| du + K_5 \int_{-\infty}^{t} \|A(t)\| \ \|C(t,s)\| \ \|x(s)\| ds \\ \ge& \mu(A(t)) \left[\|x(t)\| - \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| ds \right] \\ & + (K_5 - 1) \int_{-\infty}^{t} \|\aleph(t,s,x(s))\| ds \end{aligned}$$

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$$
\begin{aligned}
&- K_5 \int_t^{\infty} \|\aleph(u,t,x(t))\| du - K_5 \|x(t)\| \int_t^{\infty} \|C(u,t)\| \ \|A(u)\| \, du \\
&+ K_5 \|A(t)\| \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds \\
\geq & \left[\mu(A(t)) - \aleph_0 K_5 \int_t^{\infty} \|\Omega(u,t)\| du - K_5 \int_t^{\infty} \|A(u)\| \ \|C(u,t)\| \, du \right] \|x(t)\| \\
\geq & K_6 \|x(t)\| .
\end{aligned}
$$

Hence, due to the outcomes above, we acquire

$$
\left\| D^+ x_t \right\| \geq W_1(t, x_t) \geq W_1(t_0, \phi) + K_6 \int_{t_0}^{t} \|x(s)\| \, ds.
$$

Next, in view of the definition of Dx_t, we can sate that

$$
\|Dx_t\| = \left\| x(t) - \int_{-\infty}^{t} C(t,s) x(s) ds \right\| \leq \|x(t)\| + \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds.
$$

From the above estimates, we derive that

$$
\|x(t)\| + \int_{-\infty}^{t} \|C(t,s)\| \ \|x(s)\| \, ds \geq W_1(t_0, \phi) + K_6 \int_{t_0}^{t} \|x(s)\| \, ds.
$$

Hence, we have

$$
\begin{aligned}
\|x(t)\| \geq & W_1(t_0, \phi) + K_6 \int_{t_0}^{t} \|x(s)\| \, ds - \int_{-\infty}^{t_0} \|C(t,s)\| \ \|x(s)\| \, ds - \int_{t_0}^{t} \|C(t,s)\| \ \|x(s)\| \, ds \\
\geq & W_1(t_0, \phi) + K_6 \int_{t_0}^{t} \|x(s)\| \, ds - K_1 \|\phi\| - K_1 \int_{t_0}^{t} \|x(s)\| \, ds.
\end{aligned}
$$

Next, letting arbitrary $t_0 \geq 0$, we select a continuous functions $\phi \in BC(-\infty, t_0]$ such that $W_1(t_0, \phi) > K_1 \|\phi\|$. Since $K_6 > K_1$, then

$$
\|x(t)\| \geq W_1(t_0, \phi) - K_1 \|\phi\| > 0.
$$

Hence, by the virtue of the above outcomes, we derive that

$$
\begin{aligned}
\|x(t)\| \geq & W_1(t_0, \phi) - K_1 \|\phi\| + (K_6 - K_1) \int_{t_0}^{t} \|x(s)\| \, ds \\
\geq & W_1(t_0, \phi) - K_1 \|\phi\| + (K_6 - K_1) \int_{t_0}^{t} \left[W_1(t_0, \phi) - K_1 \|\phi\| \right] ds \\
= & W_1(t_0, \phi) - K_1 \|\phi\| + (K_6 - K_1) \left[W_1(t_0, \phi) - K_1 \|\phi\| \right] (t - t_0).
\end{aligned}
$$

Hence, it is obvious that $\|x(t)\| \to \infty$ as $t \to \infty$. According to this outcome, the solution $x = 0$ of system of NVIDEs (2) is unstable. In conclusion, the proof of Theorem 3 has been finished □

4. Contributions

We now express the contributions of this study briefly.

1^0) For the best of statement from the related database, stability, uniformly stability and instability of solutions system of NVIDEs (2) of first order with infinite delay have not been investigated up to now. System of NVIDEs (2) is a new mathematical model and this study has new outcomes on these qualitative concepts to system of NVIDEs (2). Hence, this study allows new results and contributions to qualitative theory of NVIDEs.

2^0) System of NVIDEs (1) of first order containing infinite delay is linear. However, system of NVIDEs (2) of first order with infinite delay is nonlinear. Next, system of NVIDEs (2) includes system of NVIDEs (1) and extends system of NVIDEs (1) from linear case to nonlinear case. Indeed, when we take $F(t,s,x(s)) = C(t,s)x(s)$, system of NVIDEs (2) reduces to system of NVIDEs (1) of Li and Jiang [20]. Hence, this study has a contribution from the linear case to more general nonlinear cases.

3^0) To prove the stability, uniformly stability and instability results of this study, i.e. Theorem 1, Theorem 2 and Theorem 3, we defined two new LKFs as the basic tools and proved these theorems via the LKF method. Indeed, the definitions of new LKFs to obtain meaningful qualitative results is a hard task for nonlinear systems of NVIDEs. The definitions of two suitable LKFs as basic tools are new and essential contributions of this study to qualitative theory of NVIDEs.

4^0) The outcomes of this study are different from those in the sources mentioned above and those can be found in the relevant database of the literature, and they also complement to the related outcomes of the earlier literature.

5. Conclusion

This study interested in stability, uniformly stability and instability of solutions of a nonlinear system of NVIDEs of first order with infinite delay. We proved three new theorems with regard to these three qualitative concepts via two new LKFs. The main theorems of this study have new sufficient conditions which guarantee stability, uniformly stability and instability of zero solution of the system of NVIDEs of first order with infinite delay. A comparison between our results and those can be found in the present database shows that the results of this study are new, original, more general, effective and convenient. As future possible works, qualitative behaviors of linear and nonlinear systems of NVIDEs of higher order with infinite delay and Ulam type stabilities of linear and nonlinear systems of NVIDEs (1), (2) can be considered and investigated as open problems.

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