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The Relationship between the Box Dimension of Continuous Functions and Their (k,s)-Riemann-Liouville Fractional Integral

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Abstract: This article is a study on the (k,s)-Riemann-Liouville fractional integral, a generalization of the Riemann-Liouville fractional integral. Firstly, we introduce several properties of the extended integral of continuous functions. Furthermore, we make the estimation of the Box dimension of the graph of continuous functions after the extended integral. It presents that the upper Box dimension of the (k,s)-Riemann-Liouville fractional integral for any continuous functions is no more than the upper Box dimension of the functions on the unit interval $I = [0,1]$, which indicates that the upper Box dimension of the integrand $f(x)$ will not be increased by the σ-order (k,s)-Riemann-Liouville fractional integral ${}^s_kD^{-\sigma}f(x)$ where $\sigma > 0$ on I. Additionally, we prove that the fractal dimension of ${}^s_kD^{-\sigma}f(x)$ of one-dimensional continuous functions $f(x)$ is still one.

Keywords: the fractal dimension; continuous functions; the (k,s)-Riemann-Liouville fractional integral

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1. Introduction

Fractional calculus, including fractional differentiation and integration, is generally recognized. After centuries of development, it has been discovered that fractional calculus can solve some non classical problems in scientific theory and engineering applications, and it has got broad significance in mathematics, physics and somewhere else on account of the valuable results arrived when fractional calculus is applied to several practical problems. Authors [1,2] established some different types of inequalities for fractional calculus and the great practicality results are obtained in the filed of integral inequalities. Based on the fundamental tool of fractional calculus, an alternative method to improve the optical models in the area of nuclear and particle physics has been proposed in [3]. In mathematical theory and applications, scholars agree with the fact that the roughness of the graph of fractal functions will change with the end of integration or differentiation of the functions. This change can be measured by the fractal dimension of the graph of fractal functions, which will decrease after fractional integration and increase after fractional differentiation. On the basis of the widely recognized fact, work about the relationship of the fractal dimension between arbitrary fractal functions and their fractional calculus, whether from its own theoretical explorations or applications in other disciplines, has attracted more and more attention of relevant researchers. The connection between the Box dimension of linear fractal interpolation functions and the fractional order has been investigated in [4]. For Besicovitch functions, work on this corresponding relationship has been discussed in Refs. [5–8]. Additionally, Liang [9] proved that the parallel relationship between a self-affine fractal function and its fractional calculus is linear. Besides these functions with specific expressions, it is worth mentioning that the previous article [10] has already explored such relationship for general continuous functions and it believes that the fractal dimension has the same order variation as fractional calculus. After that, a summary conjecture of the expression was proposed by Liang [11] as follows:

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Conjecture 1. Suppose that $f(x)$ is a continuous function defined on the unit interval $I = [0,1]$. Let

$$\mathbb{G}(f,I) = \{(x, f(x)) \ : \ x \in I\}$$

denote to be the graph of $f(x)$ on I. Assume that $D^{-\sigma}f(x)$ and $D^{\sigma}f(x)$ be σ-order fractional integration and differentiation of $f(x)$, respectively. Then the following assertions hold:

1. If $\overline{\dim}_B\mathbb{G}(f,I) = \alpha \in (1,2)$, then

$$\overline{\dim}_B\mathbb{G}(D^{-\sigma}f,I) \leq \overline{\dim}_B\mathbb{G}(f,I) - \sigma, \ \alpha - \sigma \geq 1, \tag{1}$$

$$\overline{\dim}_B\mathbb{G}(D^{\sigma}f,I) \leq \overline{\dim}_B\mathbb{G}(f,I) + \sigma, \ \alpha + \sigma \geq 1. \tag{2}$$

2. If the Box dimension of $\mathbb{G}(f,I)$ exists and equals to $\alpha \in (1,2)$, then

$$\dim_B \mathbb{G}(D^{-\sigma}f,I) \leq \dim_B \mathbb{G}(f,I) - \sigma, \ \alpha - \sigma \geq 1, \tag{3}$$

$$\dim_B \mathbb{G}(D^{\sigma}f,I) \leq \dim_B \mathbb{G}(f,I) + \sigma, \ \alpha + \sigma \geq 1. \tag{4}$$

The main results given in [10] and [12] show that (3) and (4) hold. However, Xiao [13,14] arrives a weak answer for Conjecture 1. By using a method of interval estimation, he proved that

$$\overline{\dim}_B\mathbb{G}(D^{-\sigma}f,I) \leq \overline{\dim}_B\mathbb{G}(f,I) \tag{5}$$

for arbitrary continuous functions on I. In the past 20 years, most of the researches about the fractal dimension of fractional calculus concentrate on the Weierstrass function class and the Hölder continuous function class. For the Weierstrass types, there are two cases to be studied where (3) and (4) are confirmed separately. We refer the readers to Refs. [15–17] for more details. For another, work about the fractal dimension of fractional calculus can be seen in [18]. After that, Refs. [19,20] obtained a conclusion that satisfies (1).

