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Ricci Vector Fields

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Abstract: We introduce a special vector field ω on a Riemannian manifold (N^m, g), such that the Lie derivative of the metric g with respect to ω is equal to ρRic, where Ric is the Ricci curvature of (N^m, g) and ρ is a smooth function on N^m and call this vector field a ρ-Ricci vector field. We use ρ-Ricci vector field on a Riemannian manifold (N^m, g) and find two characterizations of m-sphere $S^m(\alpha)$. In first result, we show that an m-dimensional compact and connected Riemannian manifold (N^m, g) with nonzero scalar curvature admits a ρ-Ricci vector field ω such that ρ is nonconstant function and the integral of $Ric(\omega, \omega)$ has a suitable lower bound is necessary and sufficient for (N^m, g) to be isometric to m-sphere $S^m(\alpha)$. In second result, we show that an m-dimensional complete and simply connected Riemannian manifold (N^m, g) of positive scalar curvature admits a ρ-Ricci vector field ω such that ρ is a nontrivial solution of Fischer-Marsden equation and the squared length of the covariant derivative of ω has an appropriate upper bound, if and only if, (N^m, g) to be isometric to m-sphere $S^m(\alpha)$.

Keywords: ρ-Ricci vector fields; Fischer-Marsden equation; m-sphere; Ricci curvature

1. Introduction

An m-dimensional complete simply connected Riemannian manifold of constant curvature α is isometric to one of the spaces the m-sphere $S^m(\alpha)$, the Euclidean space R^m or the hyperbolic space $H^m(\alpha)$ according as $\alpha > 0$, $\alpha = 0$ or $\alpha < 0$ respectively (cf. [2]). Since, this classification, there has been an interest in obtaining necessary and sufficient conditions on complete Riemannian manifolds so that they are isometric to one of the three model spaces $S^m(\alpha)$, R^m and $H^m(\alpha)$ respectively. In that one of most sought questions is in obtaining different characterizations of spheres $S^m(\alpha)$ among complete Riemannian manifolds. In obtaining these characterizations most of the times conformal and Killing vector fields are used on an m-dimensional complete Riemannian manifold (N^m, g) (cf. [1], [4]-[11], [14], [15]). A vector field $\mathbf{u}$ on m-Riemannian manifold (N^m, g) is a conformal vector field if the Lie derivative $\pounds_{\mathbf{u}} g$ has expression

$$\pounds_{\mathbf{u}} g = 2fg,$$

where f is a smooth function called the conformal factor. If $f = 0$ in above definition, then $\mathbf{u}$ is called a Killing vector field.

In this paper, we are interested in a vector field ω on an m-dimensional Riemannian manifold (N^m, g) that satisfies

$$\frac{1}{2}\pounds_{\omega} g = \rho Ric, \tag{1.1}$$

where $\pounds_{\omega} g$ is the Lie-derivative of the metric g with respect to ω, ρ is a smooth function and Ric is the Ricci tensor of (N^m, g). We call ω satisfying equation (1.1) a ρ-Ricci vector field on (N^m, g). Naturally, if (N^m, g) is an Einstein manifold, then a ρ-Ricci vector field ω is a conformal vector field on (N^m, g) (cf. [4]- [9]). If in the equation (1.1), we take $\rho = 0$, then 0-Ricci vector field ω on (N^m, g) is a Killing vector field on (N^m, g) (cf. [10]). A ρ-Ricci vector field on (N^m, g) is also a particular form of potential field of a generalized soliton (cf. [12]), with $\alpha = -\rho$ and $\beta = \gamma = 0$.

