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Double-Scale Expansions for a Logarithmic Type Solution to a q-Analog of a Singular Initial Value Problem

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Abstract: We examine a linear $q-$difference differential equation which is singular in complex time t at the origin. Its coefficients are polynomial in time and bounded holomorphic on horizontal strips in one complex space variable. The equation under study represents a $q-$analog of a singular partial differential equation, recently investigated by the author, which comprises Fuchsian operators and entails a forcing term that combines polynomial and logarithmic type functions in time. A sectorial holomorphic solution to the equation is constructed as a double complete Laplace transform in both time t and its complex logarithm $\log t$ and Fourier inverse integral in space. For a particular choice of the forcing term, this solution turns out to solve some specific nonlinear $q-$difference differential equation with polynomial coefficients in some positive rational power of t. Asymptotic expansions of the solution relatively to time t are investigated. A Gevrey type expansion is exhibited in a logarithmic scale. Furthermore, a formal asymptotic expansion in power scale is displayed, revealing a new fine structure involving remainders with both Gevrey and $q-$Gevrey type growth.

Keywords: asymptotic expansion; Borel-Laplace transform; Fourier transform; initial value problem; formal power series

MSC: 35C10; 35C20

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1. Introduction

In this work, we draw attention to a family of singular linear $q-$difference differential equations modelled as

$$Q(\partial_z)u(t,z) = t^{d_D}\sigma_{q;t}^{\frac{d_D}{k_1}} R_D(\partial_z)u(t,z) + P(t,z,\sigma_{q;t},\partial_z)u(t,z) + f(t,z) \quad (1)$$

for vanishing initial data $u(0,z) \equiv 0$, where $d_D, k_1 \geq 1$ from the leading term of (1) are integers, $Q(X)$, $R_D(X)$ represent polynomials with complex coefficients and $P(t,z,V_1,V_2)$ stands for a polynomial in its arguments t, V_1, V_2 with holomorphic coefficients relatively to the space variable z on a horizontal strip in $\mathbb{C}$ designed as $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$, for some prescribed width $2\beta > 0$. The forcing term $f(t,z)$ is a logarithmic type function represented as a polynomial in both complex time variable t and inverse of its complex logarithm $1/\log t$ with coefficients that are bounded holomorphic on the strip H_β.

This paper is a natural sequel of the recent study [9] by the author. Indeed, in [9], we focused on the next singularly perturbed linear partial differential equations shaped as

$$Q(\partial_z)y(t,z,\epsilon) = (\epsilon t)^{d_D}(t\partial_t)^{\frac{d_D}{k_1}} R_D(\partial_z)y(t,z,\epsilon) + H(t,z,\epsilon,t\partial_t,\partial_z)y(t,z,\epsilon) + h(t,z,\epsilon) \quad (2)$$

for given initial data $y(0,z,\epsilon) \equiv 0$, for integers $d_D, k_1 \geq 1$ appearing in the principal term of (2), complex polynomials $Q(X)$, $R_D(X)$ as above and where $H(t,z,\epsilon,V_1,V_2)$ represents a polynomial in t, V_1, V_2 whose coefficients are bounded holomorphic w.r.t z on the strip H_β and relatively to a complex parameter ϵ on some fixed disc D_{ϵ_0} centered at 0 for some radius $\epsilon_0 > 0$. The forcing term $h(t,z,\epsilon)$ comprises coefficients that rely polynomially on complex time t, analytically in ϵ on D_{ϵ_0} and

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holomorphically in z on H_{β}. This term also entails logarithmic type functions stated as truncated Laplace transforms along a fixed segment $[-a,0]$ for some radius $a > 0$ that involve the inverse complex logarithm $1/\log(\epsilon t)$. When the radius $a > 0$ is taken large, the expression of the forcing term h becomes proximate to maps that are similar to the forcing term $f(t,z)$ of (1) described above, namely a polynomial in both ϵt and $1/\log(\epsilon t)$ with bounded holomorphic coefficients on $D_{\epsilon_0} \times H_{\beta}$.

Under suitable constraints set on the profile of (2), we were able to construct a set of genuine bounded holomorphic solutions y_p to (2), for p in a finite subset I_1 of the natural numbers $\mathbb{N}$, expressed as a complete Laplace transform of integer order k_1 in the monomial ϵt, a truncated Laplace transform of order 1 in the inverse $1/\log(\epsilon t)$ and inverse Fourier integral in the space variable z,

$$y_p(t,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p}} \int_{[-a,0]} \int_{-\infty}^{+\infty} w_p(\tau_1,\tau_2,m,\epsilon) \times \exp\left(-(\frac{\tau_1}{\epsilon t})^{k_1} - \log(\epsilon t)\tau_2 \right) e^{\sqrt{-1}zm} dm \frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1} \quad (3)$$

where the so-called Borel-Fourier map $w_p(\tau_1,\tau_2,m,\epsilon)$ is

- analytic near $\tau_1 = 0$ and relatively to $\tau_2 \in D_a$ and $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ and has (at most) exponential growth of order k_1 along some well chosen unbounded sector S_{d_p} centered at 0 and containing the halfline $L_{d_p} = [0,+\infty)e^{\sqrt{-1}d_p}$ for $d_p \in \mathbb{R}$, with respect to τ_1.
- continuous and subjected to exponential decay in phase $m \in \mathbb{R}$.

As a result, these functions $y_p(t,z,\epsilon)$ define bounded holomorphic maps on domains $\mathcal{T} \times H_{\beta} \times \mathcal{E}_p$, for well selected bounded sector $\mathcal{T}$ edged at 0 and where $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{p\in I_1}$ is an appropriate set of bounded sectors centered at 0. At this point, it is crucial to notice that these solutions y_p cannot be represented as complete Laplace transform in the map $1/\log(\epsilon t)$. It turns out that the radii $a, \epsilon_0 > 0$ are related by a rule of the form $\epsilon_0^{n_0} a^{n_1} \leq M$, for some suitable constant $M > 0$ and positive integers $n_0, n_1 \geq 1$.

Besides, asymptotic features of these solutions have been examined in [9]. It appears that the family $\{y_p\}_{p\in I_1}$ owns asymptotic expansions of Gevrey type in two distinguished scales of functions. Indeed, for each $p \in I_1$, the partial map $(t,\epsilon) \mapsto y_p(t,z,\epsilon)$ holds a generalized asymptotic formal expansion (in a sense defined in the classical textbooks [5] and [14])

$$\hat{y}_p^1(t,z,\epsilon) = \sum_{n\geq 0} G_{n,p}^1(t,z,\epsilon) \frac{(1/\log(\epsilon t))^n}{n!} \quad (4)$$

on the domain $\mathcal{T} \times \mathcal{E}_p$, in the scale of logarithmic functions $\{(1/\log(\epsilon t))^n\}_{n\geq 0}$ with bounded holomorphic coefficients $G_{n,p}^1$ on $\mathcal{T} \times H_{\beta} \times \mathcal{E}_p$. These asymptotic expansions are revealed to be of Gevrey 1 on $\mathcal{T} \times \mathcal{E}_p$, giving rise to constants $K^1, M^1 > 0$ for which the error bounds

$$\sup_{z\in H_{\beta}} |y_p(t,z,\epsilon) - \sum_{n=0}^{N} G_{n,p}^1(t,z,\epsilon) \frac{(1/\log(\epsilon t))^n}{n!}| \leq K^1 (M^1)^{N+1} \Gamma(N+2) |1/\log(\epsilon t)|^{N+1} \quad (5)$$

occur for all integers $N \geq 0$, provided that $\epsilon \in \mathcal{E}_p$, $t \in \mathcal{T}$, where $\Gamma(x)$ stands for the Gamma function in x. On the other hand, all the partial maps $\epsilon \mapsto y_p(t,z,\epsilon)$, $p \in I_1$, share a common generalized asymptotic formal expansion

$$\hat{y}^2(t,z,\epsilon) = \sum_{n\geq 0} G_n^2(t,z,\epsilon) \frac{\epsilon^n}{n!} \quad (6)$$

on $\mathcal{E}_p$, in the scale of monomial $\{\epsilon^n\}_{n\geq 0}$ with bounded holomorphic coefficients G_n^2 on $\mathcal{T} \times H_{\beta} \times \mathcal{D}_{\epsilon_0}$, for some open domain $\mathcal{D}_{\epsilon_0}$ containing all the sectors $\mathcal{E}_p$, $p \in I_1$. Moreover, these asymptotic expansions happen to be of Gevrey order $1/k_1$ on each sector, meaning that constants $K_p^2, M_p^2 > 0$ can be pinpointed for which the error estimates

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$$\sup_{t \in \mathcal{T}, z \in H_{\beta}} |y_p(t,z,\epsilon) - \sum_{n=0}^{N} G_n^2(t,z,\epsilon)\frac{\epsilon^n}{n!}| \leq K_p^2 (M_p^2)^{N+1}\Gamma(1 + \frac{N+1}{k_1})|\epsilon|^{N+1} \quad (7)$$

hold for all integers $N \geq 0$, whenever $\epsilon \in \mathcal{E}_p$. At last, we proved in [9] that the coefficients $G_{n,p}^1$ and G_n^2 of both formal expansions $\hat{y}_p^1$ and $\hat{y}^2$ solve explicit differential recursion relations with respect to $n \geq 0$ that might be handy for effective computations.

In the present investigation of the problem (1), we plan to follow a similar roadmap as in [9]. Namely, we plan to build up genuine sectorial solutions to (1) and describe their asymptotic expansions as time t borders the origin, instead of a perturbation parameter ϵ which does not appear in (1). We notice that our main problem (1) can be viewed as a $q-$analog of (2) where the Fuchsian operator $t\partial_t$ is substituted by the discret dilation operator $\sigma_{q;t}$. This terminology stems from the plain observation that the quotient $(f(qt) - f(t))/(qt - t)$ neighbors the derivative $f'(t)$ as q tends to 1. The problem (2) involves at first sight only powers of the basic differential operator of Fuchsian type $t\partial_t$. However, the conditions imposed on (2) allows to express it also by means of powers of the basic differential operator of so-called irregular type $t^{k_1+1}\partial_t$. The same fact is acknowledged for the problem (1) under study for which $q-$difference operators of the form $t^{l_0}\sigma_{q;t}^{l_1}$ where $l_0 \geq k_1 l_1$ appear, see (22). These operators are labeled of irregular type in the literature by analogy with the differential case. We quote the classical textbooks [2] and [3] as references for analytic aspects of differential equations wih irregular type and the book [15] for analytic and algebraic features of $q-$difference equations with irregular type. This suggests that in the building process of the solutions to (1), the classical Laplace transform of order k_1 ought to be supplanted by a $q-$Laplace transform of order k_1 similarly to our earlier work [11] where some related initial value $q-$difference differential problem was handled.

We now describe a little more precisely the main statements of this paper achieved in Theorem 1 and Theorem 2. Namely, under fitting restrictions on the shape of (1) listed in Subsection 2.2 and complemented in the statement of Theorem 1 in Subsection 4.3, we can establish the existence of a bounded holomorphic solution $u(t,z)$ to (1) on a domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta}$, for some small radius $R_1 > 0$, where $\mathcal{R}_{d_1,\Delta_1}$ stands for an open sector centered at 0 with large opening that does not contain the halfline $L_{d_1+\pi} = [0, +\infty)e^{\sqrt{-1}(d_1+\pi)}$, see (19), for thoroughly chosen directions $d_1 \in \mathbb{R}$. In addition, the map $u(t,z)$ is modelled through a triple integral which entails a Fourier inverse, a $q-$Laplace and a complete Laplace transforms

$$u(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1, \tau_2, m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (8)$$

where the Borel-Fourier map $\omega_{d_1,\pi}(\tau_1, \tau_2, m)$ is

- analytic on a unbounded sector S_{d_1} centered at 0 containing the halfline L_{d_1} with respect to τ_1 where it has (at most) $q-$exponential growth of order k_1.
- analytic relatively to τ_2 on some open halfstrip

$$H_{\pi} = \{\tau \in \mathbb{C} / \mathrm{Re}(\tau) < 0 \ , \ |\mathrm{Im}(\tau)| < \eta_2\}$$

  with small width $\eta_2 > 0$ and on a small disc D_{ρ}.
- continuous and submitted to exponential decay in phase $m \in \mathbb{R}$.

At this stage, we emphasize that the geometry of the Borel space in the variable (τ_1, τ_2) for the map $\omega_{d_1,\pi}$ differs significantly from the one of the Borel-Fourier map w_p in (3). Indeed, the map $\omega_{d_1,\pi}(\tau_1, \tau_2, m)$ is in general not analytic near $\tau_1 = 0$ while $w_p(\tau_1, \tau_2, m, \epsilon)$ possesses this property. As we will realize later on, this will be the root of the dissemblances observed between the asymptotic properties of the solutions y_p of (2) and the solution u of (1). Besides, the partial map

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$\tau_2 \mapsto w_p(\tau_1, \tau_2, m, \epsilon)$ is only holomorphic on some fixed disc D_a but $\tau_2 \mapsto \omega_{d_1,\pi}(\tau_1, \tau_2, m)$ is analytic on a full halfstrip H_π which allows the solution $u(t,z)$ to be expressed as a complete Laplace transform in $1/\log t$ in direction π while $y_p(t,z,\epsilon)$ is represented as a truncated Laplace transform along the segment $[-a, 0]$. A direct by-product of this observation is that the forcing term $f(t,z)$ of (1) can be presented as an exact polynomial in both time t and inverse complex logarithm $1/\log t$ while the forcing term $h(t,z,\epsilon)$ has to be only considered as proximate to such a polynomial in t and $1/\log t$. Some interesting aftermath is reached when $f(t,z)$ is chosen a mere monomial in t and $1/\log t$ since in that case $f(t,z)$ solves an explicit nonlinear ordinary differential equation with polynomial coefficients in some positive rational power t^α, $\alpha \in \mathbb{Q}_+$, displayed in (34). As a result, $u(t,z)$ turns out to be an exact holomorphic solution to some specific nonlinear $q-$difference differential equation with bounded holomorphic coefficients with respect to z on H_β and polynomial in t^α, stated in (36). Contrastingly, the equation (2) becomes close to some nonlinear partial differential equation as $a \rightarrow +\infty$ but no information can be extracted about the existence of an exact genuine solution to the limit nonlinear problem.

It is worthwhile noting that in the recent years much attention has been drawn on on nonlinear $q-$difference equations and especially on those related to the so-called $q-$Painlevé equations. For a comprehensive overview on major studies for $q-$Painlevé equations and more generally for integrable discrete dynamical systems, we refer to the book [7]. In this trend of research we quote the novel paper [6] where the authors construct convergent generalized power series with complex exponents on sectors that are solutions to nonlinear algebraic $q-$difference equations. In the context of nonlinear $q-$difference differential equations we mention an important result by H. Yamazawa obtained in [17]. Indeed, he considers equations with the shape

$$u(qt, x) = u(t, x) + F(t, x, \{\partial_x^\alpha u\}_{|\alpha| \leq m}) \tag{9}$$

for $t \in \mathbb{C}$, $x \in \mathbb{C}^n$, for some integers $n, m \geq 1$, some real number $q > 1$, where F is a well prepared analytic function in its arguments. Under non resonance conditions of the so-called characteristic exponent $\rho(x)$ associated to (9) at $x = 0$, he has constructed convergent logarithmic type solutions of the form

$$u(t,x) = \sum_{i=1}^{+\infty} u_i(x) t^i + \sum_{\substack{i \geq 1, j \geq 1 \\ 0 \leq k \leq i + 2m(j-1)}} \varphi_{i,j,k}(x) t^{i + j\rho_q(x)} (\log t)^k$$

where the coefficients $u_i(x)$ and $\varphi_{i,j,k}(x)$ are holomorphic on a common disc D_R and where $\rho_q(x) = \log(1 + (q-1)\rho(x))/\log(q)$ stands for a $q-$analog of the characteristic exponent $\rho(x)$.

In the second part of Theorem 1, we exhibit for the solution $u(t,z)$ of (1) a generalized asymptotic expansion of Gevrey type in a logarithmic scale for t in the vicinity of 0. The statement is similar to the one reached in [9] for the solutions $y_p(t,z,\epsilon)$ of (2). Indeed, the partial map $t \mapsto u(t,z)$ is shown to possess a generalized formal series

$$\hat{u}(t,z) = \sum_{n \geq 0} G_n(t,z) \frac{(1/\log t)^n}{n!} \tag{10}$$

with bounded holomorphic coefficients $G_n(t,z)$ on the domain $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times H_\beta$ as asymptotic expansion of Gevrey order 1 with respect to t on $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$, leading to estimates of the form

$$\sup_{z \in H_\beta} |u(t,z) - \sum_{n=0}^{N} G_n(t,z) \frac{(1/\log t)^n}{n!}| \leq K M^{N+1} \Gamma(N+2) |1/\log t|^{N+1} \tag{11}$$

for some constants $K, M > 0$, for all integers $N \geq 0$, whenever $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$. Furthermore, in Section 4.4, Proposition 6, we provide explicit and simple $q-$difference and differential recursion relations displayed in (154) and (155) for the coefficients $G_n(t,z)$, $n \geq 0$, intended for practical

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use. The existence of such a formal expression (10) is shown in a comparable way as (4) for the partial maps $(t,\epsilon) \mapsto y_p(t,z,\epsilon)$ in the problem (2). Namely, it is based on sharp estimates of some exponential decay for the differences of neighboring analytic solutions $\{U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)\}_{0 \leq p \leq \varsigma-1}$, disclosed in (120), to some related $q-$difference differential equation which comprises an homography action, see (116) and (118) in Proposition 4. In the process, we use a classical result known as the Ramis-Sibuya theorem (see Theorem (R.S.) in the subsection 4.2) which ensures the existence of a common Gevrey asymptotic expansion for families of sectorial holomorphic functions.

In the second main result of this paper, stated in Theorem 2, a generalized asymptotic expansion of the solution $u(t,z)$ is established in the scale of monomials $\{t^n\}_{n \geq 0}$. This statement differs notably from the one obtained for the partial maps $\epsilon \mapsto y_p(t,z,\epsilon)$ in the problem (2). Namely, the holomorphic solution $u(t,z)$ to (1) can be split into a sum $u(t,z) = u_1(t,z) + u_2(t,z)$ where

- the map $u_1(t,z)$ owns a formal expression
$$\hat{u}_1(t,z) = \sum_{n \geq 0} b_n(t,z) t^n \quad (12)$$
with bounded holomorphic coefficients $b_n(t,z)$ on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_\beta$ as generalized asymptotic expansion of so-called $q-$Gevrey order k_1. It means that two constants $B_1, B_2 > 0$ can be found with the error bounds
$$\sup_{z \in H_\beta} |u_1(t,z) - \sum_{n=0}^{N} b_n(t,z) t^n| \leq B_1 (B_2)^{N+1} q^{\frac{N(N+1)}{2k_1}} |t|^{N+1} \quad (13)$$
for all integers $N \geq 0$, all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$.
- the map $u_2(t,z)$ has the null formal series as asymptotic expansion of order 1 in a logarithmic scale as t tends to 0. Indeed, two constants $B_3, B_4 > 0$ can be sorted with the estimates
$$\sup_{z \in H_\beta} |u_2(t,z)| \leq B_3 (B_4)^{N+1} \Gamma(N+2) |1/\log t|^{N+1} \quad (14)$$
for all integers $N \geq 0$, provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$.