All the aforementioned studies involve several well-known forms of fractional integrals, such as the Riemann-Liouville fractional integral [21] and the Hadamard fractional integral [22], which are two integrals that have been widely used in the field of mathematics for many years. On the basis of those integrals, people have further provided their extended integral forms, including the (k,s)-Riemann-Liouville fractional integral studied in the present article, and obtained practical results. In early research, Katugampola [23] generalized the two widely used integrals into a new fractional integral form, which is applied to the Lebesgue measurable space as a generalized fractional integration operator. After that, Mubeen [24] provided a metamorphosis of fractional integrals called the k-fractional integral that degenerates into the Riemann-Liouville fractional integral since k tends to one. In addition, an extension of the Hadamard fractional integral called the k-Hadamard fractional integral can be seen in [25]. These new constructions can be traced back to the discussion on the extension of the classical Gamma function named the k-Gamma function in [26], which is defined as

$$\Gamma_k(x) = \int_0^{\infty} \tau^{x-1} e^{\frac{-\tau^k}{k}} d\tau \tag{6}$$

where $x > 0$ and $k > 0$. Recently, Sarikaya [27] established a new fractional integral that extends all the fractional integrals mentioned above, which is named the (k,s)-Riemann-Liouville fractional integral (for short, (k,s)-RLFI) and defined as follows:

Definition 1 ([27]). *The (k,s)-Riemann-Liouville fractional integral of a continuous function $f(x)$ of order $\sigma > 0$ is given by*

$${}^s_kD^{-\sigma}f(x) = \frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \tag{7}$$

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where $k > 0$, $s \in \mathbb{R} \setminus \{-1\}$ *and* $x \in I$.

Up to now, (k,s)-RLFI has demonstrated wide applicability and flexibility in various fields. Some new extended concepts of functions after (k,s)-RLFI has been shown in [28], such as the (k,s,h)-Riemann-Liouville integral, the (k,s)-Hadamard integral and the (k,s,h)-Hadamard integral. The most remarkable feature of them is the result of applications on inequalities. Furthermore, a weighted version of the (k,s)-Riemann-Liouville fractional operator has been given in [29]. Significantly, a weighted Laplace transform is proposed, and the solution of the fractional kinetic equation model is obtained by using some properties of the new operator. Based on (k,s)-RLFI, Tomar [30] put forward some new definitions in the field of probability theory, like (k,s)-Riemann–Liouville fractional variance and expectation functions, meanwhile some generalized integral inequalities are presented and applied.

Except for those, another basic problem involved the fractal dimension of the graph of functions under (k,s)-RLFI has been noticed by researchers. Recently, Priya and Uthayakumar studied some analytical properties of (k,s)-RLFI of a function in [31]. Furthermore, the paper observed that the Hausdorff dimension and the Box dimension of the graph of a continuous function under (k,s)-RLFI are both one. Besides, the linear relationship between the fractal dimension of (k,s)-RLFI of the Weierstrass functions and the fractional order has been discussed in [32]. In this work, it is proved that the upper Box dimension of the graph of (k,s)-RLFI of any continuous functions $f(x)$ will be no more than the upper Box dimension of $f(x)$ and get a rough result of Conjecture 1. Simply speaking, our main results hold (5) for (k,s)-RLFI.

The operation of the present paper is organized followed: Section 1 mainly covers the development process of (k,s)-RLFI, a new type of fractional integral and the main work of relevant literature. Then in Section 2, we begin with some preparations and illustrations and recall some fundamental definitions. Significantly, a key lemma are proved for the main result. After that, we investigated some analysis properties about (k,s)-RLFI in Section 3. On the basis of Scetion 2, we demonstrate the main results of this paper in Section 4. Finally, we summarize the conclusion we draw in our article in Section 5.

