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*Article*

# Lump Waves in a Spatial Symmetric Nonlinear Dispersive Wave Model in (2+1)-Dimensions

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**Abstract:** This paper aims to search for lump waves in a spatial symmetric (2+1)-dimensional dispersive wave model. Through an ansatz on positive quadratic functions, we conduct symbolic computations with Maple to generate lump waves for the proposed nonlinear model. A line of critical points of the lump waves is computed, whose two spatial coordinates travel at constant speeds. The corresponding maximum and minimum values are evaluated in terms of the wave numbers, and interestingly, all those extreme values do not change with time, either. The last section is the conclusion.

**Keywords:** lump wave; hirota bilinear form; soliton; symbolic computation; nonlinearity; dispersion

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## 1. Introduction

Applied sciences rely heavily on closed-form mathematical theories. Much of such studies represents mathematical intuitions and skills of high order, challenging even for advanced researchers of today. A kind of multiple wave solutions, called soliton solutions, are examples of closed-form solutions to integrable models of nonlinear dispersive waves. The nonlinearity and the dispersion play together in generating such nonlinear dispersive wave solutions.

In soliton theory, there are two powerful techniques, the inverse scattering transform [1] and the Hirota bilinear method [2], to soliton solutions. The inverse scattering transform was developed initially for solving Cauchy problems of nonlinear model equations, genearted from Lax pairs of matrix spectral problems [3,4], It is a nonlinear version of the Fourier transform.

The Hirota bilinear method is the other direct but powerful technique to soliton waves. Hirota bilinear forms are the starting point to generate closed-form solutions [7,8]. In the (2+1)-dimensional case, take a polynomial $R$ in time $t$ and two space variables $x, y$. A (2+1)-dimensional Hirota bilinear differential equation is defined by

$$R(D_t, D_x, D_y) f \cdot f = 0, \tag{1.1}$$

where $D_t, D_x$ and $D_y$ are three Hirota bilinear derivatives given as follows [2]:

$$D_t^m D_x^n D_y^k f \cdot f = \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^m \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^n \Big(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\Big)^k f(t,x,y) f(t',x',y')\big|_{t'=t,x'=x,y'=y'} \tag{1.2}$$

in which $m, n, k$ are nonnegative integers. Associated with a Hirota bilinear equation, a nonlinear partial differential equation

$$X(u, u_t, u_x, u_y, \cdots) = 0 \tag{1.3}$$

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with a dependent variable $u$ is usually presented by the logarithmic derivative transformations

$$u = 2(\ln f)_{xx}, \ u = 2(\ln f)_{yy} \text{ or } u = 2(\ln f)_{xy}. \tag{1.4}$$

For multi-component integrable models (see, e.g., [5,6]), other kinds of transformations need to be introduced and implemented. Within the Hirota bilinear theory, an $N$-soliton solution to a nonlinear equation can be presented by solving its corresponding Hirota bilinear equation (see, e.g., [7]-[11]).

Remarkably similar to solitons, lump waves (and rogue waves) are another kind of closed-form solutions to nonlinear integrable models [12]. Lump waves are expressed in terms of analytic rational functions, which are localized in all directions in the spatial space (see, e.g., [12,13]):

$$\lim_{x^2+y^2\to\infty,\ ax+by+c=0} u(x,y,t) = 0, \ t \in \mathbb{R}, \tag{1.5}$$

where $a, b, c$ are arbitrary constants and $a^2 + b^2 \neq 0$. The KPI equation possesses abundant lump waves (see, e.g., [8]), and taking long wave limits of its soliton solutions can yield particular lump waves [14]. Lump waves can exist in nonlinear nonintegrable models as well, and illustrative examples include generalized KP, BKP, Jimbo-Miwa and Bogoyavlensky-Konopelchenko equations [15]-[19]. There also exist lump waves in linear models in higher dimensions (see, e.g., [20]).

Quadratic functions are used to present exact solutions to Hirota bilinear equations and formulate lump wave solutions to nonlinear model equations [8,12]. The logarithmic derivative transformations are taken to link nonlinear model eqautions to bilinear equations. In this paper, we would like to search for lump waves in a spatial symmetric (2+1)-dimensional nonlinear dispersive wave model via such an ansatz using quadratic functions. The proposed spatial symmetric (2+1)-dimensional dispersive wave model contains three nonlinear terms. We will conduct symbolic computations with Maple to determine its lump waves. Characteristic properties, such as critical points and extreme values, will be analyzed for the resulting lump waves. Concluding remarks will be given in the last section.