We could also approach to equation (1.1) in other context (cf. [3]). On the m-dimensional Riemannian manifold (N^m, g), take a smooth function ρ and consider 1-parameter family of metrics $g(t)$ satisfying generalized Ricci flow (or ρ-Ricci flow) equation

$$\partial_t g = 2\rho Ric, \quad g(0) = g. \tag{1.2}$$

To reach a solution of above flow, we take a 1-parameter family of diffeormorphisms $\varphi_t : N^m \to N^m$ generated by the family of vector fields $\mathbf{W}(t)$ and $\sigma(t)$ be a scale factor. Then we are interested in a solution of flow (1.2) of the form

$$g(t) = \sigma(t)\varphi_t^*(g).$$

Differentiating above equation with respect to t and substituting $t = 0$, while assuming $\sigma(0) = 1$, $\dot{\sigma}(0) = 0$, $\mathbf{W}(0) = \omega$ and using $\varphi_0 = id$, we get

$$\pounds_\omega g - 2\rho Ric = 0,$$

which is equation (1.1). Thus, a ρ-Ricci vector field ω on (N^m, g) can be considered as stable solution of the flow (1.2).

We see that as a trivial example on the Euclidean space R^m, a constant vector field $\mathbf{a}$ is a ρ-Ricci vector field for any smooth function ρ on R^m. Similarly on the complex Euclidean space C^m with complex structure J and the vector field

$$\xi = \sum_{i=1}^{m} z^i \frac{\partial}{\partial z^i},$$

where $z^1, .., , z^m$ are Euclidean coordinates, the vector field $\omega = J\xi$ is a ρ-Ricci vector field for any smooth function ρ on C^m.

Next, we show that on the sphere $S^m(\alpha)$ of constant curvature α, there are many ρ-Ricci vector fields. With the imbedding $i : S^m(\alpha) \to R^{m+1}$ and unit normal ξ and shape operator $-\sqrt{\alpha}I$, on taking a nonzero constant vector field $\mathbf{b}$ on the Euclidean space R^{m+1}, we have $\mathbf{b} = \omega + f\xi$, where $f = \langle \mathbf{b}, \xi \rangle$ and ω is the tangential component of $\mathbf{b}$ to the sphere $S^m(\alpha)$. Denote the induced metric on the sphere $S^m(\alpha)$ by g and the Riemannian connection by D. Then differentiating above equation with respect to the vector field X on $S^m(\alpha)$, we have

$$D_X\omega = -\sqrt{\alpha} f X, \quad \nabla f = \sqrt{\alpha}\omega, \tag{1.3}$$

where ∇f is the gradient of f. Using the first equation in (1.3), it follows that

$$\pounds_\omega g = -2\sqrt{\alpha} f g$$

and the Ricci tensor of the sphere $S^m(\alpha)$ is given by

$$Ric = (m-1)\alpha g.$$

Thus, we see that the vector field ω on the sphere $S^m(\alpha)$ satisfies

$$\frac{1}{2}\pounds_\omega g = \rho Ric, \quad \rho = -\frac{1}{(m-1)\sqrt{\alpha}} f, \tag{1.4}$$

that is, ω is a ρ-Ricci vector field on the sphere $S^m(\alpha)$. Indeed. for each nonzero constant vector field on the Euclidean space R^{m+1}, there is a ρ-Ricci vector field on the sphere $S^m(\alpha)$.

Above example naturally leads to a question: Under what conditions a compact and connected m-dimensional Riemannian manifold (N^m, g) admitting a ρ-Ricci vector field ω is isometric to a m-sphere $S^m(\alpha)$?

There are two well known differential equations on a Riemannian manifold (N^m, g), the first is Obata's differential equation namely (cf. [14], [15]),

$$Hess(\sigma) = -\alpha\sigma g, \tag{1.5}$$

where σ is a non-constant smooth function, α is a positive constant and $Hess(\sigma)$ is the Hessian of σ defined by

$$Hess(\sigma)(X, Y) = g\left(D_X \nabla\sigma, Y\right),$$

for smooth vector fields X, Y on N^m. Obata proved that a necessary and sufficient condition for a complete and simply connected Riemannian manifold (N^m, g) to admit a nontrivial solution of differential equation (1.5) is that (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [14], [15]). The other differential equation on (N^m, g) is Fischer-Marsden equation (cf. [13])