At this point, we stress the fact that the generalized expansion of Gevrey type (7) obtained for the solutions $y_p(t,z,\epsilon)$ of (2) in the monomial scale $\{\epsilon^n\}_{n \geq 0}$ are obtained by means of the Ramis-Sibuya theorem (see Theorem (R.S.) in Subsection 4.2) through precise estimates of some exponential decay for the differences of the consecutive maps $y_{p+1} - y_p$ relatively to ϵ on the intersections $\mathcal{E}_{p+1} \cap \mathcal{E}_p$. These estimates were achieved according to the fact that the Borel-Fourier maps $\tau_1 \mapsto w_p(\tau_1,\tau_2,m,\epsilon)$ are analytic at $\tau_1 = 0$ in (3). In contrast, for the problem (1) under study, as observed earlier in this introduction, any of the partial Borel-Fourier map $\tau_1 \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ appearing in (8) for any admissible direction $d_1 \in \mathbb{R}$ is not analytic near $\tau_1 = 0$, only on sectors centered at 0. Therefore, no bounds for differences of solutions $u(t,z)$ to (1) for different directions $d_1 \in \mathbb{R}$ can be rooted out and the recipe using the Ramis-Sibuya theorem fails to be applied. Instead we introduce a new approach based on a specific splitting of the triple integral (8) defining $u(t,z)$ and on the observation that the partial map $\tau_1 \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ can be analytically continued near $\tau_1 = 0$ provided that τ_2 remains on the small disc D_ρ, see Proposition 10. Besides, whereas explicit differential recursions could be provided for the coefficients G_n^2, $n \geq 0$ of the formal expansions (6), no such relations are achieved for the coefficients $b_n(t,z)$, $n \geq 0$ of (12). However, explicit formulas (displayed in (207)) for b_n, $n \geq 0$, can be presented as double truncated $q-$Laplace, Laplace transforms and inverse Fourier integral of derivatives of the partial Borel-Fourier map $\tau_1 \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ at the origin.

2. Setup of the main initial value problem and an associated set of $q-$difference-differential problems with homography action

2.1. Accounts on $q-$Laplace transforms of order k and Fourier inverse maps

This concise subsection presents the basic material about $q-$Laplace transforms and Fourier inverse maps that will be handled to built up the solution of our main problem under study.

Let $k \geq 1$ be an integer and set $q > 1$ a positive real number. We present the definition of a $q-$Laplace transform of order k as described in our former work [10]. In the construction of this $q-$analog of the classical Laplace transform of order k, the Jacobi Theta function of order k defined as the Laurent series

$$\Theta_{q^{1/k}}(x) = \sum_{n \in \mathbb{Z}} q^{-\frac{n(n-1)}{2k}} x^n$$

for any $x \in \mathbb{C}^*$ plays a prominent role.

We remind that the set of zeros of this analytic function is given by $\{-q^{m/k}/m \in \mathbb{Z}\}$ and is contained on the real line $\mathbb{R}$. The next lower bounds for the Jacobi Theta function attesting its so-called $q-$exponential growth of order k on a domain bypassing this set of zeros are essential. Let $\Delta > 0$. A constant $C_{q,k} > 0$ relying on q, k and independent of Δ can be chosen such that

$$|\Theta_{q^{1/k}}(x)| \geq C_{q,k} \Delta \exp\left(\frac{k}{2} \frac{\log^2 |x|}{\log(q)}\right) |x|^{1/2} \tag{15}$$

provided that $x \in \mathbb{C}^*$ with $|1 + xq^{m/k}| > \Delta$ for all $m \in \mathbb{Z}$.

Definition 1. *Let D_ρ be a disc of some radius $\rho > 0$ centered at 0 and S_d be an open unbounded sector edged at 0 with bisecting direction $d \in \mathbb{R}$ in $\mathbb{C}$. Let us consider a holomorphic function $f : D_\rho \cup S_d \to \mathbb{C}$ assumed to be continuous up to the closure $\bar{D}_\rho$ and subjected to the bounds*

$$|f(x)| \leq K|x| \exp\left(\frac{k}{2} \frac{\log^2(|x| + \delta)}{\log(q)} + \alpha \log(|x| + \delta)\right) \tag{16}$$

for all $x \in D_\rho \cup S_d$, for some given positive constants $K, \alpha > 0$, $\delta > 1$ and some integer $k \geq 1$. We select some direction $\gamma \in \mathbb{R}$ such that $e^{\sqrt{-1}\gamma} \in S_d$. The $q-$Laplace transform of order k of f in direction γ is assigned as

$$\mathcal{L}^{\gamma}_{q;1/k}(f)(T) := \frac{k}{\log(q)} \int_{L_\gamma} \frac{f(u)}{\Theta_{q^{1/k}}(u/T)} \frac{du}{u} \tag{17}$$

where $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma}$ stands for a halfline in direction γ.

Let $\Delta > 0$ be some fixed real number. The integral transform $\mathcal{L}^{\gamma}_{q;1/k}(f)(T)$ represents a bounded holomorphic function on the domain $\mathcal{R}_{\gamma,\Delta} \cap D_{r_1}$, for any radius r_1 constrained by

$$0 < r_1 \leq q^{-\frac{1}{k}(\alpha+1)}/2 \tag{18}$$

and where

$$\mathcal{R}_{\gamma,\Delta} = \{T \in \mathbb{C}^*/|1 + \frac{e^{\sqrt{-1}\gamma}}{T} r| > \Delta, \text{for all } r \geq 0\}. \tag{19}$$

In the special case $f : \mathbb{C} \to \mathbb{C}$ is an entire function with Taylor expansion $f(x) = \sum_{n \geq 1} f_n x^n$ conforming to the bounds (16), its $q-$Laplace transform of order k, (17) does not depend on the direction $\gamma \in \mathbb{R}$ and defines a bounded holomorphic function on D_{r_1} under the restriction (18) which possesses a Taylor expansion given by the convergent series $\sum_{n \geq 1} f_n q^{\frac{n(n-1)}{2k}} T^n$.

The next Banach space of continuous function on $\mathbb{R}$ with exponential decay was introduced in [4].

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Definition 2. *Let β, μ be real numbers. We denote $E_{(\beta,\mu)}$ the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that*

$$||h(m)||_{(\beta,\mu)} = \sup_{m\in\mathbb{R}} (1 + |m|)^{\mu} \exp(\beta|m|)|h(m)|$$

is finite. The space $E_{(\beta,\mu)}$ endowed with the norm $||.||_{(\beta,\mu)}$ becomes a Banach space.

We recall the definition of the inverse Fourier transform acting on the space $E_{(\beta,\mu)}$.

Definition 3. *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is given by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m) \exp(\sqrt{-1}xm)dm$$

for all $x \in \mathbb{R}$. The function $\mathcal{F}^{-1}(f)$ extends to an analytic bounded function on the strips

$$H_{\beta'} = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta'\}. \tag{20}$$

for all given $0 < \beta' < \beta$.

The next lemma described how the inverse Fourier integral is transformed under the action differential operators and products.

Lemma 1. *a) Let f be an element of $E_{(\beta,\mu)}$ for $\beta > 0$, $\mu > 1$. Define the function $m \mapsto \phi(m) = \sqrt{-1}mf(m)$ which belongs to the space $E_{(\beta,\mu-1)}$. Then, the next identity*

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z)$$

occurs for all $z \in H_{\beta'}$, for any $0 < \beta' < \beta$.
b) Take $g \in E_{(\beta,\mu)}$ and set the convolution product of f and g

$$\psi(m) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m - m_1)g(m_1)dm_1.$$

Then, ψ belongs to $E_{(\beta,\mu)}$ and moreover, the next equality

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z)$$

holds for all $z \in H_{\beta'}$, provided that $0 < \beta' < \beta$.

2.2. Layout of the main problem

Throughout this subsection, we unveil the principal initial value problem under investigation in this work. It is shaped as follows,

$$Q(\partial_z)u(t,z) = t^{d_D}\sigma_{q;t}^{\frac{d_D}{k_1}} R_D(\partial_z)u(t,z) + \sum_{\underline{l}=(l_0,l_1)\in I} c_{\underline{l}}(z)t^{l_0}\sigma_{q;t}^{l_1} R_{\underline{l}}(\partial_z)u(t,z) + f(t,z) \tag{21}$$

for vanishing data $u(0,z) \equiv 0$, where $\sigma_{q;t}$ stands for the $q-$difference operator acting on t by means of $\sigma_{q;t}u(t,z) = u(qt,z)$ for some given real number $q > 1$.

The set I represents a finite subset of $\mathbb{N}^2$ and $d_D, k_1 \geq 1$ are positive integers that are subjected to the next list of technical constraints:

1. The inequality
$$\frac{l_0}{k_1} \geq l_1 \tag{22}$$
holds for all $\underline{l} = (l_0, l_1) \in I$.
2. The restrictions
$$d_D \geq k_1 l_1 \ , \ l_0 \geq d_D \tag{23}$$
are required for all $\underline{l} = (l_0, l_1) \in I$.

The maps $Q(X)$, $R_D(X)$ and $R_{\underline{l}}(X)$ for $\underline{l} \in I$ are polynomial required to fufill the next features:

1. The degrees of Q and of $R_{\underline{l}}$ are constrained by the relation
$$\deg(Q) \geq \deg(R_{\underline{l}}) \tag{24}$$
for all $\underline{l} \in I$.
2. We assume the existence of an open sectorial domain S_{Q,R_D} with inner radius r_{Q,R_D} (resp. outer radius R_{Q,R_D}) given by
$$S_{Q,R_D} = \{Z \in \mathbb{C}^* / |\arg(Z)| < \check{\alpha}_1 \ , \ r_{Q,R_D} < |Z| < R_{Q,R_D}\}$$
for some opening $\check{\alpha}_1 > 0$, which satisfies the next inclusion
$$\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)} \in S_{Q,R_D} \tag{25}$$
for all $m \in \mathbb{R}$. Furthermore, the inner and outer radii together with the aperture of S_{Q,R_D} will be suitably constrained later on in the work.

The coefficients $c_{\underline{l}}(z)$, $\underline{l} \in I$, are built up through the next procedure. For $\underline{l} \in I$, we consider maps $m \mapsto C_{\underline{l}}(m)$ that belong to the Banach space $E_{(\beta,\mu)}$, for given real numbers $\beta > 0$ and $\mu > 1$ constrained to
$$\mu > \deg(R_{\underline{l}}) + 1 \tag{26}$$
for all $\underline{l} \in I$. We introduce the constants
$$\mathbf{C}_{\underline{l}} := ||C_{\underline{l}}(m)||_{(\beta,\mu)} \tag{27}$$
for all $\underline{l} \in I$ on which restrictions will be set in due course of the paper. We define the coefficient $c_{\underline{l}}(z)$ as the inverse Fourier transform
$$c_{\underline{l}}(z) := \mathcal{F}^{-1}(m \mapsto C_{\underline{l}}(m))(z)$$
for all $\underline{l} \in I$, provided that $z \in H_{\beta}$. According to Definition 3, the maps $z \mapsto c_{\underline{l}}(z)$ stand for bounded holomorphic functions on the strips $H_{\beta'}$ for any prescribed $0 < \beta' < \beta$.

The forcing term is described in term of the next construction. Let J_1, J_2 be finite subsets of the positive natural numbers $\mathbb{N}^*$. For all $j_1 \in J_1$, $j_2 \in J_2$, we deem some maps $m \mapsto \mathcal{F}_{j_1,j_2}(m)$ which appertain to $E_{(\beta,\mu)}$ for $\beta > 0$ and $\mu > 1$ given above. We introduce the next polynomial
$$\mathcal{F}(\tau_1, \tau_2, m) = \sum_{j_1 \in J_1, j_2 \in J_2} \mathcal{F}_{j_1,j_2}(m) \tau_1^{j_1} \tau_2^{j_2} \tag{28}$$
in the variables τ_1,τ_2 with coefficients in $E_{(\beta,\mu)}$. We bring in the map

$$F_{\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(\tau_1,\tau_2,m) \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1} \frac{d\tau_2}{\tau_2} dm \quad (29)$$

where $L_{d_1} = [0,+\infty)e^{\sqrt{-1}d_1}$ stands for a halfline in some given direction $d_1 \in \mathbb{R}$ and $L_{\pi} = [0,+\infty)e^{\sqrt{-1}\pi}$ is the negative real axis.

Owing to Definition 1 and Definition 3, this map $F_{\pi}(u_1,u_2,z)$ is well defined provided that

- the variable u_1 belongs to $\mathcal{R}_{d_1,\Delta_1} \cap D_{r_1}$, for any fixed $\Delta_1 > 0$ and radius $r_1 > 0$ subjected to (18) where $k = k_1$, for any given $\alpha > 0$,
- the variable u_2 is not vanishing and obeys the constraint $\cos(\arg(u_2)) < -\Delta_2$, for some $\Delta_2 > 0$,
- the variable z is kept in the strip $H_{\beta'}$ for any $0 < \beta' < \beta$.

However, we can further simplify the expression of F_{π}. Taking heed of Definition 1, we notice that $F_{\pi}(u_1,u_2,z)$ turns out to be a polynomial in u_1,

$$F_{\pi}(u_1,u_2,z) = \sum_{j_1 \in J_1} F_{\pi,j_1}(u_2,z) q^{\frac{j_1(j_1-1)}{2k_1}} u_1^{j_1}$$

whose coefficients F_{π,j_1} are expressed through sums over J_2 of Laplace transforms in direction π,

$$F_{\pi,j_1}(u_2,z) = \sum_{j_2 \in J_2} \mathcal{F}^{-1}(m \mapsto \mathcal{F}_{j_1,j_2}(m))(z) \int_{L_{\pi}} \tau_2^{j_2-1} \exp\left(-\frac{\tau_2}{u_2}\right) d\tau_2$$

with bounded holomorphic coefficients on $H_{\beta'}$. Besides, according to the definition of the Gamma function and Cauchy's theorem, we acknowledge that

$$\int_{L_{\pi}} \tau_2^{j_2-1} \exp\left(-\frac{\tau_2}{u_2}\right) d\tau_2 = \Gamma(j_2) u_2^{j_2}$$

provided that $u_2 \in \mathbb{C}^*$ with $\cos(\arg(u_2)) < 0$. On that account, it follows that $F_{\pi}(u_1,u_2,z)$ can be expanded as a polynomial in both variables u_1 and u_2 with bounded coefficients on $H_{\beta'}$, for $0 < \beta' < \beta$. Namely, we get

$$F_{\pi}(u_1,u_2,z) = \sum_{j_1 \in J_1, j_2 \in J_2} F_{j_1,j_2}(z) u_1^{j_1} u_2^{j_2} \quad (30)$$

where we define

$$F_{j_1,j_2}(z) := \mathcal{F}^{-1}(m \mapsto \mathcal{F}_{j_1,j_2}(m))(z) q^{\frac{j_1(j_1-1)}{2k_1}} \Gamma(j_2) \quad (31)$$

for all $j_1 \in J_1$ and $j_2 \in J_2$. At last, we configure the forcing term $f(t,z)$ as the logarithmic type function

$$f(t,z) = F_{\pi}\left(t, \frac{1}{\log(t)}, z\right). \quad (32)$$

Here $\log(t)$ stands for the principal value of logarithm, namely $\log(t) = \ln(|t|) + \sqrt{-1}\arg(t)$ provided that $-\pi < \arg(t) < \pi$. Furthermore, we observe that

$$\cos\left(\arg\left(\frac{1}{\log(t)}\right)\right) = \cos(\arg(\log(t))) < -\Delta_2 \quad (33)$$

for some $\Delta_2 > 0$, whenever $t \notin (-\infty, 0]$ and close enough to 0.

In the particular case $J_1 = \{j_1\}$ and $J_2 = \{j_2\}$ for some positive integers $j_1, j_2 \geq 1$, we make the noteworthy remark that the solution $u(t,z)$ of the linear main equation (21) actually solves a special *nonlinear $q-$difference-differential equation* with polynomial coefficients in some positive rational power of time t stated in (36). Indeed, let the forcing term $f(t,z)$ have the particular shape

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$$f_{j_1,j_2}(t,z) = F_{j_1,j_2}(z)\frac{t^{j_1}}{\log^{j_2}(t)}$$

where $F_{j_1,j_2}(z)$ is given by the expression (31). By direct computation, we check that the forcing term $f_{j_1,j_2}(t,z)$ satisfies the next *nonlinear ordinary differential equation* with polynomial coefficients in t^{1/j_2}

$$(F_{j_1,j_2}(z))^{1/j_2}t^{1+\frac{j_1}{j_2}}\partial_t f_{j_1,j_2}(t,z) = j_1 t^{j_1/j_2}(F_{j_1,j_2}(z))^{1/j_2} f_{j_1,j_2}(t,z) - j_2(f_{j_1,j_2}(t,z))^{1+\frac{1}{j_2}} \quad (34)$$

Let us recast the main equation (21) in the form

$$P(t,z,\partial_z,\sigma_{q;t})u(t,z) = f_{j_1,j_2}(t,z) \quad (35)$$

where the $q-$difference-differential operator P is polynomial in t, with bounded holomorphic coefficients in z on the strip $H_{\beta'}$, for $0 < \beta' < \beta$. The combination of (34) and (35) gives rise to the next nonlinear equation

$$(F_{j_1,j_2}(z))^{1/j_2}t^{1+\frac{j_1}{j_2}}\partial_t\big(P(t,z,\partial_z,\sigma_{q;t})u(t,z)\big)$$
$$= j_1 t^{j_1/j_2}(F_{j_1,j_2}(z))^{1/j_2}P(t,z,\partial_z,\sigma_{q;t})u(t,z) - j_2\big(P(t,z,\partial_z,\sigma_{q;t})u(t,z)\big)^{1+\frac{1}{j_2}}. \quad (36)$$

2.3. *A set of related $q-$difference-differential equations with an homography action*

In this subsection, the main problem is embedded in a set of auxiliary problems which comprise three independent complex variables which will be the object of study in the forthcoming sections.

We seek for solutions $u(t,z)$ to (21) for prescribed vanishing initial data at $t=0$ of the form

$$u(t,z) = U_\pi(t,\frac{1}{\log(t)},z)$$

for some expression $U_\pi(u_1,u_2,z)$ in the three independent variables u_1,u_2 and z.

The next computations hold for any given rational number $h>0$,

$$\sigma_{q;t}^h u(t,z) = U_\pi(q^h t,\frac{1}{\log(q^h t)},z) = U_\pi(q^h t,\frac{1}{h\log(q)+\log(t)},z)$$
$$= (\sigma_{q;u_1}^h \circ \mathbb{H}_{h\log(q);u_2}U_\pi)(t,\frac{1}{\log(t)},z)$$

where

- the dilation $\sigma_{q;u_1}^h$ acts on U_π relatively to u_1 through $(\sigma_{q;u_1}U_\pi)(u_1,u_2,z) = U_\pi(q^h u_1,u_2,z)$,
- the homography $\mathbb{H}_{h\log(q);u_2}$ is applied on U_π with respect to the variable u_2 by means of

$$(\mathbb{H}_{h\log(q);u_2}U_\pi)(u_1,u_2,z) = U_\pi(u_1,\frac{u_2}{1+u_2 h\log(q)},z)$$

As a result, it follows that the expression $u(t,z)$ (formally) solves the equation (21) under the condition $u(0,z)\equiv 0$ **if** the expression $U_\pi(u_1,u_2,z)$ fulfills the next equation

$$Q(\partial_z)U_\pi(u_1,u_2,z) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}} \circ \mathbb{H}_{\frac{d_D}{k_1}\log(q);u_2}R_D(\partial_z)U_\pi(u_1,u_2,z)+$$
$$\sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0}\sigma_{q;u_1}^{l_1}\circ\mathbb{H}_{l_1\log(q);u_2}c_{\underline{l}}(z)R_{\underline{l}}(\partial_z)U_\pi(u_1,u_2,z) + F_\pi(u_1,u_2,z) \quad (37)$$

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under the constraint $U_{\pi}(0,0,z) \equiv 0$. Later on, we plan to build a genuine solution to (21) and in order to investigate its asymptotic expansion in some particular scale described in Subsection 4.3, we are required to complement the above single equation (37) by a family of auxiliary problems stated underneath.

For any given direction $d_2 \in \mathbb{R}$ with $d_2 \neq \pi$ (modulo 2π) and given positive real radius $a > 0$, we define a new forcing term

$$F_{d_2,a}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{d_2,a}} \int_{-\infty}^{+\infty} \mathcal{F}(\tau_1,\tau_2,m) \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (38)$$

where $L_{d_2,a} = [0,a]e^{\sqrt{-1}d_2}$ stands for a segment of length $a > 0$ in direction d_2 and L_{d_1} is the halfline appearing in the formula (29). Owing to Definition 1, we notice that the map $F_{d_2,a}$ does not rely on the direction d_1. However, it hinges on the direction d_2 and radius $a > 0$. We display the next problem

$$Q(\partial_z)U_{d_2}(u_1,u_2,z) = u_1^{d_D} \sigma_{q;u_1}^{\frac{d_D}{k_1}} \circ \mathbb{H}_{\frac{d_D}{k_1}\log(q);u_2} R_D(\partial_z) U_{d_2}(u_1,u_2,z) + \sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0} \sigma_{q;u_1}^{l_1} \circ \mathbb{H}_{l_1 \log(q);u_2} c_{\underline{l}}(z) R_{\underline{l}}(\partial_z) U_{d_2}(u_1,u_2,z) + F_{d_2,a}(u_1,u_2,z) \quad (39)$$

for given vanishing initial data $U_{d_2}(0,0,z) \equiv 0$.