2. Preliminaries

This section retrospects some definitions of the fractal dimension. Significantly, a key lemma has been proved for the main results of this paper. In order to simplify the proof of this paper, we begin with some preparations and illustrations as follow:

(1) Any functions mentioned in this article are continuous, and we denote all of them as $C(I)$ on I;
(2) For any function $f(x) \in C(I)$, it is reasonable to assume $f(x) \geq 0$ according to Proposition 1;
(3) For convenience, all C, mentioned in this article are constants, which can represent different positive values without causing objection;
(4) For any $\delta > 0$, assume that I is divided into $m = [\delta^{-1}]$ sub-intervals with equal width δ, i.e, $m = \inf\{M \in \mathbb{N} : M \geq \delta^{-1}\}$;
(5) Set $\Delta_p = [p\delta, (p+1)\delta],\ p = 0, 1, 2, \ldots, m-1$. Sometimes, write

$$\int_{\Delta_p} f(x)dx = \int_{p\delta}^{(p+1)\delta} f(x)dx; \tag{8}$$

(6) For any continuous function $f(x)$ and a closed interval $[x_1, x_2]$, we write $R_{f,[x_1,x_2]}$ for the maximum range of $f(x)$ over the interval as

$$\mathcal{R}_{f,[x_1,x_2]} = \sup_{x_1 \leq x < y \leq x_2} |f(x) - f(y)|; \tag{9}$$

Now, we recall two widely used definitions of fractal dimensions, which are defined as follows:

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Definition 2. *[[33]] Suppose that E is a non-empty subset of* $\mathbb{R}^2$ *and let* $N_\delta(E)$ *be the smallest number of sets of diameter at most* δ *which can cover E. Then the lower and upper Box dimensions of E are defined as*

$$\underline{\dim}_B E = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \tag{10}$$

and

$$\overline{\dim}_B E = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{11}$$

If these two equations are equal we refer to the common value as the Box dimension of E

$$\dim_B E = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{12}$$

Definition 3 ([33])**.** *Suppose that E is any subset of* $\mathbb{R}^2$ *and h is a non-negative real number. For any* $\delta > 0$*, the h-dimensional Hausdorff measure of E is defined as*

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E) \tag{13}$$

where

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^h : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } E \right\}.$$

Remark 1. *The diameter of E is the greatest distance apart of any pair of points in E, i.e.*

$$|E| = \sup_{x,y \in E} \|x - y\|;$$

Remark 2. *Let* $\{U_i\}$ *is a countable collection of sets of diameter at most* δ *that cover E for each i, that is,* $\{U_i\}$ *is a* δ*-cover of E.*

Definition 4 ([33])**.** *Let* $A \subset \mathbb{R}^2$ *and* $h \geq 0$*. The Hausdorff dimension of E is defined as*

$$\dim_H(E) = \inf \left\{ h : \mathcal{H}^h(E) = 0 \right\} = \sup \left\{ h : \mathcal{H}^h(E) = \infty \right\}. \tag{14}$$

Based on the above definitions of fractal dimensions, we can obtain the basic following lemmas for continuous functions by [33]:

Lemma 1. *For any* $f(x) \in C[0,1]$*, we have*

$$1 \leq \dim_H \mathbb{G}(f, I) \leq \underline{\dim}_B \mathbb{G}(f, I) \leq \overline{\dim}_B \mathbb{G}(f, I).$$

Lemma 2. *Let* $f(x) \in C(I)$*. If* $N_{f,\delta}$ *is the smallest number of squares of the* δ*-mesh that intersect* $\mathbb{G}(f, I)$*, then*

$$\delta^{-1} \sum_{p=0}^{m-1} \mathcal{R}_{f,\Delta_p} \leq N_{f,\delta} \leq 2m + \delta^{-1} \sum_{p=0}^{m-1} \mathcal{R}_{f,\Delta_p}. \tag{15}$$

In the light of Lemma 2, the upcoming proposition can be :

Proposition 1 ([13])**.** *Suppose that* $f(x) \in C(I)$ *and* $n \in \mathbb{R} \setminus \{0\}$*.*

(1) $\dim_B \mathbb{G}(nf, I) = \dim_B \mathbb{G}(f, I)$*;*
(2) $\dim_B \mathbb{G}(f + n, I) = \dim_B \mathbb{G}(f, I)$*;*
(3) If $f(x)$ *is a constant function, then* $\dim_B \mathbb{G}(f + n, I) = \dim_B \mathbb{G}(f, I) = 1$*;*

(4) $1 \le \underline{\dim}_B \mathbb{G}(f,I) \le \overline{\dim}_B \mathbb{G}(f,I) \le 2$ *or* $1 \le \dim_B \mathbb{G}(f,I) \le 2$.

More recent work about fractal dimensions of the graph of continuous functions can be found in [34–38].

Lemma 3. *Let $0 < \delta < 1$, and m is the least integer no less than δ^{-1}. Assume $x,\ y \in \Delta_p = [p\delta, (p+1)\delta]$ $(p = 0, 1 \cdots, m-1)$ and $\tau \in \Delta_q = [q\delta, (q+1)\delta]$ $(q = 0, 1 \cdots, p-1)$, then*

$$\int_{q\delta}^{(q+1)\delta} \left[x^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s d\tau \le C\delta^{\frac{\sigma}{k}(s+1)} (p-q)^{\frac{\sigma}{k}(s+1)-1} \tag{16}$$

when $\sigma,\ k,\ s$ are positive numbers and $\frac{\sigma}{k} \in (0,1)$.