$$(\Delta\sigma)\, g + \sigma Ric = Hess(\sigma), \tag{1.6}$$

where σ is a smooth function on N^m and $\Delta\sigma = div\left(\nabla\sigma\right)$ is the Laplacian of σ. We shall use the abbreviation for the above Fischer-Marsden equation as FM-equation. Taking trace in the FM-equation (1.6), we get

$$\Delta\sigma = -\frac{\tau}{m-1}\sigma, \tag{1.7}$$

where $\tau = TrRic$ is the scalar curvature of the Riemannian manifold (N^m, g). It is known that if (N^m, g) admits a nontrivial solution of FM-equation, then the scalar curvature τ is necessarily constant (cf. [13]).

Note that by equation (1.3), the smooth function f on the sphere $S^m(\alpha)$ has Hessian

$$Hess(f)\,(X, Y) = g\left(D_X \nabla f, Y\right) = \sqrt{\alpha} g\left(D_X \omega, Y\right) = -\alpha f g\,(X, Y),$$

the Laplacian $\Delta f = div\left(\sqrt{\alpha}\omega\right) = -m\alpha f$ and $Ric = (m-1)\alpha g$. Consequently, on $S^m(\alpha)$, we see that

$$(\Delta f)\, g + f Ric = Hess(f), \tag{1.7}$$

that is, f is a solution of FM-equation on the sphere $S^m(\alpha)$. If we combine the two, namely a Riemannian manifold (N^m, g) admits a ρ-Ricci vector field ω such that ρ is a nontrivial solution of the FM-equation on (N^m, g) and seek additional condition under which (N^m, g) is isometric to $S^m(\alpha)$? Notice that the ρ-Ricci vector field ω on the sphere $S^m(\alpha)$ is a closed vector field. Therefore, in this paper, we use the closed ρ-Ricci vector field ω on a Riemannian manifold (N^m, g) and answer these two question in section-3, where we find two characterizations of the sphere $S^m(\alpha)$.

2. Preliminaries

Let ω be a closed ρ-Ricci vector field on an m-dimensional Riemannian manifold (N^m, g). If β is the 1-form dual to ω, that is,

$$\beta\,(X) = g\,(\omega, X)\,, \quad X \in \Theta\,(TN^m)\,, \tag{2.1}$$

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where $\Theta\left(TN^{m}\right)$ is the space of smooth sections of the tangent bundle TN^{m}, then we have $d\beta=0$. We denote by ∇_{X} the covariant derivative operator with respect to the Riemannian connection on $\left(N^{m},g\right)$ and notice that for the closed ρ-Ricci vector field ω, we have

$$\begin{aligned} 2g\left(\nabla_{X}\omega,Y\right) &= g\left(\nabla_{X}\omega,Y\right)+g\left(\nabla_{Y}\omega,X\right)+g\left(\nabla_{X}\omega,Y\right)-g\left(\nabla_{Y}\omega,X\right) \\ &= \left(\pounds_{\omega}g\right)\left(X,Y\right)+d\beta\left(X,Y\right)=2\rho Ric\left(X,Y\right). \end{aligned}$$

Thus, for a closed ρ-Ricci vector field ω, we have

$$\nabla_{X}\omega=\rho TX,\quad X\in\Theta\left(TN^{m}\right), \tag{2.2}$$

where T is a symmetric operator called Ricci operator given by

$$Ric\left(X,Y\right)=g\left(TX,Y\right).$$

Using the expression for the curvature tensor field R of $\left(N^{m},g\right)$

$$R\left(X,Y\right)Z=\left[\nabla_{X},\nabla_{Y}\right]Z-\nabla_{[X.Y]}Z,\quad X,Y,Z\in\Theta\left(TN^{m}\right),$$

and equation (2.2), we get

$$R\left(X,Y\right)\omega=X\left(\rho\right)TY-Y\left(\rho\right)TX+\rho\left(\left(\nabla_{X}T\right)\left(Y\right)-\left(\nabla_{Y}T\right)\left(X\right)\right), \tag{2.3}$$