3. Analytic solutions to the associated set of $q-$difference and differential problems under homography action

In this section, we intend to exhibit analytic solutions to the problems (37) and (39) we came up with in Subsection 2.3.

3.1. Profile of the analytic solutions and joint convolution $q-$difference equations

We search for a solution to (37) (resp. (39) for $d_2 \neq \pi$ modulo 2π) in the form of a double $q-$Laplace, Laplace transform and inverse Fourier integral with the shape

$$U_{d_1,\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (40)$$

(resp.

$$U_{d_1,d_2}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{d_2,a}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (41)$$

Here it is assumed (this fact will be justified later on in the work) that the so-called Borel-Fourier map $\omega_{d_1,\pi}$ appertains to a Banach space labelled $\mathrm{Exp}^{q;1}_{(k,\alpha,\delta,\nu,\beta,\mu,\rho)}$ which consists in functions with so-called $q-$exponential growth of order k_1 w.r.t τ_1, exponential growth in τ_2 and exponential decay relatively to the mode m. This space is described in the next

Definition 4. *We consider the constants β, μ, k_1, a as prescribed in Section 2. Let $\alpha, \nu > 0$ and $\rho > a$, $\delta > 1$ be real numbers. We set S_{d_1} as an unbounded sector edged at 0 with bisecting direction $d_1 \in \mathbb{R}$. We introduce the open half strip*

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$$H_{\pi} = \{ \tau \in \mathbb{C} / \mathrm{Re}(\tau) < 0 \ , \ |\mathrm{Im}(\tau)| < \eta_2 \} \quad (42)$$

for some given real width $\eta_2 > 0$. *We denote* $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ *the vector space of all* $\mathbb{C}-$*valued continuous maps* $(\tau_1,\tau_2,m) \mapsto h(\tau_1,\tau_2,m)$ *on the domain* $S_{d_1} \times (H_{\pi} \cup D_{\rho}) \times \mathbb{R}$, *holomorphic w.r.t* (τ_1,τ_2) *on the product* $S_{d_1} \times (H_{\pi} \cup D_{\rho})$, *such that the norm*

$$||h(\tau_1,\tau_2,m)||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} := \sup_{\substack{\tau_1 \in S_{d_1} \\ \tau_2 \in H_{\pi} \cup D_{\rho}, m \in \mathbb{R}}} (1+|m|)^{\mu} e^{\beta|m|} \frac{1}{|\tau_1|} \exp\Big(-\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} - \alpha \log(|\tau_1|+\delta)\Big) \times \frac{1}{|\tau_2|} e^{-\nu|\tau_2|} |h(\tau_1,\tau_2,m)| \quad (43)$$

is finite. The vector space $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ *equipped with the norm* $||.||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ *represents a Banach space.*

Our main objective is to establish some convolution $q-$difference equation that the Borel-Fourier map $\omega_{d_1,\pi}$ is asked to obey. On the way, we need some additional features on the $q-$Laplace transforms under multiplication by a monomial and action of $q-$difference operators. These properties have already been discussed in our past work [10]. Besides, we describe the action of the homography $\mathbb{H}_{s;u_2}$ relatively to the variable u_2 on both expressions (40) and (41).

Lemma 2. *Let the map* $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ *supposed to belong to the Banach space* $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$. *Then, the next identities hold.*

1. For prescribed integers $l_0, l_1 \geq 0$, *the* $q-$*difference operator* $u_1^{l_0}\sigma_{q;u_1}^{l_1}$ *acts on the integral representations* (40) *and* (41) *through the formulas*

$$u_1^{l_0}\sigma_{q;u_1}^{l_1} U_{d_1,\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \Big[\frac{\tau_1^{l_0}}{(q^{1/k_1})^{l_0(l_0-1)/2}} \sigma_{q;\tau_1}^{l_1 - \frac{l_0}{k_1}} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \Big] \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2} \big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (44)$$

and

$$u_1^{l_0}\sigma_{q;u_1}^{l_1} U_{d_1,d_2}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{d_2,a}} \int_{-\infty}^{+\infty} \Big[\frac{\tau_1^{l_0}}{(q^{1/k_1})^{l_0(l_0-1)/2}} \sigma_{q;\tau_1}^{l_1 - \frac{l_0}{k_1}} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \Big] \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2} \big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm. \quad (45)$$

2. For a given rational number $h > 0$, *the homography* $\mathbb{H}_{h\log(q);u_2}$ *applies on the triple integrals* (40) *and* (41) *by means of*

$$\mathbb{H}_{h\log(q);u_2} U_{d_1,\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} [\omega_{d_1,\pi}(\tau_1,\tau_2,m)\exp(-\tau_2 h \log(q))] \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2} \big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (46)$$

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along with

$$\mathbb{H}_{h\log(q);u_2}U_{d_1,d_2}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}}\int_{L_{d_1}}\int_{L_{d_2,a}}\int_{-\infty}^{+\infty}[\omega_{d_1,\pi}(\tau_1,\tau_2,m)\exp(-\tau_2 h\log(q))] \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)}\exp\big(-\frac{\tau_2}{u_2}\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm \quad (47)$$

In line with the above technical lemma together with Lemma 1, the next statement follows.

Lemma 3. *The map $U_{d_1,\pi}(u_1,u_2,z)$ solves (37) under the constraint $U_{d_1,\pi}(0,0,z)\equiv 0$ and $U_{d_1,d_2}(u_1,u_2,z)$ obeys (39) for $d_2\neq\pi$ (modulo 2π) with vanishing data $U_{d_1,d_2}(0,0,z)\equiv 0$* **if** *the Borel-Fourier map $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ fulfills the next convolution $q-$difference equation*

$$Q(\sqrt{-1}m)\omega_{d_1,\pi}(\tau_1,\tau_2,m) = R_D(\sqrt{-1}m)\frac{\tau_1^{d_D}}{(q^{1/k_1})^{d_D(d_D-1)/2}}\exp\big(-\tau_2\frac{d_D}{k_1}\log(q)\big)\omega_{d_1,\pi}(\tau_1,\tau_2,m) + \sum_{\underline{l}=(l_0,l_1)\in I}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\frac{\tau_1^{l_0}}{(q^{1/k_1})^{l_0(l_0-1)/2}}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega_{d_1,\pi}(\tau_1,\tau_2,m_1) \times\exp(-\tau_2 l_1\log(q))dm_1 + \mathcal{F}(\tau_1,\tau_2,m) \quad (48)$$

provided that $\tau_1\in S_{d_1}$, $\tau_2\in H_\pi\cup D_\rho$ and $m\in\mathbb{R}$.

3.2. Solving the convolution $q-$difference equation (48) on unbounded sectors and half strips

In the course of this subsection, we prove the existence and unicity of a solution to the convolution $q-$difference reached in Lemma 3.

Our scheme consists in reorganizing the equation (48) as a fixed point equation (displayed later on in (98)). On the way, we are asked to divide our equation by the next Fourier mode depending map with two complex variables

$$P_m(\tau_1,\tau_2) = Q(\sqrt{-1}m) - R_D(\sqrt{-1}m)\frac{\tau_1^{d_D}}{(q^{1/k_1})^{d_D(d_D-1)/2}}\exp\big(-\tau_2\frac{d_D}{k_1}\log(q)\big) \quad (49)$$

provided that $\tau_1\in S_{d_1}$ and $\tau_2\in H_\pi\cup D_\rho$. An essential factorisation of the above map is provided in the next lemma.

Lemma 4. *For a convenient choice of the inner radius r_{Q,R_D}, outer radius R_{Q,R_D} and aperture $\check{\alpha}_1>0$ of S_{Q,R_D} set up in (25), one can distinguish an unbounded sector S_{d_1} edged at 0 with suitable bisecting direction $d_1\in\mathbb{R}$ along with an appropriate strip H_π and a small radius ρ for which the next splitting of the map $P_m(\tau_1,\tau_2)$ holds. Let $\tau_1\in S_{d_1}$ written in the factorized form*

$$\tau_1 = r\tau_1^0 \quad (50)$$

for some radius $r>0$ and complex number $\tau_1^0\in S_{d_1}$ with $|\tau_1^0|=1$. Let us take $\tau_2\in H_\pi\cup D_\rho$. Then, one can decompose τ_2 in the form

$$\tau_2 = \tau_2^0 - s + \sqrt{-1}\psi \quad (51)$$

for some well chosen complex number τ_2^0 (depending on τ_1^0 and m and which remains bounded relatively to m), for some $\psi\neq 0$, close to 0 and some $s\geq -A$ (for some fixed constant $A>0$). With the above factorisations (50), (51), one can express the map P_m in the form of a non vanishing product

$$P_m(\tau_1,\tau_2) = Q(\sqrt{-1}m)\Big[1 - r^{d_D}\exp\left((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right)\Big] \tag{52}$$

for given $\tau_1 \in S_{d_1}$ *and* $\tau_2 \in H_\pi \cup D_\rho$.

Proof. We choose appropriately the sectorial domain S_{Q,R_D} given in Subsection 2.2 and select an unbounded sector S_{d_1} edged at 0 with bisecting direction d_1 chosen in a way that the next constraint

$$0 < \alpha_1 < \left|\arg\Big(\frac{R_D(\sqrt{-1}m)}{Q(\sqrt{-1}m)}\tau_1^{d_D}\frac{1}{(q^{1/k_1})^{d_D(d_D-1)/2}}\Big)\right| < \alpha_2 \tag{53}$$

holds for all $m \in \mathbb{R}$, all $\tau_1 \in S_{d_1}$ for some small positive numbers $0 < \alpha_1 < \alpha_2$. Let $\tau_1 \in S_{d_1}$ be given. We can factorize it in the form (50). We set

$$\tau_2^0 = \frac{1}{\frac{d_D}{k_1}\log(q)}\Big[\log\left|\frac{R_D(\sqrt{-1}m)}{Q(\sqrt{-1}m)}(\tau_1^0)^{d_D}\frac{1}{(q^{1/k_1})^{d_D(d_D-1)/2}}\right| + \sqrt{-1}\Big(\arg\Big(\frac{R_D(\sqrt{-1}m)}{Q(\sqrt{-1}m)}(\tau_1^0)^{d_D}\frac{1}{(q^{1/k_1})^{d_D(d_D-1)/2}}\Big)\Big)\Big] \tag{54}$$

Notice that τ_2^0 remains bounded and penned in a small domain we denote $\mathcal{T}_2^0$ which is located at some small positive distance of the real axis, when m spans the real numbers according to the condition (25) imposed.

In the next step, we select the strip H_π and the disc D_ρ in a way that

$$(H_\pi \cup D_\rho) \cap \mathcal{T}_2^0 = \emptyset \tag{55}$$

As a result, when one takes some element $\tau_2 \in H_\pi \cup D_\rho$, we can write it in the form (51) for some real number $s > -A$ for some $A > 0$ and some real number $\psi \neq 0$ that can be chosen close to 0.

By construction of τ_2^0, we get in particular that

$$\exp(\tau_2^0\frac{d_D}{k_1}\log(q)) = \frac{R_D(\sqrt{-1}m)}{Q(\sqrt{-1}m)}(\tau_1^0)^{d_D}\frac{1}{(q^{1/k_1})^{d_D(d_D-1)/2}} \tag{56}$$

In consequence of the combined factorisations (50) and (51) together with the above identity (56), the next computations hold

$$P_m(\tau_1,\tau_2) = Q(\sqrt{-1}m) - R_D(\sqrt{-1}m)\frac{r^{d_D}(\tau_1^0)^{d_D}}{(q^{1/k_1})^{d_D(d_D-1)/2}}\exp(-\tau_2^0\frac{d_D}{k_1}\log(q)) \times \exp\left((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right) = Q(\sqrt{-1}m)\Big[1 - r^{d_D}\exp\left((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right)\Big] \tag{57}$$

which is exactly the announced expression (52). In particular, this product is non vanishing since $Q(\sqrt{-1}m) \neq 0$, for all $m \in \mathbb{R}$ owing to (25) and considering that $\psi \neq 0$ but close to the origin, the piece enclosed by brackets in (52) cannot vanish. □

Let us consider the next linear map

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$$\mathcal{H}(\omega(\tau_1,\tau_2,m)) := \sum_{\underline{l}=(l_0,l_1)\in I} \frac{1}{P_m(\tau_1,\tau_2)} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1) \times \frac{\tau_1^{l_0}}{(q^{1/k_1})^{l_0(l_0-1)/2}} \sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}} \omega(\tau_1,\tau_2,m_1) \times \exp\left(-\tau_2 l_1 \log(q)\right) dm_1 + \frac{\mathcal{F}(\tau_1,\tau_2,m)}{P_m(\tau_1,\tau_2)} \quad (58)$$

In the next proposition it is stated that the map $\mathcal{H}$ stands for a shrinking map on some fittingly chosen ball of the Banach space discussed in Definition 4.

Proposition 1. *We select the sectorial domain S_{Q,R_D}, the unbounded sector S_{d_1} together with the strip H_π and the disc D_ρ as in Lemma 4. Then, provided that the constants $\mathbf{C}_{\underline{l}} > 0$ displayed in (27) are small enough, for $\underline{l} \in I$, an adequate radius $\varpi > 0$ can be chosen for which the map $\mathcal{H}$ enjoys the next two properties*

- *The inclusion*
$$\mathcal{H}(\bar{B}_\varpi) \subset \bar{B}_\varpi \quad (59)$$
is granted, where $\bar{B}_\varpi$ denotes the closed ball of radius ϖ centered at 0 in the space $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$.
- *The $1/2-$Lipschitz condition*
$$||\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \leq \frac{1}{2}||\omega_1-\omega_2||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \quad (60)$$
holds for all $\omega_1,\omega_2 \in \bar{B}_\varpi$.

Proof. We first aim our attention to the inclusion (59). Let us prescribe some real number $\varpi > 0$ and take some element $\omega(\tau_1,\tau_2,m)$ of $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ subjected to the condition

$$||\omega(\tau_1,\tau_2,m)||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \leq \varpi.$$

We plan to disclose norm estimates for each piece of the map $\mathcal{H}$. We first focus on norm upper bounds for the elements involved in the sum over I. The next technical lemma is crucial in this respect.

Lemma 5. *Under the imposed constraints (22), (23) together with (24), (25) and (26), one can find a constant $C_1 > 0$ such that*

$$||\frac{1}{P_m(\tau_1,\tau_2)} \int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1) \times \exp\left(-\tau_2 l_1\log(q)\right)dm_1||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \leq C_1\mathbf{C}_{\underline{l}}||\omega(\tau_1,\tau_2,m)||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \quad (61)$$

for all $\omega(\tau_1,\tau_2,m) \in \mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$.

Proof. According to Definition 4, we next upper bounds hold for the element ω,

$$|\omega(\tau_1,\tau_2,m_1)| \leq ||\omega||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \times |\tau_1|\exp\left(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\right)|\tau_2|e^{\nu|\tau_2|} \quad (62)$$

for all $\tau_1 \in S_{d_1}$, all $\tau_2 \in H_\pi \cup D_\rho$, $m_1 \in \mathbb{R}$. We deduce first upper bounds

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$$
\begin{aligned}
&\Big|\frac{1}{P_m(\tau_1,\tau_2)}\int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1)\\
&\times \exp\big(-\tau_2 l_1\log(q)\big)dm_1\Big| \leq \frac{1}{|P_m(\tau_1,\tau_2)|}\int_{-\infty}^{+\infty}|C_{\underline{l}}(m-m_1)||R_{\underline{l}}(\sqrt{-1}m_1)||\tau_1|^{l_0}||\omega||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}\\
&\quad\times(1+|m_1|)^{-\mu}e^{-\beta|m_1|}|q^{l_1-\frac{l_0}{k_1}}\tau_1|\exp\Big(\frac{k_1}{2}\frac{\log^2(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)}{\log(q)}+\alpha\log(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)\Big)\\
&\quad\times|\tau_2|e^{\nu|\tau_2|}|\exp\big(-\tau_2 l_1\log(q)\big)|dm_1 \qquad (63)
\end{aligned}
$$

whenever $\tau_1\in S_{d_1}$, $\tau_2\in H_\pi\cup D_\rho$ and $m\in\mathbb{R}$. The resulting bounds (61) will be reached after several steps of computations. Namely,

1) We provide upper bounds for the function

$$\mathcal{A}_1(m) := \frac{1}{|Q(\sqrt{-1}m)|}\int_{-\infty}^{+\infty}|C_{\underline{l}}(m-m_1)||R_{\underline{l}}(\sqrt{-1}m_1)|(1+|m_1|)^{-\mu}e^{-\beta|m_1|}dm_1 \qquad (64)$$

for $m\in\mathbb{R}$. Since $R_{\underline{l}}(X)$ are polynomials, we get a constant $\mathbf{R}_{\underline{l}}>0$ with

$$|R_{\underline{l}}(\sqrt{-1}m_1)| \leq \mathbf{R}_{\underline{l}}(1+|m_1|)^{\deg(R_{\underline{l}})} \qquad (65)$$

for all $m_1\in\mathbb{R}$. Besides, owing to the assumption (25), a constant $\mathbf{Q}>0$ can be pinpointed with the lower bounds

$$|Q(\sqrt{-1}m)| \geq \mathbf{Q}(1+|m|)^{\deg(Q)} \qquad (66)$$

for all $m\in\mathbb{R}$ and from the definition of the constants $\mathbf{C}_{\underline{l}}>0$, we know that

$$|C_{\underline{l}}(m-m_1)| \leq \mathbf{C}_{\underline{l}}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|} \qquad (67)$$

for all $m,m_1\in\mathbb{R}$. The collection of bounds (65), (66) and (67) together with the triangular inequality $|m|\leq|m-m_1|+|m_1|$ enable the next estimates

$$
\begin{aligned}
\mathcal{A}_1(m) &\leq \frac{\mathbf{C}_{\underline{l}}\mathbf{R}_{\underline{l}}}{\mathbf{Q}(1+|m|)^{\deg(\mathrm{Q})}}\int_{-\infty}^{+\infty}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|}(1+|m_1|)^{\deg(\mathrm{R}_{\underline{l}})}\\
&\qquad\times(1+|m_1|)^{-\mu}e^{-\beta|m_1|}dm_1\\
&\leq \frac{\mathbf{C}_{\underline{l}}\mathbf{R}_{\underline{l}}}{\mathbf{Q}}\Big\{(1+|m|)^{\mu-\deg(Q)}\int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu-\deg(R_{\underline{l}})}}dm_1\Big\}\times(1+|m|)^{-\mu}e^{-\beta|m|} \qquad (68)
\end{aligned}
$$

At last, according to Lemma 2.2 from [4] or Lemma 4 from [12], we call to mind that the quantity

$$\sup_{m\in\mathbb{R}}(1+|m|)^{\mu-\deg(Q)}\int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu-\deg(R_{\underline{l}})}}dm_1$$

is finite under the assumption (24) and (26). Therefore, a constant $C_{1.1}>0$ can be singled out with

$$\mathcal{A}_1(m) \leq \frac{\mathbf{C}_{\underline{l}}\mathbf{R}_{\underline{l}}}{\mathbf{Q}}C_{1.1}(1+|m|)^{-\mu}e^{-\beta|m|} \qquad (69)$$

for all $m\in\mathbb{R}$.