Proof. Applying the Mean Value Theorem,

$$\begin{aligned}
&\int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&= \frac{k}{\sigma(s+1)} \left\{ \left[x^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right\} \\
&= \frac{k}{\sigma(s+1)} \left[((q+1)\delta)^{s+1} - (q\delta)^{s+1}\right] \xi^{\frac{\sigma}{k}-1} \\
&\le C\delta^{s+1}(q+1)^s \left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}-1} \\
&\le C\delta^{s+1}(q+1)^s \left[x - (q+1)\delta\right]^{\frac{\sigma}{k}-1} \beta^{s(\frac{\sigma}{k}-1)} \\
&\le C\delta^{\frac{\sigma}{k}+s}(q+1)^s (p-q-1)^{\frac{\sigma}{k}-1} \left[(q+1)\delta\right]^{s(\frac{\sigma}{k}-1)} \\
&\le C\delta^{\frac{\sigma}{k}(s+1)}(q+1)^{s\frac{\sigma}{k}} (p-q-1)^{\frac{\sigma}{k}-1} \\
&\le C\delta^{\frac{\sigma}{k}(s+1)} (p-q)^{\frac{\sigma}{k}(s+1)-1}
\end{aligned}$$

where

$$\begin{cases}
\xi \in \left[x^{s+1} - ((q+1)\delta)^{s+1}, x^{s+1} - (q\delta)^{s+1}\right], \\
\beta \in [(q+1)\delta, x], \\
\frac{\sigma}{k} \in (0,1), \\
s > 0.
\end{cases}$$

So Lemma 3 holds. □

3. Analysis properties of (k,s)-RLFI

This section shows several analysis properties of continuous functions under (k,s)-RLFI, such as boundedness and continuity etc.

Theorem 1. *Let $f(x)$ is bounded on I, then the (k,s)-Riemann-Liouville fractional integral of $f(x)$ shows the boundedness where $k > 0$ and $s > -1$.*

Proof. Since $f(x)$ is bounded on I, there exists a positive constant Q such that $|f(x)| \le Q$. Then we have

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$$\begin{aligned}
\left|{}_k^s D^{-\sigma} f(x)\right| &= \left|\frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&\leq \frac{(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left|\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s\right| |f(\tau)| \, d\tau \\
&\leq \frac{Q(s+1)^{1-\frac{\sigma}{k}}}{k\Gamma_k(\sigma)} \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&\leq C x^{(s+1)\frac{\sigma}{k}}.
\end{aligned}$$

Since $x \in I$, then (k,s)-RLFI of $f(x)$ is bounded. □

Theorem 2. *Suppose that $k > 0$, $s > 0$ and $\sigma > 0$. For any functions $f(x) \in C(I)$, its (k,s)-Riemann-Liouville fractional integral ${}_k^s D^{-\sigma} f(x)$ is continuous on I where $0 < \frac{\sigma}{k} < 1$.*

Proof. Since $f(x)$ is continuous on I, there exists a positive constant number Q such that $|f(x)| \leq Q$. Let $0 \leq x < x + \varepsilon \leq 1$ where ε is a positive number that tends to 0. Then

$$\begin{aligned}
&\frac{k\Gamma_k(\sigma)}{(s+1)^{1-\frac{\sigma}{k}}} \left|{}_k^s D^{-\sigma} f(x+\varepsilon) - {}_k^s D^{-\sigma} f(x)\right| \\
&= \left|\int_0^{x+\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&= \left|\int_0^{\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau + \int_{\varepsilon}^{x+\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right. \\
&\quad \left. - \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&\leq Q \int_0^{\varepsilon} \left|\left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s\right| d\tau + \int_0^x \left|\left[(x+\varepsilon)^{s+1} - (\tau+\varepsilon)^{s+1}\right]^{\frac{\sigma}{k}-1} (\tau+\varepsilon)^s \right. \\
&\quad \left. \times f(\tau+\varepsilon) d\tau - \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&=: I_1 + I_2.
\end{aligned}$$

Since $f(x)$ is continuous and $\varepsilon \to 0$, then $x + \varepsilon \to x$, $\tau + \varepsilon \to \tau$ and $f(x+\varepsilon) \to f(x)$.

