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A Fourier Transform Differentiator Operator

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Abstract: By utilizing the Fourier transform, we present a practical method for evaluating a function of a derivative applied to any other arbitrary function of a single complex variable. As an illustration of this approach, we compute the actions of the displacement and squeeze operators on an arbitrary function, as well as the propagation of paraxial fields, Airy and number states, without the need for algebraic quantum operator techniques or the Fresnel integral.

Keywords: Fourier transform; number states

Introduction

Fourier optics is a theoretical and computational framework used to simulate and understand the propagation of light in free space and optical elements [1]. Moreover, quantum optics and its mathematical methods can explain the connections between geometrical and wave optics, classical mechanics and quantum mechanics. In addition, it offers new connections and operating procedures for solving problems in classical optics. These ideas were based on coherent and squeezed states [2], systematically including group theory associated with displacement and squeeze operators [3,4].

The main purpose of this work is to address the following question: Is it possible to calculate the actions of quantum operators on arbitrary functions using Fourier transforms? As illustrative examples, we will determine the actions of the displacement and squeeze operators on arbitrary functions and compute the value of a freely propagating paraxial field using the quantum propagator for a free particle acting on Airy functions and number states. These propagations will be carried out through the use of Fourier transforms.

Fourier transform differentiator operator

The standard 1D Fourier transform and inverse Fourier of an integrable function $f(x)$ are defined as follows:

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)\,, \tag{1}$$

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} F(k)\,, \tag{2}$$

respectively. Indeed, if the Fourier transform of $f(x)$ exists, it is not difficult to establish the differentiation property

$$\mathcal{F}\left\{\frac{d^n}{dx^n} f(x)\right\} = (ik)^n F(k)\,. \tag{3}$$

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If we assume that $g(x)$ is an analytic function, which admits a power-series expansion, and we write $g(x) = \sum_{n=0}^{\infty} A_n x^n$, then the extension to any function of the derivative is given by

$$\mathcal{F}\left\{ g\left(\frac{d}{dx}\right) f(x) \right\} = g(ik)F(k) \,. \tag{4}$$

Furthermore, assuming that inverse Fourier transform exists, we can obtain

$$g\left(\frac{d}{dx}\right) f(x) = \mathcal{F}^{-1}\left\{ g(ik)F(k) \right\} \,. \tag{5}$$

We therefore call this function a Fourier transform differentiator operator.

Displacement and squeeze operators

Coherent and squeezed states are fundamental components of the theoretical framework of modern optics. One way to define coherent states is by using the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where $\alpha = (x_0 + ip_0)/\sqrt{2}$ represents a point in the optical phase space, and $\hat{a}^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}$ and $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ are the creation and annihilation operators. In the position representation, we have

$$\langle x|\hat{D}(\alpha)|f\rangle = \mathrm{e}^{ip_0 x} \exp\left(-x_0 \frac{d}{dx}\right) f(x) \,. \tag{6}$$

By comparing equation (5) with (6), we obtain that $g(x) = \exp(-x_0 x)$. Using the relation (5), we can express the action of the displacement operator on an arbitrary function $f(x)$ as:

$$\begin{aligned} \exp\left(-x_0 \frac{d}{dx}\right) f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \mathrm{e}^{ik(x-x_0)} F(k) \\ &= f(x - x_0) \,. \end{aligned} \tag{7}$$

Another class of minimum-uncertainty states are the squeezed states, which can be obtained using the squeeze operator $\hat{S}(r) = \exp\left[\frac{1}{2} r(\hat{a}^2 - \hat{a}^{\dagger 2})\right]$, where r is an arbitrary real number. In the position representation we have,

$$\langle x|\hat{S}(r)|f\rangle = \mathrm{e}^{\frac{r}{2}} \exp\left(rx \frac{d}{dx}\right) f(x) \,. \tag{8}$$

By changing $x = \exp(u)$, we obtain

$$\langle x|\hat{S}(r)|f\rangle = \mathrm{e}^{\frac{r}{2}} \exp\left(r \frac{d}{du}\right) f(\mathrm{e}^u) = \mathrm{e}^{\frac{r}{2}} f(\mathrm{e}^r x) \,. \tag{9}$$

This expression was stated without proof in equation (4.106) of the reference [5].