2) We focus on upper estimates for the quantity

$$\mathcal{A}_2(\tau_1) := \exp\Big(\frac{k_1}{2}\frac{\log^2(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)}{\log(q)}+\alpha\log(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)\Big) \qquad (70)$$

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provided that $\tau_1 \in S_{d_1}$ with $|\tau_1| > r_1$ for some fixed real number $r_1 > 1$. We need to perform the next expansions

$$\log^2(q^{l_1-\frac{l_0}{k_1}}|\tau_1|+\delta) = \Big(\log(q^{l_1-\frac{l_0}{k_1}}|\tau_1|) + \log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|})\Big)^2$$
$$= \log^2(|\tau_1|) + 2\log(|\tau_1|)(l_1-\frac{l_0}{k_1})\log(q) + \log^2(q^{l_1-\frac{l_0}{k_1}}) + 2\log(q^{l_1-\frac{l_0}{k_1}})\log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|})$$
$$+ 2\log(|\tau_1|)\log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|}) + \log^2(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|}) \quad (71)$$

together with

$$\log(q^{l_1-\frac{l_0}{k_1}}|\tau_1|+\delta) = \log(q^{l_1-\frac{l_0}{k_1}}) + \log(|\tau_1|) + \log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|}) \quad (72)$$

Owing to the freshman classical limit $\lim_{x\to+\infty}\log(x)/x = 0$ and equivalence relation $\log(1+x) \sim x$ as x tends to 0, we reach a two constants $A_1, A_2 > 0$ with

$$\log(|\tau_1|)\log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|}) \le A_1 \ , \ 0 < \log(1+q^{\frac{l_0}{k_1}-l_1}\frac{\delta}{|\tau_1|}) \le A_2 \quad (73)$$

provided that $|\tau_1| > r_1 > 1$. Furthermore, since $x \mapsto \log^2(x)$ and $x \mapsto \log(x)$ are both increasing maps on $[1,+\infty)$, we observe the inequalities

$$\log^2(|\tau_1|) \le \log^2(|\tau_1|+\delta) \ , \ \log(|\tau_1|) \le \log(|\tau_1|+\delta) \quad (74)$$

whenever $|\tau_1| > r_1 > 1$. From the two expansions (71), (72) and the bounds (73), (74) together with the assumption (22), we arrive at the next estimates

$$\mathcal{A}_2(\tau_1) \le \exp\Big(\frac{k_1}{2\log(q)}\big[\log^2(|\tau_1|+\delta) + 2\log(|\tau_1|)(l_1-\frac{l_0}{k_1})\log(q)$$
$$+\log^2(q^{l_1-\frac{l_0}{k_1}}) + 2A_1 + A_2^2\big] + \alpha\big[\log(q^{l_1-\frac{l_0}{k_1}}) + \log(|\tau_1|+\delta) + A_2\big]\Big)$$
$$\le C_{1.2}|\tau_1|^{k_1(l_1-\frac{l_0}{k_1})}\exp\Big(\frac{k_1}{2}\frac{1}{\log(q)}\log^2(|\tau_1|+\delta) + \alpha\log(|\tau_1|+\delta)\Big) \quad (75)$$

provided that $\tau_1 \in S_{d_1}$ with $|\tau_1| > r_1$, for some constant $C_{1.2} > 0$.

3) We supply upper bounds for the quantity

$$\mathcal{A}_3(\tau_1,\tau_2) = \frac{|\tau_1|^{l_0}|\tau_1|^{k_1(l_1-\frac{l_0}{k_1})}|\exp\big(-\tau_2 l_1\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|} \quad (76)$$

for $\tau_1 = r\tau_1^0 \in S_{d_1}$ with $|\tau_1| > r_1$ and $\tau_2 = \tau_2^0 - s + \sqrt{-1}\psi \in H_\pi \cup D_\rho$ where $r > 0$, $|\tau_1^0| = 1$, $\psi \neq 0$ close to 0 and $s > -A$ for some constant $A > 0$, according to the decompositions (50) and (51). We recast $\mathcal{A}_3$ in the form

$$\mathcal{A}_3(\tau_1,\tau_2) = r^{l_0+k_1(l_1-\frac{l_0}{k_1})-d_D}|\exp\big((s-\sqrt{-1}\psi)(l_1-\frac{d_D}{k_1})\log(q)\big)| \times |\exp\big(-\tau_2^0 l_1\log(q)\big)|$$
$$\times \frac{r^{d_D}|\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|} \quad (77)$$

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Taking heed of our assumption (23), we obtain a constant $C_{1.3} > 0$ with

$$r^{l_0+k_1(l_1-\frac{l_0}{k_1})-d_D}|\exp\big((s-\sqrt{-1}\psi)(l_1-\frac{d_D}{k_1})\log(q)\big)| \leq C_{1.3} \tag{78}$$

for all $r > r_1$ and $s > -A$, for $\psi \neq 0$ close to 0. Furthermore, a constant $C_{1.4} > 0$ can be singled out with

$$\frac{r^{d_D}|\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|} \leq C_{1.4} \tag{79}$$

as long as $r > r_1$ and $s > -A$, for $\psi \neq 0$ close to 0. From (78) and (79), we deduce a constant $C_{1.5} > 0$ such that

$$\mathcal{A}_3(\tau_1,\tau_2) \leq C_{1.5} \tag{80}$$

whenever $\tau_1 \in S_{d_1}$ with $|\tau_1| > r_1$ and all $\tau_2 \in H_\pi \cup D_\rho$.

4) We establish bounds for the quantity $\mathcal{A}_2(\tau_1)$ displayed in (70) provided that $\tau_1 \in S_{d_1}$ with $|\tau_1| \leq r_1$, where $r_1 > 1$ has been fixed in 2). A mere observation yields a constant $C_{1.6} > 0$ with

$$\begin{aligned}
&\mathcal{A}_2(\tau_1)\\
&\leq \Big\{\exp\Big(\frac{k_1}{2}\frac{\log^2(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)-\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\{\log(|q^{l_1-\frac{l_0}{k_1}}\tau_1|+\delta)-\log(|\tau_1|+\delta)\}\Big)\Big\}\\
&\qquad\times\exp\Big(\frac{k_1}{2\log(q)}\log^2(|\tau_1|+\delta)+\alpha\log(|\tau_1|+\delta)\Big)\\
&\qquad\leq C_{1.6}\exp\Big(\frac{k_1}{2\log(q)}\log^2(|\tau_1|+\delta)+\alpha\log(|\tau_1|+\delta)\Big)
\end{aligned} \tag{81}$$

for all $\tau_1 \in S_{d_1}$ with $|\tau_1| \leq r_1$.

5) We present bounds for the piece

$$\mathcal{A}_4(\tau_1,\tau_2) = \frac{|\tau_1|^{l_0}|\exp\big(-\tau_2 l_1\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|} \tag{82}$$

for $\tau_1 = r\tau_1^0 \in S_{d_1}$ with $|\tau_1| \leq r_1$ and $\tau_2 = \tau_2^0 - s + \sqrt{-1}\psi \in H_\pi \cup D_\rho$ where $r > 0$, $|\tau_1^0| = 1$, $\psi \neq 0$ close to 0 and $s > -A$ for some constant $A > 0$, according to the decompositions (50) and (51). We rearrange $\mathcal{A}_4$ as follows

$$\begin{aligned}
\mathcal{A}_4(\tau_1,\tau_2) &= r^{l_0-d_D}|\exp\big((s-\sqrt{-1}\psi)(l_1-\frac{d_D}{k_1})\log(q)\big)| \times |\exp\big(-\tau_2^0 l_1\log(q)\big)|\\
&\qquad\times\frac{r^{d_D}|\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|}
\end{aligned} \tag{83}$$

Bearing in mind the condition (23), we get a constant $C_{1.7} > 0$ with

$$r^{l_0-d_D}|\exp\big((s-\sqrt{-1}\psi)(l_1-\frac{d_D}{k_1})\log(q)\big)| \leq C_{1.7} \tag{84}$$

provided that $0 < r \leq r_1$ and $s > -A$, for $\psi \neq 0$ close to 0. On the other hand, a constant $C_{1.8} > 0$ can be set with

$$\frac{r^{d_D}|\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|}{|1-r^{d_D}\exp\big((s-\sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\big)|} \leq C_{1.8} \tag{85}$$

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as long as $0 < r \leq r_1$ and $s > -A$, for $\psi \neq 0$ close to 0. Due to (84) and (85), a constant $C_{1.9} > 0$ can be picked out such that

$$\mathcal{A}_4(\tau_1, \tau_2) \leq C_{1.9} \tag{86}$$

for all $\tau_1 \in S_{d_1}$ with $|\tau_1| \leq r_1$ and all $\tau_2 \in H_\pi \cup D_\rho$.

6) As a consequence of the list of estimates (75), (80), (81) and (86), we obtain a constant $C_{1.10} > 0$ with

$$\frac{1}{|1 - r^{d_D} \exp\left((s - \sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right)|}|\tau_1|^{l_0} \times \exp\left(\frac{k_1}{2}\frac{\log^2(|q^{l_1 - \frac{l_0}{k_1}}\tau_1| + \delta)}{\log(q)} + \alpha\log(|q^{l_1-\frac{l_0}{k_1}}\tau_1| + \delta)\right) \times |\exp\left(-\tau_2 l_1 \log(q)\right)| \leq C_{1.10}\exp\left(\frac{k_1}{2\log(q)}\log^2(|\tau_1| + \delta) + \alpha\log(|\tau_1| + \delta)\right) \tag{87}$$

for $\tau_1 = r\tau_1^0 \in S_{d_1}$ and $\tau_2 = \tau_2^0 - s + \sqrt{-1}\psi \in H_\pi \cup D_\rho$ where $r > 0$, $|\tau_1^0| = 1$, $\psi \neq 0$ close to 0 and $s > -A$ for some constant $A > 0$, according to the decompositions (50) and (51).

In conclusion, on the basis of the factorization (52) for the map $P_m(\tau_1, \tau_2)$ together with the bounds (69) and (87) combined with the bounds (63), we arrive at the next inequality

$$\left|\frac{1}{P_m(\tau_1, \tau_2)}\int_{-\infty}^{+\infty} C_{\underline{l}}(m - m_1)R_{\underline{l}}(\sqrt{-1}m_1)\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1 - \frac{l_0}{k_1}}\omega(\tau_1, \tau_2, m_1) \times \exp\left(-\tau_2 l_1 \log(q)\right)dm_1\right| \leq \left[\frac{\mathbf{C}_{\underline{l}}\mathbf{R}_{\underline{l}}}{\mathbf{Q}}C_{1.1}C_{1.10}q^{l_1-\frac{l_0}{k_1}}||\omega||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}\right] \times (1 + |m|)^{-\mu}e^{-\beta|m|}|\tau_1|\exp\left(\frac{k_1}{2\log(q)}\log^2(|\tau_1| + \delta) + \alpha\log(|\tau_1| + \delta)\right) \times |\tau_2|e^{\nu|\tau_2|} \tag{88}$$

for all $\tau_1 \in S_{d_1}$, all $\tau_2 \in H_\pi \cup D_\rho$. Notice that this last inequality is tantamount to the awaited bounds (61) for the constant

$$C_1 = \frac{\mathbf{R}_{\underline{l}}}{\mathbf{Q}}C_{1.1}C_{1.10}q^{l_1-\frac{l_0}{k_1}}.$$

□

We need control on the norm of the last term of $\mathcal{H}$ related to the forcing term of the equation (48).

Lemma 6. *There exists a constant $\mathbf{F}_{\mathcal{F}} > 0$ such that*

$$||\frac{\mathcal{F}(\tau_1, \tau_2, m)}{P_m(\tau_1, \tau_2)}||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \leq \mathbf{F}_{\mathcal{F}}. \tag{89}$$

Proof. In view of the factorization (52) and the definition (28) of $\mathcal{F}$, we notice that

$$\left|\frac{\mathcal{F}(\tau_1, \tau_2, m)}{P_m(\tau_1, \tau_2)}\right| \leq \frac{\sum_{j_1 \in J_1, j_2 \in J_2}|\mathcal{F}_{j_1,j_2}(m)||\tau_1|^{j_1}|\tau_2|^{j_2}}{|Q(\sqrt{-1}m)| \times |1 - r^{d_D}\exp\left((s - \sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right)|} \tag{90}$$

for all $\tau_1 \in S_{d_1}$ and $\tau_2 \in H_\pi \cup D_\rho$ for which the splittings (50) and (51) hold, and all $m \in \mathbb{R}$. Besides, by Definition of $\mathcal{F}$, constants $\mathbf{F}_{j_1,j_2} > 0$ can be found such that

$$|\mathcal{F}_{j_1,j_2}(m)| \leq \mathbf{F}_{j_1,j_2}(1 + |m|)^{-\mu}e^{-\beta|m|} \tag{91}$$

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for all $m \in \mathbb{R}$. Furthermore, we can pinpoint a constant $K_{d_D,k_1,q} > 0$ for which

$$|1 - r^{d_D} \exp\left((s - \sqrt{-1}\psi)\frac{d_D}{k_1}\log(q)\right)| \geq K_{d_D,k_1,q} \tag{92}$$

hold where $\tau_1 \in S_{d_1}$ and $\tau_2 \in H_\pi \cup D_\rho$ with the decompositions (50) and (51). As a result, combining (90), (91) and (92) gives rise to the next upper estimates

$$\begin{aligned}
\left|\frac{\mathcal{F}(\tau_1,\tau_2,m)}{P_m(\tau_1,\tau_2)}\right| &\leq \frac{1}{\min_{m\in\mathbb{R}}|Q(\sqrt{-1}m)|K_{d_D,k_1,q}}(1+|m|)^{-\mu}e^{-\beta|m|} \\
&\times \Big[\sum_{j_1\in J_1,j_2\in J_2} \mathbf{F}_{j_1,j_2}|\tau_1|^{j_1}|\tau_2|^{j_2}\Big\{\frac{1}{|\tau_1|}\exp\left(-\frac{k_1}{2\log(q)}\log^2(|\tau_1|+\delta) - \alpha\log(|\tau_1|+\delta)\right) \\
&\times \frac{1}{|\tau_2|}e^{-\nu|\tau_2|}\Big\}\Big] \times |\tau_1|\exp\left(\frac{k_1}{2\log(q)}\log^2(|\tau_1|+\delta) + \alpha\log(|\tau_1|+\delta)\right)|\tau_2|e^{\nu|\tau_2|} \\
&\leq \mathbf{F}_{\mathcal{F}}(1+|m|)^{-\mu}e^{-\beta|m|}|\tau_1|\exp\left(\frac{k_1}{2\log(q)}\log^2(|\tau_1|+\delta) + \alpha\log(|\tau_1|+\delta)\right)|\tau_2|e^{\nu|\tau_2|}
\end{aligned} \tag{93}$$

for all $\tau_1 \in S_{d_1}$ and $\tau_2 \in H_\pi \cup D_\rho$, where

$$\begin{aligned}
\mathbf{F}_{\mathcal{F}} &= \frac{1}{\min_{m\in\mathbb{R}}|Q(\sqrt{-1}m)|K_{d_D,k_1,q}} \\
&\times \sum_{j_1\in J_1,j_2\in J_2} \mathbf{F}_{j_1,j_2}\Big\{\sup_{x\geq 0} x^{j_1-1}\exp\left(-\frac{k_1}{2\log(q)}\log^2(x+\delta) - \alpha\log(x+\delta)\right)\Big\} \times \Big\{\sup_{y\geq 0} y^{j_2-1}e^{-\nu y}\Big\}
\end{aligned}$$

keeping in mind that $0 \notin J_k \subset \mathbb{N}^*$, for $k = 1,2$. At last, it remains to notice that the due inequality (89) results from (93) by taking heed of Definition 4. □

We select the constant $\varpi > 0$ suitably together with the constants $\mathbf{C}_{\underline{l}} > 0$, for $\underline{l} \in I$, taken close enough to 0 in a way that the next inequality

$$\sum_{\underline{l}=(l_0,l_1)\in I} \frac{1}{(2\pi)^{1/2}(q^{1/k_1})^{l_0(l_0-1)/2}} C_1\mathbf{C}_{\underline{l}}\varpi + \mathbf{F}_{\mathcal{F}} \leq \varpi \tag{94}$$

holds where $C_1 > 0$ appears in Lemma 5 and $\mathbf{F}_{\mathcal{F}} > 0$ stems from Lemma 6. Eventually, the expected inclusion (59) prompts from the bounds (61) and (89) under the restriction (94).

We discuss the second item addressing the shrinking feature (60). We take two elements ω_1, ω_2 in the closed ball $\bar{B}_\varpi$ from $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ whose radius $\varpi > 0$ has been prescribed in the first item (59). According to Lemma 5, under the conditions (22), (23), (24), (25) and (26) listed in Subsection 2.2, the next inequality

$$\begin{aligned}
||\frac{1}{P_m(\tau_1,\tau_2)}\int_{-\infty}^{+\infty} & C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}(\omega_1(\tau_1,\tau_2,m_1)-\omega_2(\tau_1,\tau_2,m_1)) \\
&\times \exp\left(-\tau_2 l_1\log(q)\right)dm_1||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \\
&\leq C_1\mathbf{C}_{\underline{l}}||\omega_1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}
\end{aligned} \tag{95}$$

holds for the constant $C_1 > 0$ introduced in Lemma 5. We set the constants $\mathbf{C}_{\underline{l}} > 0$, for $\underline{l} \in I$, small enough allowing the next inequality

$$\sum_{\underline{l}=(l_0,l_1)\in I} \frac{1}{(2\pi)^{1/2}(q^{1/k_1})^{l_0(l_0-1)/2}} C_1\mathbf{C}_{\underline{l}} \leq \frac{1}{2} \tag{96}$$

to hold. The Lipschitz property (60) is a straight consequence of (95) under the requirement (96).

In conclusion, we properly choose the constants $\mathbf{C}_{\underline{l}} > 0$, $\underline{l} \in I$ and the radius $\varpi > 0$ in order to impose both constraints (94) and (96) at once, which triggers the two properties (59) and (60) for the map $\mathcal{H}$. □

In the forthcoming proposition, we provide a solution to the convolution $q-$difference equation (48) established in Lemma 3.

Proposition 2. *Let us prescribe the sectorial domain S_{Q,R_D}, the unbounded sector S_{d_1} together with the strip H_π and the disc D_ρ as in Lemma 4. Then, the constants $\mathbf{C}_{\underline{l}} > 0$ defined in (27) and a constant $\varpi > 0$ can be fittingly chosen in a manner that a unique solution $\omega_{d_1,\pi}$ to the convolution $q-$difference equation (48) can be built up in the space $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ under the condition*

$$||\omega_{d_1,\pi}||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)} \leq \varpi. \tag{97}$$

Proof. We select $\varpi > 0$ as in Proposition 1. We mind the closed ball $\bar{B}_\varpi$ in the Banach space $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ which represents a complete metric space for the distance $d(x,y) = ||x-y||_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ deduced from the norm. The proposition 1 states that $\mathcal{H}$ induces a contractive map from the metric space $(\bar{B}_\varpi, d)$ into itself. According to the classical Banach fixed point theorem, it follows that $\mathcal{H}$ owns a unique fixed point inside the ball $\bar{B}_\varpi$, we denote $\omega_{d_1,\pi}$. It means that

$$\mathcal{H}(\omega_{d_1,\pi}(\tau_1,\tau_2,m)) = \omega_{d_1,\pi}(\tau_1,\tau_2,m) \tag{98}$$

for all $\tau_1 \in S_{d_1}$, $\tau_2 \in H_\pi \cup D_\rho$ and $m \in \mathbb{R}$. By transfering the term

$$R_D(\sqrt{-1}m)\frac{\tau_1^{d_D}}{(q^{1/k_1})^{d_D(d_D-1)/2}}\exp\big(-\tau_2\frac{d_D}{k_1}\log(q)\big)\omega_{d_1,\pi}(\tau_1,\tau_2,m)$$

from the right to the left handside of (48) and dividing the resulting equation by the map $P_m(\tau_1,\tau_2)$ displayed in (49), we observe that (48) can be rearranged into the fixed point equation (98). On that account, the unique fixed point $\omega_{d_1,\pi}$ obtained in $\bar{B}_\varpi$ precisely solve (48), which yields Proposition 2. □

3.3. Analytic solutions to the auxiliary equations (37) and (39)

In the next proposition, we craft analytic solutions to the associated set of $q-$difference and differential problems under the action of homographic maps established in Subsection 2.3.