$X,Y\in\Theta\left(TN^{m}\right)$, where $\left(\nabla_{X}T\right)\left(Y\right)=\nabla_{X}TY-T\left(\nabla_{X}Y\right)$. The scalar curvature τ of $\left(N^{m},g\right)$ is given by $\tau=TrT$, where TrT is the trace of the symmetric operator T. Choosing a local frame $\left\{F_{1},..,F_{m}\right\}$ and using the definition of the Ricci tensor Ric

$$Ric\left(X,Y\right)=\sum_{j=1}^{m}g\left(R\left(F_{j},X\right)Y,F_{j}\right),$$

together with equation (1.3), we conclude that

$$Ric\left(Y,\omega\right)=Ric\left(Y,\nabla\rho\right)-\tau Y\left(\rho\right)+\rho g\left(Y,\sum_{j=1}^{m}\left(\nabla_{F_{j}}T\right)\left(F_{j}\right)\right)-\rho Y\left(\tau\right), \tag{2.4}$$

where $\nabla\rho$ is the gradient of ρ. It is known that the gradient of scalar curvature τ satisfies (cf. [2])

$$\frac{1}{2}\nabla\tau=\sum_{j=1}^{m}\left(\nabla_{F_{j}}T\right)\left(F_{j}\right). \tag{2.5}$$

Consequently, equation (2.4) takes the form

$$Ric\left(Y,\omega\right)=Ric\left(Y,\nabla\rho\right)-\tau Y\left(\rho\right)-\frac{1}{2}\rho Y\left(\tau\right) \tag{2.6}$$

and we have

$$T\left(\omega\right)=T\left(\nabla\rho\right)-\tau\nabla\rho-\frac{1}{2}\rho\nabla\tau. \tag{2.7}$$

3. Characterizing spheres via æ-Ricci fields

Let ω be a closed ρ-Ricci vector field on an m-dimensional Riemannian manifold $\left(N^{m},g\right)$. We shall use ρ-Ricci vector field and find two characterizations of m-sphere $\mathbf{S}^{m}\left(\alpha\right)$. In our first result, we prove the following result:

Theorem 1. *A closed ρ-Ricci vector field ω on an m-dimensional compact and connected Riemannian manifold (N^m, g), $m > 2$ with scalar curvature $\tau \neq 0$ and nonzero nonconstant function ρ satisfies*

$$\int_M Ric(\omega, \omega) \geq \frac{m-1}{m} \int_M (div\omega)^2,$$

if and only if, τ is a positive constant $m(m-1)\alpha$, and (N^m, g) is isometric to $S^m(\alpha)$.

Proof. Let (N^m, g) be an m-dimensional compact and connected Riemannian manifold, $m > 2$ with scalar scalar curvature $\tau \neq 0$ and ! be a closed ρ-Ricci vector field defined on (N^m, g) with nonzero and nonconstant function ρ satisfying

$$\int_M Ric(\omega, \omega) \geq \frac{m-1}{m} \int_M (div\omega)^2. \tag{3.1}$$

Then using equation (2.2), we have

$$div\omega = \rho\tau. \tag{3.2}$$

Choosing a local orthonormal frame $\{F_1, .., F_m\}$ and using

$$\|T\|^2 = \sum_{j=1}^{m} g(TF_j, TF_j)$$

and an outcome of equation (2.2) as

$$(£_\omega g)(X, Y) = 2\rho g(TX, Y), \quad X, Y \in \Theta(TN^m),$$

we conclude

$$\frac{1}{2}|£_\omega g|^2 = 2\rho^2 \|T\|^2. \tag{3.3}$$

Note that, we have

$$\begin{aligned} \left\|T - \frac{\tau}{m}I\right\|^2 &= \sum_{j=1}^{m} g\left(\left(TE_j - \frac{\tau}{m}E_j\right), \left(TE_j - \frac{\tau}{m}E_j\right)\right) \\ &= \|T\|^2 + \frac{1}{m}\tau^2 - 2\sum_{j=1}^{m} g\left(TE_j, \frac{\tau}{m}E_j\right), \end{aligned}$$

that is,

$$\left\|T - \frac{\tau}{m}I\right\|^2 = \|T\|^2 - \frac{1}{m}\tau^2. \tag{3.4}$$