Freely propagating paraxial field

Airy function

The Airy function is defined by the inverse Fourier transform as

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \exp\left(\frac{i}{3} k^3 + ikx\right) \,. \tag{10}$$

If we consider the Airy function as an initial state for the paraxial equation, the free coordinate evolution reads as follows:

$$\begin{aligned}\mathrm{Ai}(x,z) &= \exp\left(-\frac{i}{\kappa}\frac{z}{2}\hat{p}^2\right)\mathrm{Ai}(x,0)\,,\\ &= \exp\left[i\left(\frac{\kappa z}{2}\right)\frac{d^2}{dx^2}\right]\mathrm{Ai}(x,0)\,,\end{aligned} \tag{11}$$

where $\kappa = \lambda/2\pi$, and z represents the propagation coordinate, (for time evolution, we set $\kappa = \hbar$ and $z = t$). By comparing equation (5) with (11), we obtain that $g(x) = \exp[i(\kappa z/2)x^2]$. Coordinate free evolution of Airy function, reads

$$\mathrm{Ai}(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ \exp\left(\frac{i}{3}k^3 - i\frac{\kappa z}{2}k^2 + ikx\right)\,. \tag{12}$$

After rearranging terms and using the definition of the Airy function, we have

$$\mathrm{Ai}(x,z) = \mathrm{Ai}\left[x - \left(\frac{\kappa z}{2}\right)^2\right]\exp\left[i\frac{\kappa z}{2}\left(x - \frac{\kappa^2 z^2}{6}\right)\right]\,, \tag{13}$$

which is the same simple analytical expression reported in reference [6].

Number states

The free evolution of the field $E(x,z)$ can be expressed in terms of the eigenfunctions of the quantum harmonic oscillator $\psi_n(x)$ as follows:

$$E(x,z) = \sum_{n=0}^{\infty} E_n \exp\left[i\left(\frac{\kappa z}{2}\right)\frac{d^2}{dx^2}\right]\psi_n(x)\,, \tag{14}$$

where

$$\psi_n(x) = N_n \exp\left(-\frac{1}{2}x^2\right)H_n(x)\,. \tag{15}$$

with $N_n = 1/\sqrt{2^n n!\sqrt{\pi}}$. The amplitudes E_m are determined by the initial field, such that

$$E_m = \int_{-\infty}^{\infty} dx\ \psi_m^*(x)E(x,0)\,. \tag{16}$$

By comparing equation (14) with (5), we can deduce that

$$\exp\left[i\left(\frac{\kappa z}{2}\right)\frac{d^2}{dx^2}\right]\psi_n(x) = \mathcal{F}^{-1}\left\{g(-ik)F(k)\right\} = P_n(x,z)\,, \tag{17}$$

where $g(x) = \exp[i(\kappa z/2)x^2]$ and $f(x) = \psi_n(x)$. We have evaluated the direct and inverse Fourier transforms using Cauchy's integral formula, considering a suitable contour containing a real axis, (see Gradshteyn and Ryzhik, Eq. 7.375.8) [7], we write

$$F(k) = (-i)^n\sqrt{2\pi}N_n \exp\left(-\frac{1}{2}k^2\right)H_n(k)\,, \tag{18}$$

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and therefore,

$$P_n(x,z) = \frac{N_n}{\sqrt{1+i\kappa z}} \left(\frac{1-i\kappa z}{1+i\kappa z} \right)^{\frac{n}{2}} \exp\left(-\frac{1}{1+i\kappa z} \frac{x^2}{2} \right) H_n\left(\frac{x}{\sqrt{1+\kappa^2 z^2}} \right) , \tag{19}$$

and we have

$$E(x,z) = \sum_{n=0}^{\infty} E_n P_n(x,z) , \tag{20}$$

which is the same simple analytical expression for free propagation of any initial field, based on the eigenstates of the quantum harmonic oscillator, as that obtained by operators algebra in reference [8].

Conclusions

We have presented a method for evaluating a one-derivative function applied to any other arbitrary function of a single complex variable. This approach is based only on Fourier transforms and does not revolve around an algebra of creation and annihilation operators.

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