Proposition 3. *The sectorial domain S_{Q,R_D}, the unbounded sector S_{d_1} together with the strip H_π and the disc D_ρ are prescribed as in Lemma 4.*

- *We define the map*

$$U_{d_1,\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}}\int_{L_{d_1}}\int_{L_\pi}\int_{-\infty}^{+\infty}\omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)}\exp\big(-\frac{\tau_2}{u_2}\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm \tag{99}$$

  where the Borel-Fourier map $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ is built up in Proposition 2 and solves the convolution $q-$difference equation (48). The map (99) boasts the next two qualities

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– *It defines a bounded holomorphic function on the product* $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,\pi} \times H_{\beta'}$ *for some given* $\Delta_1 > 0$*, where* $\mathcal{R}_{d_1,\Delta_1}$ *stands for the set (19) and* D_{R_1} *is a disc centered at 0 with radius subjected to the constraint*

$$0 < R_1 < q^{-\frac{1}{k_1}(\alpha+1)}/2 \tag{100}$$

and $0 < \beta' < \beta$*. Besides,* $U_{2,\pi}$ *represents a bounded sector edged at 0 with bisecting direction* π *with radius* $R_2 > 0$*, submitted to the next condition: there exists some real number* $\Delta_{2,\pi} > 0$ *with*

$$\cos(\pi - \arg(u_2)) > \Delta_{2,\pi} \tag{101}$$

for all $u_2 \in U_{2,\pi}$*, where* $0 < R_2 < \Delta_{2,\pi}/\nu$*, for* $\nu > 0$ *fixed in Definition 4.*

– *It solves the auxiliary equation (37) for prescribed initial data* $U_{d_1,\pi}(0,0,z) \equiv 0$*.*

- *For a direction* $d_2 \neq \pi$ *(modulo* 2π*), we shape the map*

$$U_{d_1,d_2}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{d_2,a}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{102}$$

where $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ *is the Borel-Fourier map mentioned in the above item. The map (102) enjoys the next two properties*

– *It represents a bounded holomorphic function on the product* $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,d_2} \times H_{\beta'}$*, for the domain* $\mathcal{R}_{d_1,\Delta_1}$*, disc* D_{R_1} *and constant* $0 < \beta' < \beta$ *given in the first item. Furthermore,* U_{2,d_2} *stands for a bounded sector centered at 0 with bisecting direction* d_2 *and with radius* R_2 *chosen as in the first item and subjected to the next restriction : some positive real number* $\Delta_{2,d_2} > 0$ *can be found with*

$$\cos(d_2 - \arg(u_2)) > \Delta_{2,d_2} \tag{103}$$

for all $u_2 \in U_{2,d_2}$*.*

– *It obeys the auxiliary equation (39) for given vanishing initial data* $U_{d_1,d_2}(0,0,z) \equiv 0$*.*

Proof. We discuss the first item. We parametrize $\tau_1 \in L_{d_1}$ and $\tau_2 \in L_\pi$ in the form $\tau_1 = r_1 e^{\sqrt{-1}d_1}$ and $\tau_2 = r_2 e^{\sqrt{-1}\pi}$ for $r_1, r_2 \geq 0$. Then, owing to (15) and (97), we get

$$\begin{aligned} |\omega_{d_1,\pi}(\tau_1,\tau_2,m)| \left| \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \right| \left| \exp\left(-\frac{\tau_2}{u_2}\right) \right| \frac{1}{|\tau_1||\tau_2|} |e^{\sqrt{-1}zm}| \\ \leq \varpi(1+|m|)^{-\mu} e^{-\beta|m|} \exp\left(\frac{k_1}{2\log(q)} \log^2(r_1+\delta) + \alpha \log(r_1+\delta)\right) e^{\nu r_2} \\ \times \frac{1}{C_{q,k_1}\Delta_1} \exp\left(-\frac{k_1}{2\log(q)} \log^2(r_1/|u_1|)\right) \frac{1}{(r_1/|u_1|)^{1/2}} \\ \times \exp\left(-\frac{r_2}{|u_2|}\cos(\pi - \arg(u_2))\right) \times e^{-m\mathrm{Im}(z)} \end{aligned} \tag{104}$$

for all $u_1 \in \mathbb{C}^*$ with $|1 + \frac{\tau_1}{u_1} r| > \Delta_1$ for all $r \geq 0$ and $u_2 \in U_{2,\pi}$. In order to provide upper bounds for the right handside of (104), we propose the next alternative.

Assume that $0 \leq r_1 < 1$ with $u_1 \in \mathcal{R}_{d_1,\Delta_1}$ as above under the constraint $|u_1| \leq R_1$. Then, one can single out a constant $M_{k_1,q,\delta,\alpha} > 0$ such that

$$\exp\left(\frac{k_1}{2\log(q)}\log^2(r_1+\delta)+\alpha\log(r_1+\delta)\right)\times\exp\left(-\frac{k_1}{2\log(q)}\log^2(r_1/|u_1|)\right)\frac{1}{(r_1/|u_1|)^{1/2}} \leq M_{k_1,q,\delta,\alpha}\sup_{x>0}\exp\left(-\frac{k_1}{2\log(q)}\log^2(x)\right)\frac{1}{x^{1/2}} \quad (105)$$

for all $0 \leq r_1 < 1$ with $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$.

Assume that $r_1 \geq 1$ and $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$ for a radius $R_1 > 0$ under the constraint (100). The next three expansions are useful. Namely,

$$\log^2(r_1/|u_1|) = \log^2(r_1) - 2\log(r_1)\log(|u_1|) + \log^2(|u_1|) \quad (106)$$

together with

$$\log^2(r_1+\delta) = \log^2(r_1) + 2\log(r_1)\log(1+\frac{\delta}{r_1}) + \log^2(1+\frac{\delta}{r_1}) \quad (107)$$

and

$$\log(r_1+\delta) = \log(r_1) + \log(1+\frac{\delta}{r_1}). \quad (108)$$

Since $\log(1+x) \sim x$ holds as x is close to 0 and owing to the classical limit $\lim_{x\to+\infty}\log(x)/x = 0$, we get from (107) and (108) two constants $M_{\delta,1}, M_{\delta,2} > 0$ with

$$\log^2(r_1+\delta) \leq \log^2(r_1) + M_{\delta,1} \ , \ \log(r_1+\delta) \leq \log(r_1) + M_{\delta,2} \quad (109)$$

for all $r_1 \geq 1$. As a result, we get from the computation (106) and bounds (109) that

$$\exp\left(\frac{k_1}{2\log(q)}\log^2(r_1+\delta)+\alpha\log(r_1+\delta)\right)\times\exp\left(-\frac{k_1}{2\log(q)}\log^2(r_1/|u_1|)\right)\frac{1}{(r_1/|u_1|)^{1/2}} \leq \exp\left(\frac{k_1}{2\log(q)}M_{\delta,1}+\alpha M_{\delta,2}\right)\times\exp\left(-\frac{k_1}{2\log(q)}\log^2(|u_1|)\right)|u_1|^{1/2} \times\exp\left(\alpha\log(r_1)+\frac{k_1}{\log(q)}\log(r_1)\log(|u_1|)\right)\frac{1}{r_1^{1/2}} \quad (110)$$

At last, from the assumption (100) and requirement $|u_1| \leq R_1$, the next bounds

$$\exp\left(\frac{k_1}{2\log(q)}\log^2(r_1+\delta)+\alpha\log(r_1+\delta)\right)\times\exp\left(-\frac{k_1}{2\log(q)}\log^2(r_1/|u_1|)\right)\frac{1}{(r_1/|u_1|)^{1/2}} \leq \exp\left(\frac{k_1}{2\log(q)}M_{\delta,1}+\alpha M_{\delta,2}\right)\times\exp\left(-\frac{k_1}{2\log(q)}\log^2(|u_1|)\right)|u_1|^{1/2}\times\frac{1}{r_1^{3/2}} \quad (111)$$

are deduced from (110).

On the other hand, taking heed of (101), we observe that

$$e^{-\beta|m|}e^{\nu r_2}\exp\left(-\frac{r_2}{|u_2|}\cos(\pi-\arg(u_2))\right)\times e^{-m\mathrm{Im}(z)} \leq e^{-(\beta-\beta')|m|}\exp\left(r_2(\nu-\frac{\Delta_{2,\pi}}{|u_2|})\right) \quad (112)$$

provided that $z \in H_{\beta'}$, where $0 < \beta' < \beta$ and $\nu - \frac{\Delta_{2,\pi}}{|u_2|} < 0$ according to the claim that $|u_2| < R_2 < \Delta_{2,\pi}/\nu$.

As a consequence of the above bounds (105) along with (111) and (112), we deduce that the map $(u_1,u_2,z) \mapsto U_{d_1,\pi}(u_1,u_2,z)$ is well defined and represents a bounded holomorphic function on the product $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,\pi} \times H_{\beta'}$ under the above requirements (100) and (101).

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Recall that the Borel-Fourier map $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ has been constructed as a solution of the associated convolution $q-$difference equation (48) in Proposition 2. From Lemma 3, we deduce that $U_{d_1,\pi}(u_1,u_2,z)$ obeys the auxiliary equation (37) on the domain $(\mathcal{R}_{d_1,\Delta_1}\cap D_{R_1})\times U_{2,\pi}\times H_{\beta'}$ for prescribed initial data $U_{d_1,\pi}(0,0,z)\equiv 0$.

We turn to the second item. Let $\tau_1\in L_{d_1}$ and $\tau_2\in L_{d_2,a}$ be parametrized as follows $\tau_1=r_1e^{\sqrt{-1}d_1}$ and $\tau_2=r_2e^{\sqrt{-1}d_2}$ with $r_1\geq 0$, $0\leq r_2\leq a$. Bearing in mind (15) and (97), we obtain a constant $\varpi>0$ such that the next inequality

$$
\begin{aligned}
|\omega_{d_1,\pi}(\tau_1,\tau_2,m)|\Big|\frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)}\Big||\exp\big(-\frac{\tau_2}{u_2}\big)|\frac{1}{|\tau_1||\tau_2|}|e^{\sqrt{-1}zm}| \\
\leq \varpi(1+|m|)^{-\mu}e^{-\beta|m|}\exp\Big(\frac{k_1}{2\log(q)}\log^2(r_1+\delta)+\alpha\log(r_1+\delta)\Big)e^{\nu r_2} \\
\times\frac{1}{C_{q,k_1}\Delta_1}\exp\Big(-\frac{k_1}{2\log(q)}\log^2(r_1/|u_1|)\Big)\frac{1}{(r_1/|u_1|)^{1/2}} \\
\times\exp\Big(-\frac{r_2}{|u_2|}\cos(d_2-\arg(u_2))\Big)\times e^{-m\mathrm{Im}(z)} \quad (113)
\end{aligned}
$$

holds provided that $u_1\in\mathcal{R}_{d_1,\Delta_1}$ and $u_2\in U_{2,d_2}$. According to (103), we notice that

$$
e^{\nu r_2}\exp\Big(-\frac{r_2}{|u_2|}\cos(d_2-\arg(u_2))\Big)\leq e^{\nu a}\exp\Big(-\frac{r_2}{|u_2|}\Delta_{2,d_2}\Big)\leq e^{\nu a}\exp\Big(-\frac{r_2}{R_2}\Delta_{2,d_2}\Big) \quad (114)
$$

under the restriction $|u_2|<R_2$. By dint of the upper bounds (105) in a row with (111), (112) and (114), we acknowledge the fact that $(u_1,u_2,z)\mapsto U_{d_1,d_2}(u_1,u_2,z)$ is bounded and stands for a holomorphic map on the product $(\mathcal{R}_{d_1,\Delta_1}\cap D_{R_1})\times U_{2,d_2}\times H_{\beta'}$ under the assumptions (100) and (103). Since the Borel-Fourier map $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ solves the convolution $q-$difference equation (48) as shown in Proposition 2, we deduce from Lemma 3 that $U_{d_1,d_2}(u_1,u_2,z)$ conforms the auxiliary equation (39) on the domain $(\mathcal{R}_{d_1,\Delta_1}\cap D_{R_1})\times U_{2,d_2}\times H_{\beta'}$ for given vanishing initial data $U_{d_1,d_2}(0,0,z)\equiv 0$. □

4. Construction of a holomorphic solution to the main initial value problem (21) and its Gevrey asymptotic expansion relatively to complex time t in logarithmic scale.

4.1. A finite set of genuine solutions to related initial value problems.

We restate the definition of a good covering in $\mathbb{C}^*$ as described in the textbook [8], Section XI-2.

Definition 5. *Let $\varsigma\geq 2$ be an integer. A set of bounded sectors $\underline{\mathcal{U}}=\{U_p\}_{0\leq p\leq\varsigma-1}$ edged at 0 is deemed with the next three attributes*

1. *Any two consecutive sectors U_p and U_{p+1} have non empty intersection $U_p\cap U_{p+1}$, for $0\leq p\leq\varsigma-1$, where the convention $U_\varsigma=U_0$ is assumed.*
2. *The intersection of any three sectors $U_p\cap U_q\cap U_r$ is reduced to the empty set for all distinct non negative integers p,q,r less than $\varsigma-1$.*
3. *The union $\cup_{p=0}^{\varsigma-1}U_p$ covers some punctured neighborhood of 0 in $\mathbb{C}^*$.*

Such a set $\underline{\mathcal{U}}$ is tagged a good covering in $\mathbb{C}^*$.

A notion of fitting set of sectors is discussed in the next definition.

Definition 6. *Let $\varsigma\geq 2$ be an integer. A finite set of bounded sectors $\underline{U}=\{U_{2,\mathfrak{d}_p}\}_{0\leq p\leq\varsigma-1}$ is minded with the next three constraints.*

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1. *For each* $0 \leq p \leq \varsigma - 1$*, the sector* $U_{2,\mathfrak{d}_p}$ *is edged at 0, with bisecting direction* $\mathfrak{d}_p \in \mathbb{R}$ *and is subjected to the condition that some real number* $\Delta_{2,\mathfrak{d}_p} > 0$ *can be singled out with*
$$\cos(\mathfrak{d}_p - \arg(u_2)) > \Delta_{2,\mathfrak{d}_p} \tag{115}$$
for all $u_2 \in U_{2,\mathfrak{d}_p}$*.*
2. *There exists an index* $p_1 \in \{1, \ldots, \varsigma - 1\}$ *with* $\mathfrak{d}_{p_1} = \pi$*. All the sectors* $U_{2,\mathfrak{d}_p}$*,* $0 \leq p \leq \varsigma - 1$ *have the same radius* R_2 *which obeys the restriction*
$$0 < R_2 < \frac{\Delta_{2,\pi}}{\nu}$$
where $\Delta_{2,\pi} > 0$ *is introduced in the above item and* $\nu > 0$ *is declared in Definition 4.*
3. *The set* $\underline{U}$ *forms a good covering in* $\mathbb{C}^*$ *in the sense of Definition 5.*

A set $\underline{U}$ *endowed with the above three features is called a fitting set of sectors.*

In the oncoming proposition, we exhibit analytic solutions to the auxiliary problems (37) and (39) where the directions d_2 span the set of bisecting directions of some fitting set of sectors. Furthermore, sharp estimates of their consecutive differences are provided which are essential in the study of their asymptotic expansions in the variable u_2 that will be described in the next Subsection 4.2.

Proposition 4. *Let the sectorial domain* S_{Q,R_D}*, the unbounded sector* S_{d_1} *together with the strip* H_π *and the disc* D_ρ *be arranged as in Lemma 4. Consider a fitting set of sectors* $\underline{U} = \{U_{2,\mathfrak{d}_p}\}_{0 \leq p \leq \varsigma - 1}$ *and assign a radius* a *with* $0 < a < \rho$*. Then, provided that the constants* $\mathbf{C}_{\underline{l}} > 0$ *are taken close enough to 0 in accordance with the requirements of Proposition 2, the properties described in the forthcoming three items hold.*

- *For each* $p \in \{0, \ldots, \varsigma - 1\} \setminus \{p_1\}$ *(where* p_1 *stems from Definition 6 2.) the equation*
$$Q(\partial_z)U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}} \circ \mathbb{H}_{\frac{d_D}{k_1}\log(q);u_2} R_D(\partial_z)U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) + \sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0}\sigma_{q;u_1}^{l_1} \circ \mathbb{H}_{l_1\log(q);u_2} c_{\underline{l}}(z) R_{\underline{l}}(\partial_z) U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) + F_{\mathfrak{d}_p,a}(u_1,u_2,z) \tag{116}$$
where the forcing term $F_{\mathfrak{d}_p,a}$ *is given by the triple integral formula (38), possesses a bounded holomorphic solution* $(u_1,u_2,z) \mapsto U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)$ *on the domain* $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,\mathfrak{d}_p} \times H_{\beta'}$*, where* $\mathcal{R}_{d_1,\Delta_1}$ *stands for the set (19), for a radius* $R_1 > 0$ *fulfilling (100), which observes the condition* $U_{d_1,\mathfrak{d}_p}(0,0,z) \equiv 0$*. Furthermore, the map* $U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)$ *is embodied in a Fourier inverse and a double* $q-$*Laplace, Laplace transform*
$$U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\mathfrak{d}_p,a}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{117}$$
where the Borel-Fourier map $(\tau_1,\tau_2,m) \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ *belongs to the Banach space* $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ *(introduced in Definition 4) constrained to the bounds (97).*
- *The equation*
$$Q(\partial_z)U_{d_1,\pi}(u_1,u_2,z) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}} \circ \mathbb{H}_{\frac{d_D}{k_1}\log(q);u_2} R_D(\partial_z)U_{d_1,\pi}(u_1,u_2,z) + \sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0}\sigma_{q;u_1}^{l_1} \circ \mathbb{H}_{l_1\log(q);u_2} c_{\underline{l}}(z) R_{\underline{l}}(\partial_z) U_{d_1,\pi}(u_1,u_2,z) + F_\pi(u_1,u_2,z) \tag{118}$$

with forcing term F_{π} is displayed in (29) and expressed as a polynomial in (30), holds a bounded holomorphic solution $(u_1, u_2, z) \mapsto U_{d_1,\pi}(u_1, u_2, z)$ on the domain $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,\pi} \times H_{\beta'}$ where the set $\mathcal{R}_{d_1,\Delta_1}$ and radius R_1 are given in the above item, under the vanishing condition $U_{d_1,\pi}(0,0,z) \equiv 0$. In addition, the map $U_{d_1,\pi}(u_1, u_2, z)$ is expressed through a Fourier inverse and a double $q-$Laplace, Laplace transform

$$U_{d_1,\pi}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (119)$$

where the Borel-Fourier map $(\tau_1, \tau_2, m) \mapsto \omega_{d_1,\pi}(\tau_1, \tau_2, m)$ is described in the former item.

- *The neighboring differences of the maps $U_{d_1,\mathfrak{d}_p}$ are controlled by the next bounds. For all $0 \leq p \leq \varsigma - 1$, two constants $M_{p,1}, K_{p,1} > 0$ can be found such that*

$$|U_{d_1,\mathfrak{d}_{p+1}}(u_1,u_2,z) - U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)| \leq M_{p,1} \exp\left(-\frac{K_{p,1}}{|u_2|}\right) \quad (120)$$

for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$, provided that $u_2 \in U_{2,\mathfrak{d}_p} \cap U_{2,\mathfrak{d}_{p+1}} \cap D_{\check{R}_2}$ for a well chosen radius $0 < \check{R}_2 < R_2$. Here we adopt the convention that $\mathfrak{d}_{\varsigma} = \mathfrak{d}_0$.

Proof. The first two items are direct corollaries of the statement of Proposition 3 and the definition of a fitting set of sectors $\underline{U}$ chosen at the onset of Proposition 4.

We focus on the third item which demands more labor and hinges on paths deformations arguments. We distinguish two different situations.