## 2. A spatial symmetric nonlinear model and its Hirota bilinear form

Let $\alpha$ and $\beta$ be real constants. We introduce a spatial symmetric (2+1)-dimensional nonlinear dispersive wave model equation:

$$\begin{aligned} X(u) = {} & \alpha(3u_{xx}p_y + 3u_x p_{xy} + 3u_{xy}v + 3u_y v_x + u_{xxxy} + u_{tx} - u_{yy} \\ & + 3u_{yy}q_x + 3u_y q_{xy} + 3u_{xy}w + 3u_x w_y + u_{xyyy} + u_{ty} - u_{xx}) \\ & + \beta(4uu_{xy} + 5u_x u_y + u_{yy}v + u_{xx}w + v_x w_y + u_{xxyy}) = 0, \end{aligned} \tag{2.1}$$

with $v_y = u_x, w_x = u_y, p_x = v, q_y = w$, and search for its lump waves via the indicated ansatz using quadratic functions. The example with $\alpha = 1$ ad $\beta = 0$ of this nonlinear model gives the special spatial symmetric (2+1)-dimensional model equation

$$\begin{aligned} & 3u_{xx}p_y + 3u_x p_{xy} + 3u_{xy}v + 3u_y v_x + u_{xxxy} + u_{tx} - u_{yy} \\ & + 3u_{yy}q_x + 3u_y q_{xy} + 3u_{xy}w + 3u_x w_y + u_{xyyy} + u_{ty} - u_{xx} = 0. \end{aligned} \tag{2.2}$$

Under the help fo Maple, through the logarithmic derivative transformations

$$u = 2(\ln f)_{xy}, \ v = 2(\ln f)_{xx}, \ w = 2(\ln f)_{yy}, \ p = 2(\ln f)_x, \ q = 2(\ln f)_y, \tag{2.3}$$

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the above spatial symmetric (2+1)-dimensional model equation (2.1) is put into the Hirota bilinear form:

$$\begin{aligned}R(f) &= [\alpha(D_x^3D_y + D_y^3D_x + D_tD_x + D_tD_y - D_x^2 - D_y^2) + \beta D_x^2D_y^2]f\cdot f\\ &= 2\alpha(f_{xxxy}f - 3f_{xxy}f_x + 3f_{xy}f_{xx} - f_yf_{xxx} + f_{tx}f - f_tf_x - f_{yy}f + f_y^2\\ &\quad + f_{xyyy}f - 3f_{xyy}f_y + 3f_{xy}f_{yy} - f_xf_{yyy} + f_{ty}f - f_tf_y - f_{xx}f + f_x^2)\\ &\quad + 2\beta(f_{xxyy}f - 2f_{xxy}f_y - 2f_{xyy}f_x + f_{xx}f_{yy} + 2f_{xy}^2) = 0,\end{aligned} \tag{2.4}$$

where $D_t, D_x$ and $D_y$ are the standard Hirota bilinear derivatives [2] (see also, (1.2)). By symbolic computation, a precise relation between the nonlinear model equation and the bilinear model equation can be explored to be

$$X(u) = [\frac{R(f)}{f^2}]_{xy}, \tag{2.5}$$

where the involved functions $u, v, w, p, q$ are determined through the logarithmic derivative transformations of $f$ in (2.3).

The same link also exists in a spatial symmetric KP model [21] and a spatial symmetric HSI model [22]. It is now evident that if $f$ is a solution to the bilinear model equation (2.4), then $u, v, w, p, q$ determined by (2.3) solve the spatial symmetric (2+1)-dimensional dispersive wave model equation (2.1). In the following section, we would like to look for a class of lump waves in this spatial symmetric nonlinear dispersive wave model.

### 3. Lump wave solutions

We would now like to compute lump wave solutions to the spatial symmetric (2+1)-dimensional dispersive wave model equation (2.1), through conducting symbolic computations. A direct computation can show that the above general nonlinear model equation does not pass the three-soliton test (see, e.g., [9,11] for the three-soliton test) and thus it doesn't possess an $N$-soliton solution..

Applying a general ansatz on lump waves in (2+1)-dimensions [8], we start looking for positive quadratic function solutions

$$f = \zeta_1^2 + \eta_2^2 + a_9,\ \zeta_1 = a_1x + a_2y + a_3t + a_4,\ \zeta_2 = a_5x + a_6y + a_7t + a_8, \tag{3.1}$$

to the corresponding Hirota bilinear equation (2.4), and the task will be to determine the real constant parameters $a_i$, $1 \le i \le 9$ (see, e.g., [12,15,17] for illustrative examples). It is known that this is a general ansatz for lump wave solutions of lower order in (2+1)-dimensions [12].