Now, using equation (2.2), we have

$$\rho\left(TX - \frac{\tau}{m}X\right) = \left(\nabla_X\omega - \frac{\tau}{m}\rho X\right),$$

which in view of a local frame $\{F_1, .., F_m\}$ on (N^m, g) implies

$$\begin{aligned} \rho^2\left\|T - \frac{\tau}{m}I\right\|^2 &= \sum_{j=1}^{m} g\left(\rho\left(TE_j - \frac{\tau}{m}E_j\right), \rho\left(TE_j - \frac{\tau}{m}E_j\right)\right) \\ &= \sum_{j=1}^{m} g\left(\nabla_{E_j}\omega - \frac{\tau}{m}\rho E_j, \nabla_{E_j}\omega - \frac{\tau}{m}\rho E_j\right) \\ &= \|\nabla\omega\|^2 + \frac{1}{m}\tau^2\rho^2 - \frac{2}{m}\tau\rho div\omega. \end{aligned}$$

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Using (3.2), in above equation, yields

$$\rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \|\nabla \omega\|^2 - \frac{1}{m}\tau^2\rho^2,$$

which on integration gives

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \int_{N^m} \left(\|\nabla \omega\|^2 - \frac{1}{m}\tau^2\rho^2 \right). \tag{3.5}$$

Next, we recall the following integral formula (cf. [16])

$$\int_{N^m} \left(Ric(\omega, \omega) + \frac{1}{2} |\pounds_\omega g|^2 - \|\nabla \omega\|^2 - (div\omega)^2 \right) = 0,$$

and employing it in equation (3.5), we conclude

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \int_{N^m} \left(Ric(\omega, \omega) + \frac{1}{2} |\pounds_\omega g|^2 - (div\omega)^2 - \frac{1}{m}\tau^2\rho^2 \right).$$

Using equations (3.2) and (3.3) in above equation, yields

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \int_{N^m} \left(Ric(\omega, \omega) + 2\rho^2 \|T\|^2 - \tau^2\rho^2 - \frac{1}{m}\tau^2\rho^2 \right),$$

that is,

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \int_{N^m} \left(Ric(\omega, \omega) + 2\rho^2 \left(\|T\|^2 - \frac{1}{m}\tau^2\rho^2 \right) - \tau^2\rho^2 + \frac{1}{m}\tau^2\rho^2 \right).$$

In view of equation (3.4), above equation implies

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \int_{N^m} \left(\frac{m-1}{m}\tau^2\rho^2 - Ric(\omega, \omega) \right)$$

and substituting from equation (3.2), it yields

$$\int_{N^m} \rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \frac{m-1}{m} \int_{N^m} (div\omega)^2 - \int_{N^m} Ric(\omega, \omega).$$

Employing inequality (3.1) in above equation, we conclude

$$\rho^2 \left\| T - \frac{\tau}{m} I \right\|^2 = 0.$$

However, $\rho \neq 0$ on connected N^m, gives

$$T = \frac{\tau}{m} I. \tag{3.6}$$

Taking covariant derivative in above equation, we have

$$(\nabla_X T)(Y) = \frac{1}{m} X(\tau) Y$$

and using a frame $\{F_1, .., F_m\}$ on (N^m, g) in above equation, we get

$$\sum_{j=1}^{m} \left(\nabla_{E_j} T \right)(E_j) = \frac{1}{m} \nabla \tau.$$

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Using equation (2.5) in this equation, we arrive at