Case 1. Let $p = p_1$ or $p = p_1 - 1$. We discuss only the subcase $p = p_1$ since the other alternative $p = p_1 - 1$ can be treated in a similar manner. By construction, we notice that $\mathfrak{d}_{p+1} \neq \pi$ (modulo 2π). According to Proposition 2, for any prescribed $\tau_1 \in S_{d_1}$ and $m \in \mathbb{R}$, the partial map $\tau_2 \mapsto \omega_{d_1,\pi}(\tau_1, \tau_2, m)$ is analytic on the union $H_{\pi} \cup D_{\rho}$. As a result, the oriented path $L_{\mathfrak{d}_{p+1},a} - L_{\pi}$ can be bent into the union of

- The halfline $-L_{\pi,a,\infty} = -[a, +\infty)e^{\sqrt{-1}\pi}$
- The arc of circle $C_{\pi,\mathfrak{d}_{p+1},a} = \{ae^{\sqrt{-1}\theta} / \theta \in [\pi, \mathfrak{d}_{p+1}]\}$

and the classical Cauchy's theorem enables the difference $U_{d_1,\mathfrak{d}_{p+1}} - U_{d_1,\pi}$ to be reorganized as a sum of two contributions. Namely,

$$\begin{aligned} U_{d_1,\mathfrak{d}_{p+1}}(u_1,u_2,z) - U_{d_1,\pi}(u_1,u_2,z) &= -\frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi,a,\infty}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \\ &\times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \\ &+ \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{C_{\pi,\mathfrak{d}_{p+1},a}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \\ &\times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\left(-\frac{\tau_2}{u_2}\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \end{aligned} \quad (121)$$

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for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$ and $u_2 \in U_{2,\pi} \cap U_{2,\mathfrak{d}_{p+1}}$. We need to control the first piece of (121)

$$J_1 = \Big| \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_{\pi,a,\infty}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2}\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \Big| \quad (122)$$

Drew on the bounds (104), (105) together with (111) and (112), we split the halfline L_{d_1} in the union of two segments $L_{d_1,1} = [0,1]e^{\sqrt{-1}d_1}$ and $L_{d_1,1,\infty} = [1,+\infty)e^{\sqrt{-1}d_1}$ and we are reduced to provide bounds for the next two quantities $J_{1.1}$ and $J_{1.2}$ for

$$J_1 \leq J_{1.1} + J_{1.2} \quad (123)$$

where

$$J_{1.1} = \Big| \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,1}} \int_{L_{\pi,a,\infty}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2}\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \Big|$$

and

$$J_{1.2} = \Big| \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,1,\infty}} \int_{L_{\pi,a,\infty}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\big(-\frac{\tau_2}{u_2}\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \Big|$$

Indeed,

$$J_{1.1} \leq \frac{k_1}{\log(q)(2\pi)^{1/2}} \varpi \Big(\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \Big) \frac{1}{C_{q,k_1}\Delta_1} M_{k_1,q,\delta,\alpha} \times \sup_{x>0} \exp\Big(-\frac{k_1}{2\log(q)}\log^2(x)\Big) \frac{1}{x^{1/2}} \times \int_a^{+\infty} \exp\Big(r_2(\nu - \frac{\Delta_{2,\pi}}{|u_2|}) \Big) dr_2 \quad (124)$$

and

$$J_{1.2} \leq \frac{k_1}{\log(q)(2\pi)^{1/2}} \varpi \Big(\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \Big) \frac{1}{C_{q,k_1}\Delta_1} \Big(\int_1^{+\infty} \frac{1}{r_1^{3/2}} dr_1 \Big) \times \exp\Big(\frac{k_1}{2\log(q)} M_{\delta,1} + \alpha M_{\delta,2} \Big) \times \exp\Big(-\frac{k_1}{2\log(q)} \log^2(|u_1|) \Big) |u_1|^{1/2} \times \int_a^{+\infty} \exp\Big(r_2(\nu - \frac{\Delta_{2,\pi}}{|u_2|}) \Big) dr_2 \quad (125)$$

Now, we set $0 < \check{R}_2 = (\Delta_{2,\pi} - \check{\Delta}_{2,\pi})/\nu < R_2$ for some real number $0 < \check{\Delta}_{2,\pi} < \Delta_{2,\pi}$. Hence,

$$\int_a^{+\infty} \exp\Big(r_2(\nu - \frac{\Delta_{2,\pi}}{|u_2|}) \Big) dr_2 \leq \int_a^{+\infty} \exp\Big(-\frac{\check{\Delta}_{2,\pi}}{|u_2|} r_2 \Big) dr_2 = \frac{|u_2|}{\check{\Delta}_{2,\pi}} \exp\Big(-\frac{\check{\Delta}_{2,\pi}}{|u_2|} a \Big) \quad (126)$$

provided that $|u_2| \leq \check{R}_2$. As a result of (124), (125) and (126), we deduce from the splitting (123) that

$$J_1 \leq \Big(\sup_{u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}} \mathcal{M}_1(|u_1|) \Big) \frac{|u_2|}{\check{\Delta}_{2,\pi}} \exp\Big(-\frac{\check{\Delta}_{2,\pi}}{|u_2|} a \Big) \quad (127)$$

for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$ and $u_2 \in U_{2,\pi} \cap U_{2,\mathfrak{d}_{p+1}} \cap D_{\check{R}_2}$, where

$$\mathcal{M}_1(|u_1|) = \frac{k_1}{\log(q)(2\pi)^{1/2}}\varpi\Big(\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}dm\Big)\frac{1}{C_{q,k_1}\Delta_1}M_{k_1,q,\delta,\alpha} \times \sup_{x>0}\exp\Big(-\frac{k_1}{2\log(q)}\log^2(x)\Big)\frac{1}{x^{1/2}} + \frac{k_1}{\log(q)(2\pi)^{1/2}}\varpi\Big(\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}dm\Big)\frac{1}{C_{q,k_1}\Delta_1} \times \Big(\int_1^{+\infty}\frac{1}{r_1^{3/2}}dr_1\Big) \times \exp\Big(\frac{k_1}{2\log(q)}M_{\delta,1} + \alpha M_{\delta,2}\Big) \times \exp\Big(-\frac{k_1}{2\log(q)}\log^2(|u_1|)\Big)|u_1|^{1/2} \quad (128)$$

In the next step, we display bounds for the second piece of (121)

$$J_2 = \Big|\frac{k_1}{\log(q)(2\pi)^{1/2}}\int_{L_{d_1}}\int_{C_{\pi,\mathfrak{d}_{p+1},a}}\int_{-\infty}^{+\infty}\omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)}\exp\big(-\frac{\tau_2}{u_2}\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm\Big| \quad (129)$$

According to Definition 6 1. of fitting set of sectors, we notice that the lower bounds

$$\cos(\theta - \arg(u_2)) > \tilde{\Delta}_{2,\mathfrak{d}_{p+1},\pi} = \min(\Delta_{2,\mathfrak{d}_{p+1}},\Delta_{2,\pi}) \quad (130)$$

for all $u_2 \in U_{2,\pi} \cap U_{2,\mathfrak{d}_{p+1}}$ whenever the angle θ belongs to $(\pi, \mathfrak{d}_{p+1})$. By breaking up the halfine L_{d_1} into the segments $L_{d_1,1}$ and $L_{d_1,1,\infty}$, similar computations as above yield the bounds

$$J_2 \leq \Big(\sup_{u_1 \in \mathcal{R}_{d_1,\Delta_1}\cap D_{R_1}}\mathcal{M}_1(|u_1|)\Big)\Big|\int_{\pi}^{\mathfrak{d}_{p+1}} e^{av}\exp\Big(-\frac{a}{|u_2|}\cos(\theta-\arg(u_2))\Big)ad\theta\Big| \leq \Big(\sup_{u_1 \in \mathcal{R}_{d_1,\Delta_1}\cap D_{R_1}}\mathcal{M}_1(|u_1|)\Big)|\pi - \mathfrak{d}_{p+1}|ae^{av}\exp\Big(-\frac{a}{|u_2|}\tilde{\Delta}_{2,\mathfrak{d}_{p+1},\pi}\Big) \quad (131)$$

for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$, as long as $u_2 \in U_{2,\pi} \cap U_{2,\mathfrak{d}_{p+1}}$.

In conclusion, the decomposition (121) along with the two upper bounds (127) and (131) beget the estimates (120) under the assumption that $p = p_1$.

Case 2. Assume that $p \notin \{p_1 - 1, p_1\}$. We observe that both directions $\mathfrak{d}_p$ and $\mathfrak{d}_{p+1}$ are not equal to π modulo 2π. Owing to Proposition 2, for any fixed $\tau_1 \in S_{d_1}$ and $m \in \mathbb{R}$, the partial map $\tau_2 \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ is analytic on the disc D_ρ. On these grounds, we can deform the oriented path $L_{\mathfrak{d}_{p+1},a} - L_{\mathfrak{d}_p,a}$ into a single arc of circle

$$C_{\mathfrak{d}_p,\mathfrak{d}_{p+1},a} = \{ae^{\sqrt{-1}\theta}/\theta \in [\mathfrak{d}_p,\mathfrak{d}_{p+1}]\}$$

and rewrite the difference $U_{d_1,\mathfrak{d}_{p+1}} - U_{d_1,\mathfrak{d}_p}$ as a single triple path integral

$$U_{d_1,\mathfrak{d}_{p+1}}(u_1,u_2,z) - U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}}\int_{L_{d_1}}\int_{C_{\mathfrak{d}_p,\mathfrak{d}_{p+1},a}}\int_{-\infty}^{+\infty}\omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)}\exp\big(-\frac{\tau_2}{u_2}\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm \quad (132)$$

for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$ and $u_2 \in U_{2,\mathfrak{d}_p} \cap U_{2,\mathfrak{d}_{p+1}}$. Upper bounds are asked for the quantity

$$J_3 = \Big| \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{C_{\mathfrak{d}_p,\mathfrak{d}_{p+1},a}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/u_1)} \exp\Big(-\frac{\tau_2}{u_2}\Big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \Big|. \quad (133)$$

The definition 6 of fitting sets of sectors allows the next lower bounds

$$\cos(\theta - \arg(u_2)) > \tilde{\Delta}_{2,\mathfrak{d}_{p+1},\mathfrak{d}_p} = \min(\Delta_{2,\mathfrak{d}_{p+1}}, \Delta_{2,\mathfrak{d}_p}) \quad (134)$$

to hold for all $u_2 \in U_{2,\mathfrak{d}_p} \cap U_{2,\mathfrak{d}_{p+1}}$ whenever the angle θ is taken in $(\mathfrak{d}_p, \mathfrak{d}_{p+1})$. Using the partition of the halfline L_{d_1} in two segments $L_{d_1,1} = [0,1]e^{\sqrt{-1}d_1}$ and $L_{d_1,1,\infty} = [1,+\infty)e^{\sqrt{-1}d_1}$, comparable estimates as the ones performed in the case 1. give rise to the next bounds

$$J_3 \le \Big(\sup_{u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}} \mathcal{M}_1(|u_1|) \Big) \Big| \int_{\mathfrak{d}_p}^{\mathfrak{d}_{p+1}} e^{av} \exp\Big(-\frac{a}{|u_2|}\cos(\theta - \arg(u_2))\Big) a d\theta \Big|$$
$$\le \Big(\sup_{u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}} \mathcal{M}_1(|u_1|) \Big) |\mathfrak{d}_p - \mathfrak{d}_{p+1}| a e^{av} \exp\Big(-\frac{a}{|u_2|}\tilde{\Delta}_{2,\mathfrak{d}_{p+1},\mathfrak{d}_p}\Big) \quad (135)$$

for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $z \in H_{\beta'}$, provided that $u_2 \in U_{2,\mathfrak{d}_p} \cap U_{2,\mathfrak{d}_{p+1}}$, where $\mathcal{M}_1(|u_1|)$ is given by the expression (128).

In brief, the recast expression (132) coupled with the bounds (135) prompts the awaited estimates (120) under the assumption that $p \notin \{p_1 - 1, p_1\}$. □

4.2. Gevrey asymptotic expansions for the bounded holomorphic solutions to the family of auxiliary problems (116) and (118).

In the next proposition, asymptotic expansions of Gevrey type are achieved for the maps $U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)$, that are displayed in Proposition 4, relatively to the variable u_2.

Proposition 5. *For the constants d_1, Δ_1, R_1 and β' fixed in Proposition 4, we denote $\mathbb{F}_{d_1,\Delta_1,R_1,\beta'}$ the Banach space of $\mathbb{C}-$valued bounded holomorphic functions on the product $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times H_{\beta'}$ endowed with the sup norm. Then, for all $0 \le p \le \varsigma - 1$, the partial maps $u_2 \mapsto U_{d_1,\mathfrak{d}_p}(u_1,u_2,z)$, viewed as bounded holomorphic maps from the bounded sector $U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2}$ into $\mathbb{F}_{d_1,\Delta_1,R_1,\beta'}$, share a common formal power series*

$$\hat{\mathbb{G}}(u_2) = \sum_{n \ge 0} G_n(u_1,z)\frac{u_2^n}{n!} \quad (136)$$

with coefficients G_n, $n \ge 0$, that belong to $\mathbb{F}_{d_1,\Delta_1,R_1,\beta'}$, as Gevrey asymptotic expansion of order 1 on $U_{2,\mathfrak{d}_p}$. It means that, for each $0 \le p \le \varsigma - 1$, two constants $K_{p,2}, M_{p,2} > 0$ can be chosen in a way that the next error bounds

$$\Big| U_{d_1,\mathfrak{d}_p}(u_1,u_2,z) - \sum_{n=0}^{N} G_n(u_1,z)\frac{u_2^n}{n!} \Big| \le K_{p,2}(M_{p,2})^{N+1}\Gamma(N+2)|u_2|^{N+1} \quad (137)$$

hold for all integers $N \ge 0$, all $u_2 \in U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2}$, whenever $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$ and $z \in H_{\beta'}$.

Proof. In the proof, we apply the next result known as the Ramis-Sibuya theorem that we rephrase for the sake of completeness and clarity for the reader (see Lemma XI-2-6 in [8]).

Theorem (R.S.) *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a Banach space over the field of complex numbers and let $\{U_p\}_{0 \le p \le \varsigma-1}$ be a good covering in $\mathbb{C}^*$ as outlined in Definition 5. For all $0 \le p \le \varsigma - 1$, we consider holomorphic functions $G_p : U_p \to \mathbb{F}$ that enjoy the next two features*

1. *The maps G_p are bounded on U_p for all $0 \leq p \leq \varsigma - 1$.*
2. *The difference $\Theta_p(u) = G_{p+1}(u) - G_p(u)$ stands for a holomorphic map on the intersection $Z_p = U_{p+1} \cap U_p$ which is exponentially flat of order k, for some integer $k \geq 1$, meaning that one can select two constants $C_p, A_p > 0$ for which*
$$||\Theta_p(u)||_{\mathbb{F}} \leq C_p \exp(-\frac{A_p}{|u|^k})$$
holds provided that $u \in Z_p$, for all $0 \leq p \leq \varsigma - 1$. By convention, we set $G_\varsigma = G_0$ and $U_\varsigma = U_0$.

Then, a formal power series $\hat{G}(u) = \sum_{n \geq 0} G_n u^n$ with coefficients G_n belonging to $\mathbb{F}$ can be singled out, which is the common Gevrey asymptotic expansion of order $1/k$ relatively to u on U_p for all the maps G_p, for $0 \leq p \leq \varsigma - 1$. It attests that two constants $K_p, M_p > 0$ can be chosen with the result that the error bounds

$$||G_p(u) - \sum_{n=0}^{N} G_n u^n||_{\mathbb{F}} \leq K_p M_p^{N+1} \Gamma(1 + \frac{N+1}{k})|u|^{N+1} \tag{138}$$

hold for all integers $N \geq 0$, all $u \in U_p$, all $0 \leq p \leq \varsigma - 1$.

For each $0 \leq p \leq \varsigma - 1$, we introduce the map $G_p : U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2} \to \mathbb{F}_{d_1,\Delta_1,R_1,\beta'}$ set as

$$G_p(u_2) := (u_1, z) \mapsto U_{d_1,\mathfrak{d}_p}(u_1, u_2, z).$$

In view of Proposition 4, we acknowledge that

- The set of sectors $\{U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2}\}_{0 \leq p \leq \varsigma - 1}$ forms a good covering in $\mathbb{C}^*$ owing to Definition 6 3.
- For each $0 \leq p \leq \varsigma - 1$, the map G_p is bounded holomorphic on the sector $U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2}$.
- For each $0 \leq p \leq \varsigma - 1$, the difference $\Theta_p(u_2) = G_{p+1}(u_2) - G_p(u_2)$ suffers the bounds
$$||\Theta_p(u_2)||_{\mathbb{F}_{d_1,\Delta_1,R_1,\beta'}} \leq M_{p,1} \exp\left(-\frac{K_{p,1}}{|u_2|}\right)$$
for the constants $M_{p,1}$ and $K_{p,1}$ displayed in (120), provided that $u_2 \in U_{2,\mathfrak{d}_p} \cap U_{2,\mathfrak{d}_{p+1}} \cap D_{\check{R}_2}$.

Thereupon, the claims 1. and 2. of Theorem (R.S) are matched for the family of maps $\{G_p\}_{0 \leq p \leq \varsigma - 1}$ with the constant $k = 1$. The existence of the formal power series (136) which represents the collective Gevrey asymptotic expansion of order 1 relatively to u_2 on $U_{2,\mathfrak{d}_p} \cap D_{\check{R}_2}$ for all the maps G_p, $0 \leq p \leq \varsigma - 1$ follows. As a result, the error bounds (137) are warranted. □

4.3. Statement of the first main result.

In this subsection, a bounded holomorphic solution to our main initial value problem (21) is shaped. This solution is favored with an asymptotic expansion in some logarithmic scale that reveals to be of Gevrey type. The next theorem represents the first main achievement of our work.

Theorem 1. *Let the sectorial domain S_{Q,R_D}, the unbounded sector S_{d_1} together with the strip H_π and the disc D_ρ be duly prescribed as in Lemma 4. Then, assuming that the constants $\mathbf{C}_{\underline{l}} > 0$ are in the vicinity of 0 as specified by the requirements of Proposition 2 and that the radius $R_1 > 0$ is close enough to 0, the equation*

$$Q(\partial_z)u(t,z) = t^{d_D} \sigma_{q;t}^{\frac{d_D}{k_1}} R_D(\partial_z)u(t,z) + \sum_{\underline{l}=(l_0,l_1) \in I} c_{\underline{l}}(z) t^{l_0} \sigma_{q;t}^{l_1} R_{\underline{l}}(\partial_z)u(t,z) + f(t,z) \tag{139}$$

has a bounded holomorphic solution $(t,z) \mapsto u(t,z)$ *on the domain* $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ *for vanishing initial data* $u(0,z) \equiv 0$*. In addition, the map* $u(t,z)$ *can be expressed as a triple integral comprising a Fourier inverse, a* $q-$*Laplace and Laplace transforms*

$$u(t,z) = U_{d_1,\pi}(t, \frac{1}{\log(t)}, z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1}} \int_{L_\pi} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (140)$$

where the Borel-Fourier map $(\tau_1,\tau_2,m) \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ *originates from the Banach space* $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$ *(see Definition 4) and is restrained to the bounds (97).*

The function $u(t,z)$ *enjoys a generalized asymptotic expansion of Gevrey type in a logarithmic scale as* t *tends to 0. More precisely, one can single out a formal series*

$$\hat{u}(t,z) = \sum_{n\geq 0} G_n(t,z) \frac{(1/\log(t))^n}{n!} \quad (141)$$

with bounded holomorphic coefficients $G_n(t,z)$ *on the domain* $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times H_{\beta'}$*, which stands for an asymptotic expansion of Gevrey order 1 in the scale of logarithmic functions* $\{(1/\log(t))^n\}_{n\geq 0}$ *of the map* $u(t,z)$ *with respect to* t *on the domain* $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$*. In other words, two constants* $K_{p_1,2}, M_{p_1,2} > 0$ *can be found with the aim that the next error bounds*

$$\Big| u(t,z) - \sum_{n=0}^{N} G_n(t,z) \frac{(1/\log(t))^n}{n!} \Big| \leq K_{p_1,2}(M_{p_1,2})^{N+1}\Gamma(N+2)|1/\log(t)|^{N+1} \quad (142)$$

hold for all integers $N \geq 0$*, all* $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$*, provided that* $z \in H_{\beta'}$*.*

Proof. We select a fitting set of sectors $\underline{U} = \{U_{2,\mathfrak{d}_p}\}_{0\leq p\leq \varsigma-1}$ and we take the index $p = p_1$ for which $\mathfrak{d}_{p_1} = \pi$ according to Definition 6 2. By definition of the principal value of the logarithm $\log(t) = \ln|t| + \sqrt{-1}\arg(t)$, for $\arg(t) \in (-\pi,\pi)$, whenever $t \in \mathbb{C} \setminus (-\infty,0]$, we check that

$$\frac{1}{\log(t)} \in U_{2,\pi} \cap D_{\check{R}_2} \quad (143)$$

as long as $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, provided that we take $R_1 > 0$ sufficiently close to 0, where $\check{R}_2 > 0$ has been disclosed in the third item of Proposition 4. We define

$$u(t,z) = U_{d_1,\pi}(t, \frac{1}{\log(t)}, z) \quad (144)$$

where the map $U_{d_1,\pi}(u_1,u_2,z)$ is described in the second item of Proposition 4. By construction of $U_{d_1,\pi}$, we ascertain that $u(t,z)$ represents a bounded holomorphic function on the product $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$.