$$\begin{aligned}
I_1 &= Q \int_0^{\varepsilon} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s d\tau \\
&= -\frac{kQ}{\sigma(s+1)} \left[(x+\varepsilon)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}} \Big|_0^{\varepsilon} \\
&= \frac{kQ}{\sigma(s+1)} \left\{(x+\varepsilon)^{(s+1)\frac{\sigma}{k}} - \left[(x+\varepsilon)^{s+1} - \varepsilon^{s+1}\right]^{\frac{\sigma}{k}}\right\} \\
&\to 0 \quad (\varepsilon \to 0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \int_0^x \left|\left[(x+\varepsilon)^{s+1} - (\tau+\varepsilon)^{s+1}\right]^{\frac{\sigma}{k}-1} (\tau+\varepsilon)^s f(\tau+\varepsilon) d\tau - \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \\
&\to \int_0^x \left|\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau\right| \quad (\varepsilon \to 0) \\
&= 0.
\end{aligned}$$

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Therefore, we obtain that ${}_k^sD^{-\sigma}f(x+\varepsilon) \to {}_k^sD^{-\sigma}f(x)$ when $\varepsilon \to 0$, which means ${}_k^sD^{-\sigma}f(x)$ is continuous. □

Theorem 3. *Suppose k, s , σ_1 and σ_2 are any real numbers. For any $f(x) \in C(I)$, it holds that*

$$({}_k^sD^{-\sigma_1}{}_k^sD^{-\sigma_2})f(x) = {}_k^sD^{-(\sigma_1+\sigma_2)}f(x). \tag{17}$$

Proof. From The Fubini's theorem, we have

$$\begin{aligned}({}_k^sD^{-\sigma_1}{}_k^sD^{-\sigma_2})f(x) &= \frac{(s+1)^{1-\frac{\sigma_1}{k}}}{k\Gamma_k(\sigma_1)}\int_0^x \left(x^{s+1}-\tau_2^{s+1}\right)^{\frac{\sigma_1}{k}-1}\tau_2^s \\ &\times\left(\frac{(s+1)^{1-\frac{\sigma_2}{k}}}{k\Gamma_k(\sigma_2)}\int_0^x \left(\tau_2^{s+1}-\tau_1^{s+1}\right)^{\frac{\sigma_2}{k}-1}\tau_1^s f(\tau_1)d\tau_1\right)d\tau_2 \\ &= \frac{(s+1)^{2-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}\int_0^x \tau_1^s f(\tau_1)\times \\ &\left[\int_0^x \left(x^{s+1}-\tau_2^{s+1}\right)^{\frac{\sigma_1}{k}-1}\tau_2^s\left(\tau_2^{s+1}-\tau_1^{s+1}\right)^{\frac{\sigma_2}{k}-1}d\tau_2\right]d\tau_1.\end{aligned}$$

Let $y = \frac{\tau_2^{s+1}-\tau_1^{s+1}}{x^{s+1}-\tau_1^{s+1}}$, then $(x^{s+1}-\tau_1^{s+1})du = \tau_2^s d\tau_2$. So we obtain the following changes in the internal integral of the above equation.

$$\begin{aligned}&\int_0^x \left(x^{s+1}-\tau_2^{s+1}\right)^{\frac{\sigma_1}{k}-1}\tau_2^s\left(\tau_2^{s+1}-\tau_1^{s+1}\right)^{\frac{\sigma_2}{k}-1}d\tau_2 \\ =&\frac{(x^{s+1}-\tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1}\int_0^1 y^{\frac{\sigma_2}{k}-1}(1-y)^{\frac{\sigma_1}{k}-1}dy \\ =&\frac{(x^{s+1}-\tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1}\frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1+\sigma_2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}({}_k^sD^{-\sigma_1}{}_k^sD^{-\sigma_2})f(x) &= \frac{(s+1)^{2-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}\int_0^x \tau_1^s f(\tau_1)\left[\frac{(x^{s+1}-\tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}}{s+1}\frac{\Gamma_k(\sigma_1)\Gamma_k(\sigma_2)}{\Gamma_k(\sigma_1+\sigma_2)}\right]d\tau_1 \\ &= \frac{(s+1)^{1-\frac{\sigma_1}{k}-\frac{\sigma_2}{k}}}{k^2\Gamma_k(\sigma_1+\sigma_2)}\int_0^x (x^{s+1}-\tau_1^{s+1})^{\frac{\sigma_1+\sigma_2}{k}-1}\tau_1^s f(\tau_1)d\tau_1 \\ &= {}_k^sD^{-(\sigma_1+\sigma_2)}f(x).\end{aligned}$$

The proof of Theorem 3 is done. □

Remark 3. *Assume that k, s , σ_1 and σ_2 are real numbers. For $f(x) \in C(I)$, it holds that*

$$({}_k^sD^{-\sigma_1}{}_k^sD^{-\sigma_2})f(x) = ({}_k^sD^{-\sigma_2}{}_k^sD^{-\sigma_1})f(x).$$

4. Main results

In this section, we obtain the main results of our article. The relationship between the Box dimension of $\mathbb{G}(f,I)$ and $\mathbb{G}({}_k^sD^{-\sigma}f,I)$ is given from Lemma 2 and Definition 2. Moreover, we calculate that the fractal dimension of one-dimensional continuous functions under (k,s)-RLFI in this section.