$$\frac{1}{2}\nabla\tau = \frac{1}{m}\nabla\tau$$

and as $m > 2$, we conclude $\nabla\tau = 0$. Hence, the scalar curvature τ is a constant and it is a nonzero constant. Now, equations (2.7) and (3.6) imply

$$\frac{\tau}{m}\omega = \frac{\tau}{m}\nabla\rho - \tau\nabla\rho,$$

that is,

$$\omega = -(m-1)\nabla\rho \tag{3.7}$$

and it gives $div\omega = -(m-1)\Delta\rho$, which in view of equation (3.2) implies $\tau\rho = -(m-1)\Delta\rho$, that is,

$$-(m-1)\rho\Delta\rho = \tau\rho^2.$$

Integrating above equation by parts, we arrive at

$$(m-1)\int_{N^m}\|\nabla\rho\|^2 = \tau\int_{N^m}\rho^2.$$

Since, ρ is a nonconstant, from above equation, we conclude the constant $\tau > 0$. We put $\tau = m(m-1)\alpha$ for a positive constant α. Now, differentiating equation (3.7) and using equations (2.2) and (3.6), we conclude

$$\nabla_X\nabla\rho = -\alpha\rho X, \quad X \in \Theta\left(TN^m\right),$$

where ρ is nonconstant function and $\alpha > 0$ is a constant. Hence, $Hess(\rho) = -\alpha\rho g$, that is, (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [14], [15]).

Conversely, suppose that (N^m, g) is isometric to the sphere $S^m(\alpha)$. Then, we know that a nonzero constant vector field **b** on the ambient Euclidean space R^{m+1} induces a vector field ω on the sphere $S^m(\alpha)$, which by equation (1.4) is a ρ-Ricci vector field. Clearly, the scalar curvature of $S^m(\alpha)$ is given by $\tau = m(m-1)\alpha \neq 0$. We claim that the function ρ is nonzero and nonconstant. If $\rho = 0$, then by equation (1.4), we have $f = 0$, which in view of equation (1.3) implies $\omega = 0$, and this in turn will imply that the constant vector field $\mathbf{b} = 0$. This is a contrary to the assumption that **b** is a nonzero constant vector field. Hence, $\rho \neq 0$. Now, suppose ρ is a constant, then by equation (1.4), f is a constant and by equation (1.3), we have $div\omega = -m\sqrt{\alpha}f$, which by Stokes's Theorem on compact $S^m(\alpha)$, would imply $f = 0$. This in turn by virtue of equation (1.4) implies $\rho = 0$, which is a contradiction as seen above. Hence, the function ρ is nonzero and nonconstant.

Next, using equations (1.3) and (1.4), we have

$$div\omega = m(m-1)\alpha\rho \tag{3.8}$$

and it gives

$$\int_{S^m(\alpha)}(div\omega)^2 = m^2(m-1)^2\alpha^2\int_{S^m(\alpha)}\rho^2. \tag{3.9}$$

Now, using equation (1.4), we have

$$\nabla\rho = -\frac{1}{(m-1)\sqrt{\alpha}}\nabla f, \tag{3.10}$$

which on using equation (1.3), gives

$$\nabla\rho = -\frac{1}{m-1}\omega.$$

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Taking divergence in above equation and using equation (3.8), we conclude $\Delta\rho = -m\alpha\rho$, that is, $\rho\Delta\rho = -m\alpha\rho^2$ integrating this equation by parts, we conclude

$$\int_{S^m(\alpha)} \|\nabla\rho\|^2 = m\alpha \int_{S^m(\alpha)} \rho^2.$$

Treating this equation with equation (3.9), we conclude

$$\int_{S^m(\alpha)} (div\omega)^2 = m(m-1)^2\alpha \int_{S^m(\alpha)} \|\nabla\rho\|^2 . \tag{3.11}$$