Besides, according to the second item of Proposition 4, we know that the map $U_{d_1,\pi}(u_1,u_2,z)$ stands for a solution to the equation (118) on the domain $(\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times U_{2,\pi} \times H_{\beta'}$. On the basis of the computations made in Subsection 2.3, we deduce that the map $u(t,z)$ solves the main equation (21) on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$, constrained to the initial value condition $u(0,z) \equiv 0$.

At last, the asymptotic expansion property (142) of the map $u(t,z)$ is a direct offspring of the expansion (137) for the particular case $p = p_1$, where u_1 is set to be the time variable t and the variable u_2 is merely replaced by the logarithmic function $1/\log(t)$ for $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$. □

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4.4. Computational features related to the formal power series (136).

In this subsection, we establish that the formal series (136) which represent the asymptotic expansion of Gevrey type for the holomorphic maps $U_{d_1,\mathfrak{d}_p}$ actually solve some functional partial differential equation. On the journey, we notice that its coefficients G_n, $n \geq 0$ fulfill some handy recursion relations that might be of interest for concrete applications.

Proposition 6. *The formal power series*

$$\hat{\mathbb{G}}(u_2) = \sum_{n\geq 0} G_n(u_1,z)\frac{u_2^n}{n!} \tag{145}$$

with coefficients G_n, $n \geq 0$ in the space $\mathbb{F}_{d_1,\Delta_1,R_1,\beta'}$, conforms the next functional partial differential equation

$$Q(\partial_z)\hat{\mathbb{G}}(u_2) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}}R_D(\partial_z)\hat{\mathbb{G}}(\frac{u_2}{1+u_2\frac{d_D}{k_1}\log(q)}) + \sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0}\sigma_{q;u_1}^{l_1}c_{\underline{l}}(z)R_{\underline{l}}(\partial_z)\hat{\mathbb{G}}(\frac{u_2}{1+u_2 l_1\log(q)}) + F_\pi(u_1,u_2,z). \tag{146}$$

In addition, the coefficients G_n, $n \geq 0$ satisfy the recursion relations (154) and (155).

Proof. We depart from the equation (118) recast in the form

$$Q(\partial_z)U_{d_1,\pi}(u_1,u_2,z) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}}R_D(\partial_z)U_{d_1,\pi}(u_1,\frac{u_2}{1+u_2\frac{d_D}{k_1}\log(q)},z) + \sum_{\underline{l}=(l_0,l_1)\in I} u_1^{l_0}\sigma_{q;u_1}^{l_1}c_{\underline{l}}(z)R_{\underline{l}}(\partial_z)U_{d_1,\pi}(u_1,\frac{u_2}{1+u_2 l_1\log(q)},z) + F_\pi(u_1,u_2,z) \tag{147}$$

provided that $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, $u_2 \in U_{2,\pi}$ and $z \in H_{\beta'}$. We remind the reader the next useful classical result which relates the coefficients of an asymptotic expansion of a holomorphic map f to its high order derivatives.

Proposition ([2], Proposition 8, p. 66) *Let $f : G \to \mathbb{F}$ be a holomorphic map from a bounded open sector G centered at 0 into a complex Banach space $\mathbb{F}$ endowed with a norm $||.||_{\mathbb{F}}$. The following two statements are equivalent*

- *There exists a formal power series $\hat{f}(z) = \sum_{n\geq 0} f_n z^n/n!$ with coefficients f_n in $\mathbb{F}$ subjected to the next feature. For all closed subsector S of G centered at 0, there exists a sequence $(c(N,S))_{N\geq 0}$ of positive real numbers such that*
$$||f(z) - \sum_{n=0}^{N-1} f_n z^n/n!||_{\mathbb{F}} \leq c(N,S)|z|^N$$
for all $z \in S$, all integers $N \geq 1$.
- *All derivatives of order n, $f^{(n)}(z)$ are continuous at the origin and there exists a sequence $(f_n)_{n\geq 0}$ of elements in $\mathbb{F}$ such that*
$$\lim_{z\to 0, z\in G} ||f^{(n)}(z) - f_n||_{\mathbb{F}} = 0$$
for all integers $n \geq 0$.

As a result of the above proposition, we deduce from the asymptotic expansion (137) in the particular case $p = p_1$ (meaning that $\mathfrak{d}_{p_1} = \pi$) the next limits

$$\lim_{\substack{u_2\in U_{2,\pi}\\ u_2\to 0}} \sup_{\substack{u_1\in\mathcal{R}_{d_1,\Delta_1}\cap D_{R_1}\\ z\in H_{\beta'}}} |\partial_{u_2}^m U_{d_1,\pi}(u_1,u_2,z) - G_m(u_1,z)| = 0 \tag{148}$$

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for all integers $m \geq 0$. On the basis of the above limits, in order to reach recursion relations for the coefficients G_m, $m \geq 0$, our strategy consists in searching for recursion relations for the related $m-$th derivatives of the map $U_{d_1,\pi}$ relatively to u_2. On the way, we need to take the $m-$th derivative with respect to u_2 of the left and right handside of the equation (147). However, the equation (147) involves composition of $U_{d_1,\pi}$ with explicit homographic maps and we are asked to explicitely compute their higher order derivatives. In order to overcome this difficulty, we will apply a rule to evaluate high order derivatives of compositions of functions which has been introduced in [16] and is suitable for Gevrey estimates. This rather new identity allows us to avoid computations with the cumbersome combinatorial classical Faa-Di-Bruno formula and enables us to present very practical recursions relations. Indeed, we recall this higher order chain rule (Theorem 2.1 in [16]) under stronger assumptions (which will be sufficient for our scope) as stated in the previous work of the author [13].

Lemma 7. *Let D, G be open sets in $\mathbb{C}$. Let $g : D \to G$ and $f : G \to \mathbb{C}$ be holomorphic functions. Then, the n-th order derivative of the composite function $f \circ g : D \to \mathbb{C}$ is given by the formula*

$$\partial_x^n (f \circ g)(x) = \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} (\partial_x^j f)(g(x)) \left\{ \partial_h^{n-j} [\int_0^1 g'(x+\theta h) d\theta]^j \right\}_{|h=0},$$

for all integers $n \geq 1$ and $x \in D$.

In the next lemma, we perform an auxiliary computation which entails the homographic maps appearing in the main equation (147).

Lemma 8. *For any integer $l \geq 0$, we set*

$$g_l(u_2) = \frac{u_2}{1 + u_2 l \log(q)}.$$

Then, for all integers $n, j \geq 1$ with $n - j \geq 0$, the next identity

$$\left\{ \partial_h^{n-j} [\int_0^1 g_l'(u_2 + \theta h) d\theta]^j \right\}_{|h=0} = \frac{1}{(1 + u_2 l \log(q))^{j+n}} (l \log(q))^{n-j} \times j(j+1) \cdots (n-1) \times (-1)^{n-j} \quad (149)$$

holds for all $u_2 \in U_{2,\pi}$, with the convention that $j(j+1) \cdots (n-1) = 1$ when $j = n$.

Proof. Direct computations show that

$$g_l'(u_2) = \frac{1}{(1 + u_2 l \log(q))^2}$$

and hence

$$\int_0^1 g_l'(u_2 + \theta h) d\theta = \int_0^1 \frac{1}{(1 + (u_2 + \theta h) l \log(q))^2} d\theta = \frac{1}{(1 + u_2 l \log(q))(1 + u_2 l \log(q) + hl \log(q))}$$

for all $u_2 \in U_{2,\pi}$. We deduce that

$$[\int_0^1 g_l'(u_2 + \theta h) d\theta]^j = \frac{1}{(1 + u_2 l \log(q))^j} \times (1 + u_2 l \log(q) + hl \log(q))^{-j} \quad (150)$$

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for all $u_2 \in U_{2,\pi}$ and all integers $j \geq 1$. It follows from (150) that

$$\left\{ [\int_0^1 g_l'(u_2 + \theta h) d\theta]^j \right\}_{|h=0} = \frac{1}{(1 + u_2 l \log(q))^{j+j}} \tag{151}$$

which coincides with the formula (149) in the case $j = n \geq 1$ under the convention that $j(j+1)\cdots(n-1) = 1$. On the other hand, when $n - j \geq 1$, we deduce from (150) that

$$\begin{aligned} &\partial_h^{n-j} [\int_0^1 g_l'(u_2 + \theta h) d\theta]^j \\ &= \frac{1}{(1 + u_2 l \log(q))^j} (l \log(q))^{n-j} j(j+1)\cdots(n-1) \times (-1)^{n-j} \times (1 + u_2 l \log(q) + hl \log(q))^{-n} \end{aligned} \tag{152}$$

which yields the awaited identity (149) by setting $h = 0$ in the formula (152). □

On the ground of the above lemmas and based on equation (147), we can derive some recursion relation on the sequence of $m-$th derivatives $\partial_{u_2}^m U_{d_1,\pi}(u_1, u_2, z)$ of $U_{d_1,\pi}$ with respect to u_2. Namely,

$$\begin{aligned} &Q(\partial_z) \partial_{u_2}^m U_{d_1,\pi}(u_1, u_2, z) \\ &= u_1^{d_D} \sigma_{q;u_1}^{\frac{d_D}{k_1}} R_D(\partial_z) \Big[\sum_{j=1}^m \frac{m!}{j!(m-j)!} (\partial_{u_2}^j U_{d_1,\pi})(u_1, \frac{u_2}{1 + u_2 \frac{d_D}{k_1} \log(q)}, z) \times \frac{1}{(1 + u_2 \frac{d_D}{k_1} \log(q))^{j+m}} \\ &\qquad \times (\frac{d_D}{k_1} \log(q))^{m-j} \times j(j+1)\cdots(m-1) \times (-1)^{m-j} \Big] \\ &\quad + \sum_{\underline{l} = (l_0, l_1) \in I} u_1^{l_0} \sigma_{q;u_1}^{l_1} c_{\underline{l}}(z) R_{\underline{l}}(z) \Big[\sum_{j=1}^m \frac{m!}{j!(m-j)!} (\partial_{u_2}^j U_{d_1,\pi})(u_1, \frac{u_2}{1 + u_2 l_1 \log(q)}, z) \\ &\quad \times \frac{1}{(1 + u_2 l_1 \log(q))^{j+m}} \times (l_1 \log(q))^{m-j} \times j(j+1)\cdots(m-1) \times (-1)^{m-j} \Big] \\ &\qquad + \partial_{u_2}^m F_\pi(u_1, u_2, z) \end{aligned} \tag{153}$$

for all $m \geq 1$, all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all $u_2 \in U_{2,\pi}$ and all $z \in H_{\beta'}$.

In the next step, we let u_2 tend to 0 on the sector $U_{2,\pi}$ in both identities (147) and (153). According to the limits (148) and bearing in mind that the maps $U_{d_1,\pi}(u_1, u_2, z)$ and $G_m(u_1, z)$ are holomorphic relatively to $(u_1, z) \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \times H_{\beta'}$, we get the next recursion relations for the coefficients G_m, $m \geq 0$. Namely,

$$\begin{aligned} Q(\partial_z) G_0(u_1, z) &= u_1^{d_D} \sigma_{q;u_1}^{\frac{d_D}{k_1}} R_D(\partial_z) G_0(u_1, z) \\ &\quad + \sum_{\underline{l} = (l_0, l_1) \in I} u_1^{l_0} \sigma_{q;u_1}^{l_1} c_{\underline{l}}(z) R_{\underline{l}}(\partial_z) G_0(u_1, z) + F_\pi(u_1, 0, z) \end{aligned} \tag{154}$$

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together with

$$
\begin{aligned}
Q(\partial_z)G_m(u_1,z) = u_1^{d_D}\sigma_{q;u_1}^{\frac{d_D}{k_1}}R_D(\partial_z)\Big[\sum_{j=1}^{m}\frac{m!}{j!(m-j)!}G_j(u_1,z)\\
\times(\frac{d_D}{k_1}\log(q))^{m-j}\times j(j+1)\cdots(m-1)\times(-1)^{m-j}\Big]\\
+\sum_{\underline{l}=(l_0,l_1)\in I}u_1^{l_0}\sigma_{q;u_1}^{l_1}c_{\underline{l}}(z)R_{\underline{l}}(z)\Big[\sum_{j=1}^{m}\frac{m!}{j!(m-j)!}G_j(u_1,z)\\
\times(l_1\log(q))^{m-j}\times j(j+1)\cdots(m-1)\times(-1)^{m-j}\Big]+(\partial_{u_2}^m F_\pi)(u_1,0,z) \quad (155)
\end{aligned}
$$

for all $m \geq 1$, all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$ and $z \in H_{\beta'}$.

In the last part of the proof, we show that the formal power series (145) obey the functional equation (146). Our approach hinges on the next technical lemma where the Taylor expansion of the composition of the formal series (145) with some homographic map is explicitely computed.

Lemma 9. *Let $l \geq 0$ be an integer. The next formal Taylor expansion*

$$
\begin{aligned}
&\hat{\mathbb{G}}(\frac{u_2}{1+u_2 l\log(q)})\\
&= G_0(u_1,z)+\sum_{m\geq 1}\Big(\sum_{1\leq j\leq m}G_j(u_1,z)(-1)^{m-j}(l\log(q))^{m-j}\frac{j(j+1)\cdots(m-1)}{j!(m-j)!}\Big)u_2^m \quad (156)
\end{aligned}
$$

holds, for all $u_1 \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$ and $z \in H_{\beta'}$.

Proof. By mere composition, we notice that

$$
\hat{\mathbb{G}}(\frac{u_2}{1+u_2 l\log(q)}) = \sum_{n\geq 0}G_n(u_1,z)\frac{1}{n!}\frac{u_2^n}{(1+u_2 l\log(q))^n} \quad (157)
$$

On the other hand, the geometric series allows to write

$$
\frac{1}{1+u_2 l\log(q)} = \sum_{h\geq 0}(-1)^h(l\log(q))^h u_2^h \quad (158)
$$

and taking its derivative of order $n \geq 0$ with respect to u_2 yields the expansion

$$
\frac{(-1)^n(l\log(q))^n n!}{(1+u_2 l\log(q))^{n+1}} = \sum_{h\geq n}(-1)^h(l\log(q))^h h(h-1)\cdots(h-(n-1))u_2^{h-n} \quad (159)
$$

with the notation $h(h-1)\cdots(h-(n-1)) = 1$ if $n = 0$ and $h(h-1)\cdots(h-(n-1)) = h$ if $n = 1$, for all $h \geq n$. From (159), for all integers $n \geq 1$, we deduce the next identity

$$
\begin{aligned}
\frac{u_2^n}{(1+u_2 l\log(q))^n} &= \frac{u_2^n}{(-1)^{n-1}(l\log(q))^{n-1}(n-1)!}\\
&\times\sum_{h\geq n-1}(-1)^h(l\log(q))^h h(h-1)\cdots(h-(n-2))u_2^{h-(n-1)}\\
&= \sum_{h\geq n-1}(-1)^{h-n+1}(l\log(q))^{h-n+1}\frac{h(h-1)\cdots(h-(n-2))}{(n-1)!}u_2^{h+1}\\
&= \sum_{h'\geq n}(-1)^{h'-n}(l\log(q))^{h'-n}\frac{(h'-1)(h'-2)\cdots(h'-(n-1))}{(n-1)!}u_2^{h'}. \quad (160)
\end{aligned}
$$

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As a result of (157) and (160), we deduce that

$$\hat{\mathbb{G}}(\frac{u_2}{1+u_2 l\log(q)}) = G_0(u_1,z) + \sum_{n\geq 1} G_n(u_1,z)\Big(\sum_{h\geq n}(-1)^{h-n}(l\log(q))^{h-n}\frac{(h-1)(h-2)\cdots(h-(n-1))}{n!(n-1)!}u_2^h\Big)$$
$$= G_0(u_1,z) + \sum_{h\geq 1}\Big(\sum_{1\leq n\leq h} G_n(u_1,z)(-1)^{h-n}(l\log(q))^{h-n}\frac{(h-1)(h-2)\cdots(h-(n-1))}{n!(n-1)!}\Big)u_2^h$$
$$= G_0(u_1,z) + \sum_{m\geq 1}\Big(\sum_{1\leq j\leq m} G_j(u_1,z)(-1)^{m-j}(l\log(q))^{m-j}\frac{(m-1)(m-2)\cdots(m-(j-1))}{j!(j-1)!}\Big)u_2^m \quad (161)$$

Besides, by straight calculus, we observe that

$$\frac{(m-1)(m-2)\cdots(m-(j-1))}{j!(j-1)!} = \frac{j(j+1)\cdots(m-1)}{j!(m-j)!} \quad (162)$$

for all $m\geq 1$ and $1\leq j\leq m$. Eventually, the combination of (161) and (162) yields the awaited formal Taylor expansion (156). □

According to the fact observed in (30) that the map F_π defines a polynomial in the variable u_2, it follows that its Taylor expansion

$$F_\pi(u_1,u_2,z) = \sum_{m\geq 0}\frac{(\partial_{u_2}^m F_\pi)(u_1,0,z)}{m!}u_2^m \quad (163)$$

is convergent (and actually a finite sum) near the origin with respect to u_2, for all $u_1\in\mathcal{R}_{d_1,\Delta_1}\cap D_{R_1}$ and $z\in H_{\beta'}$.

At the very end of the proof, we observe by plugging the expansions (156) and (163) into the equation (146) that the series $\hat{\mathbb{G}}(u_2)$ formally solves (146) if its Taylor coefficients G_m, $m\geq 0$ fulfill the recursion relations (154), (155) which has been shown to hold. Proposition 6 follows. □

5. Fine structure of Gevrey/$q-$Gevrey asymptotic expansions in combined power and logarithmic scales for the holomorphic solution to the initial value problem (21).

5.1. *Solving the convolution $q-$difference equation (48) on some neighborhood of the origin*

In order to study the equation (48) in the Borel space near the origin in $\mathbb{C}^2$ and Fourier space on $\mathbb{R}$, we introduce the next Banach space.

Definition 7. *Let $\beta,\mu,\rho>0$ be real numbers. For a given real number $b>0$, we denote $E_{(b,\rho,\beta,\mu)}$ the vector space of all continuous $\mathbb{C}-$valued functions $(\tau_1,\tau_2,m)\mapsto h(\tau_1,\tau_2,m)$ on $D_b\times D_\rho\times\mathbb{R}$, holomorphic with respect to (τ_1,τ_2) on $D_b\times D_\rho$, such that the norm*

$$||h(\tau_1,\tau_2,m)||_{(b,\rho,\beta,\mu)} := \sup_{\substack{\tau_1\in D_b,\tau_2\in D_\rho\\ m\in\mathbb{R}}}(1+|m|)^\mu e^{\beta|m|}\frac{1}{|\tau_2|}|h(\tau_1,\tau_2,m)| \quad (164)$$

is finite. The vector space $E_{(b,\rho,\beta,\mu)}$ endowed with the norm $||.||_{(b,\rho,\beta,\mu)}$ is a Banach space.