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Theorem 4. *Let $f(x) \in C(I)$. For the (k,s)-Riemann-Liouville fractional integral of $f(x)$, it holds that*

$$\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) \leq \overline{\dim}_B \mathbb{G}(f, I) \tag{18}$$

where $k > 0$, $s > 0$ and $\sigma > 0$ such that $\frac{\sigma}{k} \in (0,1)$.

Proof. Firstly, we estimate the oscillation of ${}_k^s D^{-\sigma} f(x)$ on Δ_p, indicated by $\mathcal{R}_{{}_k^s D^{-\sigma} f, \Delta_p}$. At this step, there are three parts that we need to consider separately. Since $f(x) \in C(I)$, we choose to represent the maximum and minimum values of $f(x)$ on Δ_p as $M_{p,\delta}$ and $m_{p,\delta}$, respectively. Then from (9) and $f(x) \geq 0$, we observe that

$$\mathcal{R}_{f,\Delta_p} = M_{p,\delta} - m_{p,\delta}.$$

Let $p\delta \leq x < y \leq (p+1)\delta$,

$$\begin{aligned}
&\frac{k\Gamma_k(\sigma)}{(s+1)^{1-\frac{\sigma}{k}}} \left| {}_k^s D^{-\sigma} f(x) - {}_k^s D^{-\sigma} f(y) \right| \\
&= \left| \int_0^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_0^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&= \left| \int_0^{p\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_0^{p\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right. \\
&\quad \left. + \int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&\leq \int_0^{p\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s M_{q,\delta} d\tau - \int_0^{p\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s m_{q,\delta} d\tau \\
&\quad + \left| \int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&= \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s M_{q,\delta} d\tau - \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s m_{q,\delta} d\tau \\
&\quad + \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s m_{q,\delta} d\tau - \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s m_{q,\delta} d\tau \\
&\quad + \left| \int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau - \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right| \\
&\leq \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \left(M_{q,\delta} - m_{q,\delta}\right) d\tau \\
&\quad + \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left[\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s - \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \right] m_{q,\delta} d\tau \\
&\quad + \max\left\{ \int_{p\delta}^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau, \int_{p\delta}^y \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right\} \\
&=: H_1 + H_2 + H_3.
\end{aligned}$$

From Lemma 3, we can get that

$$\begin{aligned} H_1 &= \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \left(M_{q,\delta} - m_{q,\delta}\right) d\tau \\ &\leq C\delta^{\frac{\sigma}{k}(s+1)} \sum_{q=0}^{p-1} (p-q)^{\frac{\sigma}{k}(s+1)-1} (M_{q,\delta} - m_{q,\delta}) \\ &= C\delta^{\frac{\sigma}{k}(s+1)} \sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q}. \end{aligned}$$

Since $f(x) \in C(I)$ is bounded, there exists a positive constant Q such that $0 \leq f(x) \leq Q$. For H_2, it follows from (16) that

$$\begin{aligned} H_2 &= \sum_{q=0}^{p-1} \int_{q\delta}^{(q+1)\delta} \left[\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s - \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s \right] m_{q,\delta} d\tau \\ &= \frac{k}{\sigma(s+1)} \sum_{q=0}^{p-1} \left\{ \left[\left[x^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[x^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right] \right. \\ &\quad \left. - \left[\left[y^{s+1} - (q\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[y^{s+1} - ((q+1)\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right] \right\} m_{q,\delta} \\ &\leq C \left\{ \left| (x^{s+1})^{\frac{\sigma}{k}} - (y^{s+1})^{\frac{\sigma}{k}} \right| + \left| \left[x^{s+1} - (p\delta)^{s+1}\right]^{\frac{\sigma}{k}} - \left[y^{s+1} - (p\delta)^{s+1}\right]^{\frac{\sigma}{k}} \right| \right\} \max_{0\leq q<p} m_{q,\delta} \\ &\leq C \left[2 \left| (x^{s+1})^{\frac{\sigma}{k}} - (y^{s+1})^{\frac{\sigma}{k}} \right| \right] Q \\ &\leq C \left[((p+1)\delta)^{\frac{\sigma}{k}(s+1)} - (p\delta)^{\frac{\sigma}{k}(s+1)} \right] Q \\ &\leq C\delta^{\frac{\sigma}{k}(s+1)} p^{\frac{\sigma}{k}(s+1)-1} Q. \end{aligned}$$