Also, using equations (1.3) and (3.10), we have

$$\omega = -(m-1)\nabla\rho$$

and it changes the equation (3.11) to

$$\int_{S^m(\alpha)} (div\omega)^2 = m\alpha \int_{S^m(\alpha)} \|\omega\|^2 .$$

Finally, using $Ric(\omega,\omega) = (m-1)\|\omega\|^2$ in above equation, we conclude

$$\int_{S^m(\alpha)} Ric(\omega,\omega) = \frac{m-1}{m} \int_{S^m(\alpha)} (div\omega)^2$$

and this finishes the proof. □

Next, we consider a closed ρ-Ricci vector field on a compact and connected Riemannian manifold (N^m, g) such that the smooth function ρ is a nontrivial solution of the FM-equation and find yet another characterization of the sphere $S^m(\alpha)$. Indeed we prove the following:

Theorem 2. *An m-dimensional complete and simply connected Riemannian manifold (N^m, g) with scalar curvature $\tau > 0$ admits a closed ρ-Ricci vector field ω such that the function ρ is a nontrivial solution of the FM-equation and the length of covariant derivative of ω satisfies*

$$\|\nabla\omega\|^2 \leq \frac{1}{m}\tau^2\rho^2,$$

if and only if, τ is a positive constant $\tau = m(m-1)\alpha$ and (N^m, g) is isometric to $S^m(\alpha)$.

Proof. Suppose (N^m, g) is an m-dimensional complete and simply connected Riemannian manifold with scalar curvature $\tau > 0$, and it admits a closed ρ-Ricci vector field ω, where ρ is nontrivial solution of the FM-equation (1.6) and the length of covariant derivative of ω satisfies

$$\|\nabla\omega\|^2 \leq \frac{1}{m}\tau^2\rho^2. \tag{3.12}$$

For ρ, we define the operator B_ρ by

$$B_\rho X = \nabla_X \nabla\rho, \quad X \in \Theta(TN^m),$$

then B_ρ is a symmetric operator related to $Hess(\rho)$ by

$$Hess(\rho)(X,Y) = g(B_\rho X, Y), \quad X, Y \in \Theta(TN^m). \tag{3.13}$$

As, ρ is a nontrivial solution of the FM-equation, using equations (3.13) and (1.6), we have

$$\rho T X = B_\rho X - (\Delta \rho) X,$$

which in view of equation (1.7) becomes

$$B_\rho X = \rho T X - \frac{\tau}{m-1}\rho X. \tag{3.14}$$

Note that owing to the fact that ρ is a nontrivial solution of the FM-equation on (N^m, g), the scalar curvature τ is a constant and we put $\tau = m(m-1)\alpha$ for a constant α. Using equation (3.14), we have

$$B_\rho X + \alpha\rho X = \rho T X - (m-1)\alpha\rho X, \quad X \in \Theta\left(TN^m\right).$$

Now, using equation (2.2) in above equation, we get

$$B_\rho X + \alpha\rho X = \nabla_X \omega - (m-1)\alpha\rho X, \quad X \in \Theta\left(TN^m\right).$$

Taking a local frame $\{F_1, .., F_m\}$ on (N^m, g), by above equation, we conclude

$$\begin{aligned} \left\|B_\rho + \alpha\rho I\right\|^2 &= \sum_{j=1}^m g\left(B_\rho F_j + \alpha\rho F_j, B_\rho F_j + \alpha\rho F_j\right) \\ &= \sum_{j=1}^m g\left(\nabla_{F_j}\omega - (m-1)\alpha\rho F_j, \nabla_{F_j}\omega - (m-1)\alpha\rho F_j\right) \\ &= \|\nabla\omega\|^2 + m(m-1)^2\alpha^2\rho^2 - 2(m-1)\alpha\rho\,(div\omega). \end{aligned}$$