We substitute $f$ by (3.1) into the Hirota bilinear equation (2.4) and obtain a system of algebraic equations on the involved parameters. A direct Maple computation to solve this system for $a_3, a_7$ and $a_9$ yields a set of solutions for the parameters:

$$\begin{cases} a_3 = \dfrac{(a_1+a_2)[a_1^2+a_2^2+(a_5+a_6)^2] - 2a_1a_6^2 - 2a_2a_5^2}{(a_1+a_2)^2+(a_5+a_6)^2},\\[2ex] a_7 = \dfrac{(a_5+a_6)[(a_1+a_2)^2+a_5^2+a_6^2] - 2a_2^2a_5 - 2a_1^2a_6}{(a_1+a_2)^2+(a_5+a_6)^2},\\[2ex] a_9 = \dfrac{3(a_1a_2+a_5a_6)(a_1^2+a_2^2+a_5^2+a_6^2)[(a_1+a_2)^2+(a_5+a_6)^2]}{2(a_1a_6-a_2a_5)^2}\\[2ex] \quad + \dfrac{\beta[(a_1a_2-a_5a_6)^2+2(a_1a_2+a_5a_6)^2+(a_1a_6+a_2a_5)^2][(a_1+a_2)^2+(a_5+a_6)^2]}{2\alpha(a_1a_6-a_2a_5)^2},\end{cases} \tag{3.2}$$

and all other parameters are arbitrary.

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The above two frequency parameters, $a_3$ and $a_7$, exhibit a class of dispersion relations in (2+1)-dimensional nonlinear dispersive waves, and the constant term parameter, $a_9$, tells a complicated expresssion of the wave numbers, which is crucial in formulating lump waves within the Hirota bilinear theory. Interestingly, there also exists a kind of higher-order dispersion relations appearing in lump waves of the second model equation of the integrable KP hierarchy [23].

Let us anlayze the analycity of the lump waves by observing the above simplified expressions for the wave frequencies and the constant term in (3.2). Obviously, if

$$a_1 + a_2 = a_5 + a_6 = 0 \tag{3.3}$$

then

$$a_1a_6 - a_2a_5 = 0. \tag{3.4}$$

This implies that if $a_1a_6 - a_2a_5 \neq 0$, then we have

$$(a_1 + a_2)^2 + (a_5 + a_6)^2 > 0,\ a_1^2 + a_2^2 + a_5^2 + a_6^2 > 0.$$

Therefore, to generate lump wave solutions through the logarithmic derivative transformations, we require two basic conditions:

$$a_1a_6 - a_2a_5 \neq 0, \tag{3.5}$$

and

$$3(a_1a_2 + a_5a_6) + \frac{\beta[(a_1a_2 - a_5a_6)^2 + 2(a_1a_2 + a_5a_6)^2 + (a_1a_6 + a_2a_5)^2]}{\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)} > 0. \tag{3.6}$$

Those two necessary and sufficient conditions really guarantee the fundamental properties of lump waves. First, the resulting solutions of $u, v, w$ are localized in all spatial directions, under (3.5). Second, they are analytic in the whole spatial and temporal space, under (3.5) and (3.6), which lead equivalently to that $a_9 > 0$. We will show in the next section that $a_9 > 0$ is also necessary for $u, v, w$ to be analytic in $\mathbb{R}^3$.

The second condition defined by (3.6) caontains the two coefficients, $\alpha$ and $\beta$. Clearly, if

$$a_1a_2 + a_5a_6 \geq 0,\ \alpha\beta \geq 0,\ (a_1a_2 + a_5a_6)^2 + (\alpha\beta)^2 > 0, \tag{3.7}$$

then we have $a_9 > 0$. Therefore, the nonlinearity affects the analyricity of the lump waves in the model equation (2.1), but it does not affect the speeds of the two single waves in the lumps, in view of (3.2).

One reduced case can be worked out. When $\alpha = 1$ and $\beta = 0$, we obtain

$$a_9 = \frac{3(a_1a_2 + a_5a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2]}{2(a_1a_6 - a_2a_5)^2}. \tag{3.8}$$

Then, the conditions for the existence of lump waves in this reduced case simply become

$$(a_1a_5 - a_2a_5) \neq 0,\ a_1a_2 + a_5a_6 > 0. \tag{3.9}$$

## 4. Characteristics of the lump waves

In this section, we would like to consider the characteristic behaviors of the resultant lump waves presented previously.