For H_3,

$$\begin{aligned} H_3 &= \max \left\{ \int_{p\delta}^{x} \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau, \in \tau_{p\delta}^{y} \left(y^{s+1} - \tau^{s+1}\right)^{\frac{\sigma}{k}-1} \tau^s f(\tau) d\tau \right\} \\ &\leq \int_{p\delta}^{(p+1)\delta} \left[((p+1)\delta)^{s+1} - \tau^{s+1}\right]^{\frac{\sigma}{k}-1} \tau^s M_{p,\delta} d\tau \\ &= \frac{k}{\sigma(s+1)} \left[((p+1)\delta)^{s+1} - (p\delta)^{s+1}\right]^{\frac{\sigma}{k}} M_{p,\delta} \\ &\leq C(p^s \delta^{s+1})^{\frac{\sigma}{k}} M_{p,\delta} \\ &\leq C p^{s\frac{\sigma}{k}} \delta^{\frac{\sigma}{k}(s+1)} Q. \end{aligned}$$

Hence, combining H_1 and H_2,

$$\begin{aligned} \frac{k\Gamma_k(\sigma)}{(s+1)^{1-\frac{\sigma}{k}}} \mathcal{R}_{{}_k^sD^{-\sigma} f,\Delta_p} &= \sup_{x,y\in\Delta_p} \left| {}_k^sD^{-\sigma} f(x) - {}_k^sD^{-\sigma} f(y) \right| \\ &\leq H_1 + H_2 + H_3 \\ &\leq C\delta^{\frac{\sigma}{k}(s+1)} \left[\sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q} + p^{\frac{\sigma}{k}(s+1)-1} Q + p^{s\frac{\sigma}{k}} Q \right] \\ &\leq C\delta^{\frac{\sigma}{k}(s+1)} \sum_{q=0}^{p} (p-q+1)^{\frac{\sigma}{k}(s+1)-1} \mathcal{R}_{f,\Delta_q}. \end{aligned}$$

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Secondly, we calculate $N_{{}^{s}_{k}D^{-\sigma}f,\delta}$, the size of δ-mesh squares intersecting $\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I)$. From (15) of Lemma 2, it follows that

$$\begin{aligned}
N_{{}^{s}_{k}D^{-\sigma}f,\delta} &\leq \sum_{p=0}^{m-1}\left\{2+\delta^{-1}\mathcal{R}_{{}^{s}_{k}D^{-v}f,\Delta_p}\right\} \\
&\leq \sum_{p=0}^{m-1}\left\{2+\delta^{-1}C\delta^{\frac{\sigma}{k}(s+1)}\left[\sum_{q=0}^{p}(p-q+1)^{\frac{\sigma}{k}(s+1)-1}\mathcal{R}_{f,\Delta_q}\right]\right\} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)-1}\sum_{q=0}^{m-1}q^{\frac{\sigma}{k}(s+1)-1}\sum_{p=0}^{m-1}\mathcal{R}_{f,\Delta_p} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)}m^{\frac{\sigma}{k}(s+1)}\delta^{-1}\sum_{p=0}^{m-1}\mathcal{R}_{f,\Delta_p} \\
&\leq C\delta^{\frac{\sigma}{k}(s+1)}\delta^{-\frac{\sigma}{k}(s+1)}\delta^{-1}\sum_{p=0}^{m-1}\mathcal{R}_{f,\Delta_p} \\
&\leq CN_{f,\delta}.
\end{aligned}$$

Ultimately, by Definition 2,

$$\begin{aligned}
\overline{\dim}_B\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I) &= \overline{\lim_{\delta\to 0}}\frac{\log N_{{}^{s}_{k}D^{-\sigma}f,\delta}}{-\log\delta} \leq \overline{\lim_{\delta\to 0}}\frac{\log CN_{f,\delta}}{-\log\delta} \\
&= \overline{\lim_{\delta\to 0}}\frac{\log N_{f,\delta}}{-\log\delta} = \overline{\dim}_B\mathbb{G}(f, I).
\end{aligned}$$

We complete the proof of this theorem. □

Among all the fractal functions, the most fundamental one may be a kind of continuous fractal functions with the Box dimension one. So, we provide a discussion of the functions in the following theorems.