Now, using equating (2.2), we have $div\omega = \tau\rho = m(m-1)\alpha\rho$ and inserting it in above equation, we arrive at

$$\left\|B_\rho + \alpha\rho I\right\|^2 = \|\nabla\omega\|^2 - m(m-1)^2\alpha^2\rho^2,$$

that is,

$$\left\|B_\rho + \alpha\rho I\right\|^2 = \|\nabla\omega\|^2 - \frac{1}{m}\tau^2\rho^2.$$

Using inequality (3.12) in above equation results in

$$B_\rho = -\alpha\rho I,$$

that is,

$$Hess(\rho) = -\alpha\rho g. \tag{3.15}$$

Note that as $\tau > 0$, the constant $\alpha > 0$ and ρ being a nontrivial solution, ρ is a nonconstant function. Hence, by equation (3.15), the complete and simply connected Riemannian manifold (N^m, g) is isometric to the sphere $S^m(\alpha)$ (cf. [14], [15]).

Conversely, suppose that (N^m, g) is isometric to the sphere $S^m(\alpha)$. Then, by equation (1.7), the function f is a solution of FM-equation on the sphere $S^m(\alpha)$, which has a closed ρ-Ricci vector field ω. The solution f of FM-equation is related to ρ by equation (1.4), that is,

$$f = -(m-1)\sqrt{\alpha}\rho. \tag{3.16}$$

In the proof of Theorem-1, we have seen that ρ is a nonconstant function on $S^m(\alpha)$. Moreover, using equation (3.16), we have

$$\Delta f = -(m-1)\sqrt{\alpha}\Delta\rho, \quad Hess(f) = -(m-1)\sqrt{\alpha}Hess(\rho)$$

and the equation (1.7) takes the form

$$-(m-1)\sqrt{\alpha}\,(\Delta\rho)\,g + fRic = -(m-1)\sqrt{\alpha}Hess(\rho),$$

which in view of equation (3.16) changes to

$$(\Delta\rho)\,g + \rho Ric = Hess(\rho).$$

Hence, ρ is a nontrivial solution of the FM-equation on the sphere $S^m(\alpha)$. Now, the Ricci operator T of the sphere $S^m(\alpha)$ is given by $T = (m-1)\alpha I$ and therefore equation (2.2) on $S^m(\alpha)$ is

$$\nabla_X \omega = (m-1)\alpha\rho X, \quad X \in \Theta\left(TS^m(\alpha)\right).$$

Using the expression for the scalar curvature $\tau = m(m-1)\alpha$ for the sphere $S^m(\alpha)$, we have

$$\nabla_X \omega = \frac{\tau}{m}\rho X, \quad X \in \Theta\left(TS^m(\alpha)\right).$$

This proves

$$\|\nabla\omega\|^2 = \frac{1}{m}\tau^2\rho^2$$

and completes the proof. □

4. Conclusions

In previous section, we have used a closed ρ-Ricci vector field ω on an m-dimensional Riemannian manifold (N^m, g) to find two different characterizations of a m-sphere $S^m(\alpha)$. The scope of studying ρ-Ricci vector fields on a Riemannian manifold is quite modest. We observe that, in previous section, we restricted the ρ-Ricci vector field ω to be closed that simplified the expression for the covariant derivative of ω. It will be interesting to investigate whether, we could get similar results after removing the restriction that the ρ-Ricci vector field ω is closed. It will be interesting future topic to study the geometry of an m-dimensional Riemannian manifold (N^m, g) that admits a ρ-Ricci vector field ω, which need not be closed. In order to simplify the findings on an m-dimensional Riemannian manifold (N^m, g) admitting a ρ-Ricci vector field ω, which is not necessarily closed, we could impose the restriction on the Ricci operator T of (N^m, g) to be Codazzi type tensor, namely it satisfies

$$(\nabla_X T)(Y) = (\nabla_Y T)(X), \quad X, Y \in \Theta(TN^m).$$

Note that above restriction on (N^m, g) is slightly stronger than demanding the scalar curvature is a constant.

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