### *4.1. Line of critical points*

Let us first compute critical points of $f$ defined by (3.1) as a function of $x$ and $y$. To this end, we need to determine solutions to the system

$$f_x(x(t), y(t), t) = 0,\ f_y(x(t), y(t), t) = 0. \tag{4.1}$$

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Since $f$ is a quadratic polynomial in $x$ and $y$, this system just requires

$$a_1\zeta_1 + a_5\zeta_2 = 0,\ a_2\zeta_1 + a_6\zeta_2 = 0.$$

Accordingly, based on the condition (3.5), we have $\zeta_1 = \zeta_2 = 0$, i.e.,

$$a_1x + a_2y + a_3t + a_4 = 0,\ a_5x + a_6y + a_7t + a_8 = 0, \tag{4.2}$$

This is a linear system of $x$ and $y$, and all solutions are critical points of the quadratic function $f$:

$$\begin{cases} x(t) = -\dfrac{(a_1+a_2)^2 + (a_5+a_6)^2 - 2a_2^2 - 2a_6^2}{(a_1+a_2)^2 + (a_5+a_6)^2} t + \dfrac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \\ y(t) = \dfrac{(a_1-a_2)^2 + (a_5-a_6)^2 - 2a_2^2 - 2a_6^2}{(a_1+a_2)^2 + (a_5+a_6)^2} t - \dfrac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \end{cases} \tag{4.3}$$

at an arbitrary time $t$.

Evidently, those critical points form a straight line, whose two spatial coordinates travel at constant speeds. Now, a further straightforward computation can verify that all those points $(x(t), y(t))$ determined above are also critical points of the three solution functions $u, v$ and $w$ defined by (2.3).

### 4.2. Analyticity condition

Taking advantage of (4.2), we see that the sum of two squares, i.e., the function $f - a_9 = \zeta_1^2 + \zeta_2^2$ becomes zero at all critical points defined by (4.3). Accordingly, the quadratic function $f > 0$ in $\mathbb{R}^3$ if and only if the constant term $a_9 > 0$. The sufficiency is clear, as analyzed earlier. The necessity is true, because we have that $f$ vanishes at the critical points if $a_9 = 0$, and $f$ vanishes at all pointa on the circle $\zeta_1^2 + \zeta_2^2 = -a_9$ if $a_9 < 0$.

Consequently, the three solutions $u, v, w$ defined by (2.3) are analytic in $\mathbb{R}^3$ if and only if the constant parameter $a_9$ must be positive. Further, in view of the analysis on the positiveness of $a_9$ made in the previous section, the necessary and sufficient conditions for $u, v, w$ to be analytic are the two conditions in (3.5) and (3.6) on the wave numbers $a_1, a_2, a_5, a_6$ and the coefficients $\alpha$ and $\beta$.

### 4.3. Extreme values

Applying the second partial derivative test, we can see that the both lump waves, $v$ and $w$, have a peak at the critical points $(x(t), y(t))$. This is because we have

$$\begin{cases} v_{xx} = -\dfrac{32\alpha^2(a_1^2+a_5^2)^2(a_1a_6 - a_2a_5)^4}{3[(a_1+a_2)^2 + (a_5+a_6)^2]^2 a_{10}^2} < 0, \\ v_{xx}v_{yy} - v_{xy}^2 = \dfrac{1024\alpha^4(a_1^2+a_5^2)^2(a_1a_6 - a_2a_5)^{10}}{27[(a_1+a_2)^2 + (a_5+a_6)^2]^4 a_{10}^4} > 0, \end{cases} \tag{4.4}$$

and

$$\begin{cases} w_{yy} = -\dfrac{32\alpha^2(a_2^2+a_6^2)^2(a_1a_6 - a_2a_5)^4}{3[(a_1+a_2)^2 + (a_5+a_6)^2]^2 a_{10}^2} < 0, \\ w_{xx}w_{yy} - w_{xy}^2 = \dfrac{1024\alpha^4(a_2^2+a_6^2)^2(a_1a_6 - a_2a_5)^{10}}{27[(a_1+a_2)^2 + (a_5+a_6)^2]^4 a_{10}^4} > 0, \end{cases} \tag{4.5}$$

where $a_{10}$ is defined by

$$a_{10} = \alpha(a_1a_2 + a_5a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2) + \frac{1}{3}\beta[(a_1a_2 + a_5a_6)^2 + (a_1a_6 + a_2a_5)^2 + 2a_1^2a_2^2 + 2a_5^2a_6^2]. \tag{4.6}$$