Theorem 5. *Suppose* $f(x) \in C(I)$ *and* $\overline{\dim}_B\mathbb{G}(f, I) = 1$*, then*

$$\overline{\dim}_B\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I) = 1 \tag{19}$$

where $\sigma > 0,\ s > 0, k > 0$ *and* $\frac{\sigma}{k} \in (0, 1)$*.*

Proof. By Definition 2,

$$\overline{\dim}_B\mathbb{G}(f, I) = \overline{\lim_{\delta\to 0}}\frac{\log N_{f,\delta}}{-\log\delta} = 1.$$

Therefore,

$$\begin{aligned}
\overline{\dim}_B\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I) &= \overline{\lim_{\delta\to 0}}\frac{\log N_{{}^{s}_{k}D^{-\sigma}f,\delta}}{-\log\delta} \\
&\leq \overline{\lim_{\delta\to 0}}\frac{\log CN_{f,\delta}}{-\log\delta} \\
&= \overline{\lim_{\delta\to 0}}\frac{\log N_{f,\delta}}{-\log\delta} = 1.
\end{aligned}$$

Meanwhile, from Proposition 1, we know the upper Box dimension of $\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I)$ is no less than one as

$$\overline{\dim}_B\mathbb{G}({}^{s}_{k}D^{-\sigma}f, I) \geq 1,$$

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which can lead that

$$\overline{\dim}_B \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1.$$

The proof of Theorem 5 is done. □

From Theorem 5, Lemma 1 and Proposition 1, we can derive the upcoming conclusions.

Theorem 6. *Suppose that $k > 0,\ s > 0$ and $\sigma > 0$. Let $f(x) \in C(I)$ and $\overline{\dim}_B \mathbb{G}(f, I) = 1$, we have*

$$\dim_H \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1. \tag{20}$$

Corollary 1. *If $f(x) \in C(I)$ is a one-dimensional function and $\frac{\sigma}{k} \in (0, 1)$, then*

$$\dim_B \mathbb{G}({}_k^s D^{-\sigma} f, I) = 1 \tag{21}$$

for any $\sigma > 0$, $s > 0$, and $k > 0$.

A few one-dimensional functions are constructed in Refs. [39–41], and the articles took carefully research of the fractal dimension about their Riemann-Liouville fractional integral. In [42], authors made research on the Katugampola fractional calculus of one-dimensional continuous functions which are bounded variation. Other articles about one-dimensional continuous functions can be seen in [12] and [13], which prove that the Box dimensions of the Riemann-Liouville fractional integral and the Hadamard fractional integral of one-dimensional continuous functions are still one. On the basis of fractional calculus of one-dimensional continuous functions, researchers can continue to study continuous functions with the fractal dimension greater than one. Therefore, one-dimensional continuous functions that satisfies Theorem 5 and Corollary 1 is worth further exploration.

5. Conclusions

Researches on fractional calculus have been flourishing for a long time. In the present article, we discuss a new kind of fractional integral based on the Riemann-Liouville fractional integral and the Hadamard fractional integral and further understand the importance of the integral in various fields. Simultaneously, this article presents the change of the fractal dimension of continuous functions after (k, s)-RLFI although the issue has not yet been solved completely. Therefore, it is still important to find a way to demonstrate that the fractal dimension of continuous functions after (k, s)-RLFI can reach the upper bound of (1) or (3). Besides, there are still some points that need improvement in this article. In fact, we subjectively limit the range of parameters $s,\ \sigma$ and k of (7) when estimating the Box dimension of $\mathbb{G}({}_k^s D^{-\sigma} f, I)$. Therefore, we naturally put forward a question, that is, what is the relationship between the parameters of (7) and the change of the fractal dimension of $\mathbb{G}({}_k^s D^{-\sigma} f, I)$ if these parameters are not limited, and whether this relationship is linear like Conjecture 1? Furthermore, the main result has estimated the upper Box dimension of (k, s)-RLFI of a continuous function $f(x)$. Further exploration is needed for the study of the lower Box dimension of (k, s)-RLFI of $f(x)$.

The previous articles show that better results can be obtained by studying the fractal dimension of fractional calculus of special functions, like the Weierstrass type functions or the Hölder continuous functions. For example, a better result is obtained from [32] for the Weierstrass functions. The paper investigated the relationship between the order of (k, s)-RLFI of the Weierstrass function $f(x)$ and the fractal dimension of $\mathbb{G}({}_k^s D^{-\sigma} f(x), I)$. Besides, it's reasonable to conjecture that (k, s)-RLFI, as the extension of the Riemann-Liouville fractional integral, has some properties similar to the Riemann-Liouville fractional integral. So, we speculate that the investigation of the fractal dimension of $\mathbb{G}({}_k^s D^{-\sigma} f, I)$ in the Hölder space has the similar result with [19]. Based on this, we can continue to further consider the relationship between the Box dimension of $\mathbb{G}({}_k^s D^{-\sigma} f, I)$ and $\mathbb{G}(f, I)$ in the Hölder space in the future.

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