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In a similar way, we can work out that

$$\begin{cases} u_{xx} = -\dfrac{32\alpha^2(a_1a_2+a_5a_6)(a_1^2+a_5^2)(a_1a_6-a_2a_5)^4}{3[(a_1+a_2)^2+(a_5+a_6)^2]^2a_{10}^2}, \\ u_{xx}u_{yy}-u_{xy}^2 \\ = \dfrac{1024\alpha^4[3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2](a_1a_6-a_2a_5)^{10}}{81[(a_1+a_2)^2+(a_5+a_6)^2]^4a_{10}^4}, \end{cases} \tag{4.7}$$

where $a_{10}$ is given by (4.6). Accordingly, the lump wave $u$ has the maximum (or minimum) points $(x(t), y(t))$, when $a_1a_2 + a_5a_6 > 0$ (or $a_1a_2 + a_5a_6 < 0$) and

$$3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2 > 0;$$

the lump wave $u$ has the saddle points $(x(t), y(t))$, when

$$3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2 < 0;$$

and the second partial derivative test is inconclusive, when

$$3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2 = 0.$$

A direct computation can generate the extreme values of $v$, $w$ and $u$, achieved at the critical points $(x(t), y(t))$, as follows:

$$v_{maximum} = \frac{8\alpha(a_1^2+a_5^2)(a_1a_6-a_2a_5)^2}{3[(a_1+a_2)^2+(a_5+a_6)^2]a_{10}}, \tag{4.8}$$

$$w_{maximum} = \frac{8\alpha(a_2^2+a_6^2)(a_1a_6-a_2a_5)^2}{3[(a_1+a_2)^2+(a_5+a_6)^2]a_{10}}, \tag{4.9}$$

$$u_{extremum} = \frac{8\alpha(a_1a_2+a_5a_6)(a_1a_6-a_2a_5)^2}{3[(a_1+a_2)^2+(a_5+a_6)^2]a_{10}}, \tag{4.10}$$

where $a_{10}$ is defined by (4.6). Upon observing those expressions for the extreme values, we find that all extreme values do not depend on time $t$; they are all constants on the characteristic line of critical points (see also, [21,22] for other examples). Furthermore, when $a_1a_6 - a_2a_5$ goes to zero, i.e., the two spatial directions $(a_1, a_2)$ and $(a_5, a_6)$ tends to be parallel to each other, the lump waves of $u, v, w$ may not decay in all cases of the wave numbers $a_1, a_2, a_5$ and $a_6$.

## 5. Conclusion

Through conducting symbolic computations with Maple, we have explored lump waves in a spatial symmetric (2+1)-dimensional dispersive wave model. The resulting lump waves have a line of critical points, whose spatial coordinates travel with constant velocities. The frequencies $a_3, a_7$ and the constant term $a_9$ of the lump waves were computed in terms of the wave numbers in the quadratic function $f$. Characteristic properties of the lump waves, such as critical points and extreme values, were worked out, and the effects of the nonlinear terms and the wave numbers were analyzed.

Interestingly, abundant lump waves also exist in linear wave model equations [20], besides nonlinear (2+1)-dimensional models (see, e.g., [24]-[27]) and (3+1)-dimension models (see, e.g., [28, 29]). The Hirota bilinear forms and the generalized bilinear forms are the starting points [12,30], exhibiting a great convenience in determining lump waves. Interaction solutions between lump waves and other interesting waves, including homoclinic and heteroclinic solutions, can be explored for (2+1)-dimensonal integrable model equations (see, e.g., [16,31,32]).

It is also known that $N$-soliton solutions have been systematically studied by the Riemann-Hilbert technique for local and nonlocal integrable equations generated from groups reductions of matrix spectral problems (see, e.g., [33]-[36]). It is intriguing to analyze the existence of lump waves in reduced integarble equations (see, e.g., [37,38]), both local and nonlocal. It is expected that studies of lump waves could advance our understanding of nonlinear wave phenomena and their integrability theory [39].

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**Data Availability Statement:** All data generated or analyzed during this study are included in this published article.

**Conflicts of Interest:** The author declares that there is no known competing interest that could have appeared to influence this work.

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