We plan to solve the next convolution $q-$difference equation

$$
\begin{aligned}
&Q(\sqrt{-1}m)\omega(\tau_1,\tau_2,m)\\
&\quad = R_D(\sqrt{-1}m)\frac{\tau_1^{d_D}}{(q^{1/k_1})^{d_D(d_D-1)/2}}\exp\big(-\tau_2\frac{d_D}{k_1}\log(q)\big)\omega(\tau_1,\tau_2,m)\\
&\quad + \sum_{\underline{l}=(l_0,l_1)\in I}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\frac{\tau_1^{l_0}}{(q^{1/k_1})^{l_0(l_0-1)/2}}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1)\\
&\qquad \times\exp(-\tau_2 l_1\log(q))dm_1 + \mathcal{F}(\tau_1,\tau_2,m) \qquad (165)
\end{aligned}
$$

provided that $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$, with some function ω in the Banach space $(E_{(b,\rho,\beta,\mu)}, ||.||_{(b,\rho,\beta,\mu)})$.

In preparation for achieving our goal, we rearrange the equation (165) as a fixed point equation (disclosed later on in (188)). Along the road, we need to divide our equation by the map $P_m(\tau_1,\tau_2)$ displayed in (49) whenever $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and the mode m belongs to $\mathbb{R}$. Lower bounds for the map P_m are provided in the next lemma.

Lemma 10. *Let the inner radius r_{Q,R_D}, outer radius R_{Q,R_D} and aperture of S_{Q,R_D} introduced in Subsection 2.2 be chosen as in Lemma 4. Let $\rho > 0$ be the radius fixed in Lemma 4. Then, for a proper choice of radius $b > 0$, taken close enough to 0, one can find a constant $\hat{K}_{d_D,k_1,q}$ with*

$$
|P_m(\tau_1,\tau_2)| \geq |Q(\sqrt{-1}m)|\hat{K}_{d_D,k_1,q} \qquad (166)
$$

for all $\tau_1 \in D_b$, all $\tau_2 \in D_\rho$, all $m \in \mathbb{R}$.

Proof. Take a fixed $\tilde{\tau}_2^0 \in D_\rho$. We introduce the complex number

$$
\begin{aligned}
\tilde{\tau}_1^0 &= \Big[\Big|\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)}\Big|(q^{1/k_1})^{d_D(d_D-1)/2}\big|\exp\big(\tilde{\tau}_2^0\frac{d_D}{k_1}\log(q)\big)\big|\Big]^{1/d_D}\\
&\quad\times\exp\Big(\frac{\sqrt{-1}}{d_D}\Big[\arg\Big(\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)}(q^{1/k_1})^{d_D(d_D-1)/2}\exp\big(\tilde{\tau}_2^0\frac{d_D}{k_1}\log(q)\big)\Big)\Big]\Big) \qquad (167)
\end{aligned}
$$

Observe that $\tilde{\tau}_1^0$ remains bounded and parked in a small domain we denote $\tilde{\mathcal{T}}_1^0$ which is located at some small positive distance of the origin, when m varies within the real numbers, owing to the requirement (25). We select the radius $b > 0$ accordingly to the condition

$$
D_b \cap \tilde{\mathcal{T}}_1^0 = \emptyset. \qquad (168)
$$

Now, let us take an arbitrary complex number $\tau_2 \in D_\rho$. We decompose it in the form

$$
\tau_2 = \tilde{\tau}_2^0 - (\tilde{s} + \sqrt{-1}\tilde{\psi}) \qquad (169)
$$

for some real numbers $\tilde{s}, \tilde{\psi}$ close to 0 for $\tilde{\tau}_2^0$ given above. By construction of $\tilde{\tau}_1^0$ in (167), the next identity

$$
(\tilde{\tau}_1^0)^{d_D} = \frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)}(q^{1/k_1})^{d_D(d_D-1)/2}\exp\big(\tilde{\tau}_2^0\frac{d_D}{k_1}\log(q)\big) \qquad (170)
$$

holds. Select some arbitrary $\tau_1 \in D_b$. We split it in a factorized form

$$
\tau_1 = \tilde{\tau}_1^0\tilde{r}e^{\sqrt{-1}\tilde{\theta}} \qquad (171)
$$

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for some angle $\tilde{\theta} \in \mathbb{R}$ and radius $\tilde{r}$ with the constraint $0 \leq \tilde{r} < b/|\tilde{\tau}_1^0|$. The combined splitting (169) and (171) together with the identity (170) enables the factorisation of the map

$$
\begin{aligned}
P_m(\tau_1,\tau_2) &= Q(\sqrt{-1}m) - R_D(\sqrt{-1}m)\frac{(\tilde{\tau}_1^0)^{d_D}\tilde{r}^{d_D}\exp(\sqrt{-1}d_D\tilde{\theta})}{(q^{1/k_1})^{d_D(d_D-1)/2}} \\
&\quad \times \exp(-\tilde{\tau}_2^0\frac{d_D}{k_1}\log(q)) \times \exp((\tilde{s}+\sqrt{-1}\tilde{\psi})\frac{d_D}{k_1}\log(q)) \\
&= Q(\sqrt{-1}m)\left[1 - \tilde{r}^{d_D}e^{\sqrt{-1}d_D\tilde{\theta}} \times \exp((\tilde{s}+\sqrt{-1}\tilde{\psi})\frac{d_D}{k_1}\log(q))\right]. \quad (172)
\end{aligned}
$$

Besides, provided that the radius $b > 0$ is chosen in the vicinity of the origin, we can find a constant $\hat{K}_{d_D,k_1,q} > 0$ with

$$
\left|1 - \tilde{r}^{d_D}e^{\sqrt{-1}d_D\tilde{\theta}} \times \exp((\tilde{s}+\sqrt{-1}\tilde{\psi})\frac{d_D}{k_1}\log(q))\right| \geq \hat{K}_{d_D,k_1,q} \quad (173)
$$

for all $0 \leq \tilde{r} < b/|\tilde{\tau}_1^0|$, all $\tilde{\theta} \in \mathbb{R}$, all $\tilde{s}, \tilde{\psi}$ close to 0. At last, the factorization (172) and the lower bounds (173) give rise to (166). □

In the ongoing proposition, we check that the map $\mathcal{H}$ introduced in (58) represents a shrinking map on some appropriately selected ball in the Banach space examined in Definition 7.

Proposition 7. *We fix the sectorial domain S_{Q,R_D} and the radius ρ, b as in Lemma 10. Let $\beta, \mu > 0$ be real numbers fixed as in Subsection 2.2. Then, assuming that the constants $\mathbf{C}_{\underline{l}} > 0$ presented in (27) are small enough, for $\underline{l} \in I$, for all radius $\varpi_E > 0$ chosen large enough, the map $\mathcal{H}$ given by (58) is favoured with the next two features*

- *The inclusion*

$$
\mathcal{H}(\bar{B}_{\varpi_E}) \subset \bar{B}_{\varpi_E} \quad (174)
$$

is granted, where $\bar{B}_{\varpi_E}$ denotes the closed ball of radius ϖ_E centered at 0 in the space $E_{(b,\rho,\beta,\mu)}$.

- *The $1/2-$Lipschitz condition*

$$
||\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)||_{(b,\rho,\beta,\mu)} \leq \frac{1}{2}||\omega_1 - \omega_2||_{(b,\rho,\beta,\mu)} \quad (175)
$$

holds for all $\omega_1, \omega_2 \in \bar{B}_{\varpi_E}$.

In particular, since the radius ϖ_E can be taken arbitrarily large, we observe that the map $\mathcal{H}$ turns out to be well defined on the whole space $E_{(b,\rho,\beta,\mu)}$ where the shrinking property (175) holds true.

Proof. Let us focus on the first item of the proposition. We first provide bounds for the forcing term $\mathcal{F}/P_m$ of $\mathcal{H}$ disclosed in the next

Lemma 11. *There exists a constant $\hat{\mathbf{F}}_{\mathcal{F}} > 0$ such that*

$$
||\frac{\mathcal{F}(\tau_1,\tau_2,m)}{P_m(\tau_1,\tau_2)}||_{(b,\rho,\beta,\mu)} \leq \hat{\mathbf{F}}_{\mathcal{F}}. \quad (176)
$$

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Proof. Owing to the lower bounds (166) and the definition (28) of $\mathcal{F}$ together with the bounds (91), we arrive at

$$\left|\frac{\mathcal{F}(\tau_1,\tau_2,m)}{P_m(\tau_1,\tau_2)}\right| \leq \frac{\sum_{j_1\in J_1,j_2\in J_2}|\mathcal{F}_{j_1,j_2}(m)[|\tau_1|^{j_1}|\tau_2|^{j_2}}{|Q(\sqrt{-1}m)|\hat{K}_{d_D,k_1,q}}$$
$$\leq \frac{1}{\min_{m\in\mathbb{R}}|Q(\sqrt{-1}m)|\hat{K}_{d_D,k_1,q}}(1+|m|)^{-\mu}e^{-\beta|m|}\Big[\sum_{j_1\in J_1,j_2\in J_2}\mathbf{F}_{j_1,j_2}b^{j_1}|\tau_2|^{j_2-1}\Big]\times|\tau_2|$$
$$\leq \hat{\mathbf{F}}_{\mathcal{F}}(1+|m|)^{-\mu}e^{-\beta|m|}|\tau_2| \quad (177)$$

for all $\tau_1 \in D_b$, $\tau_2 \in D_\rho$, all $m \in \mathbb{R}$, where

$$\hat{\mathbf{F}}_{\mathcal{F}} = \frac{1}{\min_{m\in\mathbb{R}}|Q(\sqrt{-1}m)|\hat{K}_{d_D,k_1,q}}\Big[\sum_{j_1\in J_1,j_2\in J_2}\mathbf{F}_{j_1,j_2}b^{j_1}\rho^{j_2-1}\Big]$$

paying regard to the fact that $0 \notin J_2 \subset \mathbb{N}^*$. At last, the expected bounds (176) follow from (177) and Definition 7. □

In the next lemma, we come up with bounds for the linear part of the map $\mathcal{H}$.

Lemma 12. *One can find a constant $C_2 > 0$ such that*

$$||\frac{1}{P_m(\tau_1,\tau_2)}\int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1)$$
$$\times\exp\big(-\tau_2 l_1\log(q)\big)dm_1||_{(b,\rho,\beta,\mu)} \leq C_2\mathbf{C}_{\underline{l}}||\omega(\tau_1,\tau_2,m)||_{(b,\rho,\beta,\mu)} \quad (178)$$

for all $\omega(\tau_1,\tau_2,m) \in E_{(b,\rho,\beta,\mu)}$.

Proof. Let us take $\omega \in E_{(b,\rho,\beta,\mu)}$. We provide bounds for the function

$$\mathcal{B}(\tau_1,\tau_2,m) := \frac{1}{P_m(\tau_1,\tau_2)}\int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1)R_{\underline{l}}(\sqrt{-1}m_1)$$
$$\times\tau_1^{l_0}\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1)\times\exp\big(-\tau_2 l_1\log(q)\big)dm_1. \quad (179)$$

By definition of the space $E_{(b,\rho,\beta,\mu)}$, we notice that

$$|\omega(\tau_1,\tau_2,m)| \leq ||\omega||_{(b,\rho,\beta,\mu)}(1+|m|)^{-\mu}e^{-\beta|m|}|\tau_2|$$

for all $\tau_1 \in D_b$, all $\tau_2 \in D_\rho$ and all $m \in \mathbb{R}$. Owing to the assumption (22), we notice that $q^{l_1-\frac{l_0}{k_1}}\tau_1 \in D_b$ provided that $\tau_1 \in D_b$. Hence,

$$|\sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}}\omega(\tau_1,\tau_2,m_1)| \leq ||\omega||_{(b,\rho,\beta,\mu)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}|\tau_2| \quad (180)$$

whenever $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and $m_1 \in \mathbb{R}$. Then, according to the lower bounds (166) together with (180), we deduce that

$$|\mathcal{B}(\tau_1,\tau_2,m)| \leq b^{l_0}\exp(\rho l_1\log(q))\frac{1}{\hat{K}_{d_D,k_1,q}}||\omega||_{(b,\rho,\beta,\mu)}|\tau_2|$$
$$\times\frac{1}{|Q(\sqrt{-1}m)|}\int_{-\infty}^{+\infty}|C_{\underline{l}}(m-m_1)||R_{\underline{l}}(\sqrt{-1}m_1)|(1+|m_1|)^{-\mu}e^{-\beta|m_1|}dm_1 \quad (181)$$

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and bearing in mind the estimates (69) where the map $\mathcal{A}_1(m)$ is introduced in (64), we reach

$$|\mathcal{B}(\tau_1,\tau_2,m)| \leq \frac{\mathbf{C}_{\underline{l}}\mathbf{R}_{\underline{l}}}{\mathbf{Q}\hat{K}_{d_D,k_1,q}} C_{1.1} b^{l_0} \exp(\rho l_1 \log(q))||\omega||_{(b,\rho,\beta,\mu)} \times \left[(1+|m|)^{-\mu} e^{-\beta|m|}|\tau_2|\right] \quad (182)$$

for all $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. At last, we arrive at some constant $C_2 > 0$ for which the norm bounds

$$||\mathcal{B}(\tau_1,\tau_2,m)||_{(b,\rho,\beta,\mu)} \leq C_2 \mathbf{C}_{\underline{l}} ||\omega||_{(b,\rho,\beta,\mu)} \quad (183)$$

holds. □

Now, we select the constants $\mathbf{C}_{\underline{l}} > 0$, for $\underline{l} \in I$, small enough and take a radius $\varpi_E > 0$ large enough in a way that the next inequality

$$\sum_{\underline{l}=(l_0,l_1)\in I} \frac{1}{(2\pi)^{1/2}(q^{1/k_1})^{l_0(l_0-1)/2}} C_2 \mathbf{C}_{\underline{l}} \varpi_E + \hat{\mathbf{F}}_{\mathcal{F}} \leq \varpi_E \quad (184)$$

holds where the constant $C_2 > 0$ appears in Lemma 12 and $\hat{\mathbf{F}}_{\mathcal{F}}$ shows up in Lemma 11. Eventually, the bounds (176) along with (178) under the restriction (184) trigger the expected inclusion (174).

In the second part of the proof, we address the shrinking property (175). Let us choose two arbitrary elements ω_1, ω_2 in the closed ball $\bar{B}_{\varpi_E}$ whose radius has been prescribed in the first item (174). Owing to Lemma 12, the following inequality

$$\begin{aligned} ||\frac{1}{P_m(\tau_1,\tau_2)} \int_{-\infty}^{+\infty} C_{\underline{l}}(m-m_1) R_{\underline{l}}(\sqrt{-1}m_1) \tau_1^{l_0} \sigma_{q;\tau_1}^{l_1-\frac{l_0}{k_1}} (\omega_1(\tau_1,\tau_2,m_1) - \omega_2(\tau_1,\tau_2,m_1)) \\ \times \exp\left(-\tau_2 l_1 \log(q)\right) dm_1||_{(b,\rho,\beta,\mu)} \\ \leq C_2 \mathbf{C}_{\underline{l}} ||\omega_1(\tau_1,\tau_2,m) - \omega_2(\tau_1,\tau_2,m)||_{(b,\rho,\beta,\mu)} \end{aligned} \quad (185)$$

holds for the constant $C_2 > 0$ stemming from Lemma 12. We prescribe the constants $\mathbf{C}_{\underline{l}} > 0$, for $\underline{l} \in I$, small enough allowing the next inequality

$$\sum_{\underline{l}=(l_0,l_1)\in I} \frac{1}{(2\pi)^{1/2}(q^{1/k_1})^{l_0(l_0-1)/2}} C_2 \mathbf{C}_{\underline{l}} \leq \frac{1}{2} \quad (186)$$

to hold. The Lipschitz property (175) is a straight consequence of (185) under the requirement (186).

In the end, we suitably select the constants $\mathbf{C}_{\underline{l}} > 0$, $\underline{l} \in I$ small enough and a radius $\varpi_E > 0$ large enough in order that both constraints (184) and (186) are granted at once. This induces the two features (174) and (175) for the map $\mathcal{H}$. □

The next proposition provides a solution to the convolution $q-$difference equation (165) inside the space $E_{(b,\rho,\beta,\mu)}$.

Proposition 8. *We prescribe the sectorial domain S_{Q,R_D} together with the radius ρ, b as in Lemma 10. Let $\beta, \mu > 0$ be real numbers fixed as in Subsection 2.2. Assume that the constants $\mathbf{C}_{\underline{l}} > 0$, $\underline{l} \in I$, are chosen small enough in a suitable way as in Proposition 7. Then, for all radius $\varpi_E > 0$ large enough, a unique solution $\omega_{b,\rho}$ to the convolution $q-$difference equation (165) can be constructed in the space $E_{(b,\rho,\beta,\mu)}$ under the requirement*

$$||\omega_{b,\rho}||_{(b,\rho,\beta,\mu)} \leq \varpi_E. \quad (187)$$

Proof. Select a radius $\varpi_E > 0$ as in Proposition 7. The closed ball $\bar{B}_{\varpi_E} \subset E_{(b,\rho,\beta,\mu)}$ stands for a complete metric space for the distance $\tilde{d}(x,y) = ||x-y||_{(b,\rho,\beta,\mu)}$. The proposition 7 claims that the map $\mathcal{H}$ induces a contractive map from the metric space $(\bar{B}_{\varpi_E}, \tilde{d})$ into itself. The classical Banach fixed point theorem allows the map $\mathcal{H}$ to possess a unique fixed point located inside de ball $\bar{B}_{\varpi_E}$ that we denote $\omega_{b,\rho}$. As a result, the next identity

$$\mathcal{H}(\omega_{b,\rho}(\tau_1,\tau_2,m)) = \omega_{b,\rho}(\tau_1,\tau_2,m) \tag{188}$$

holds provided $\tau_1 \in D_b$, $\tau_2 \in D_\rho$, for all $m \in \mathbb{R}$. At last, under the conditions imposed, we observe that the convolution $q-$difference equation (165) can exactly be rearranged after a division by the map $P_m(\tau_1,\tau_2)$ as (188). As a consequence, the unique fixed point $\omega_{b,\rho}$ obtained in $\bar{B}_{\varpi_E}$ fully solves (165). This yields Proposition 8. □

5.2. Link between the solutions $\omega_{d_1,\pi}$ and $\omega_{b,\rho}$ to the convolution $q-$difference equation (48), (165).

In order to unveil the analytic relation between the two solutions $\omega_{d_1,\pi}$ and $\omega_{b,\rho}$ to the same convolution $q-$difference equation considered in Subsection 3.2 and Subsection 5.1, we introduce a new auxiliary Banach space.

Definition 8. *Let $b,\rho > 0$ be given positive real numbers and let S_{d_1} be an unbounded sector edged at 0 with bisecting direction $d_1 \in \mathbb{R}$. We denote $E_{(b,\rho,\beta,\mu,S_{d_1})}$ the vector space of all continuous maps $(\tau_1,\tau_2,m) \mapsto h(\tau_1,\tau_2,m)$ on the product $(S_{d_1} \cap D_b) \times D_\rho \times \mathbb{R}$, holomorphic relatively to the couple (τ_1,τ_2) on the domain $(S_{d_1} \cap D_b) \times D_\rho$, for which the norm*

$$||h(\tau_1,\tau_2,m)||_{(b,\rho,\beta,\mu,S_{d_1})} := \sup_{\substack{\tau_1 \in S_{d_1} \cap D_b, \tau_2 \in D_\rho \\ m \in \mathbb{R}}} (1+|m|)^{\mu} e^{\beta|m|} \frac{1}{|\tau_2|} |h(\tau_1,\tau_2,m)| \tag{189}$$

is a finite quantity. The vector space $E_{(b,\rho,\beta,\mu,S_{d_1})}$ equipped with the norm $||.||_{(b,\rho,\beta,\mu,S_{d_1})}$ is a Banach space.

In the next proposition, we claim that the map $\mathcal{H}$ displayed in (58) is well defined on the space $E_{(b,\rho,\beta,\mu,S_{d_1})}$ where it boasts a $1/2-$Lipschitz feature.

Proposition 9. *We prescribe the sectorial domain S_{Q,R_D} and the radius b, ρ as in Lemma 10. We set the constants $\beta,\mu > 0$ as in Subsection 2.2. We select an unbounded sector S_{d_1} as in Lemma 4. Then, assuming that the constants $\mathbf{C}_{\underline{l}} > 0$ introduced in (27) are close enough to 0, for all $\underline{l} \in I$, the map $\mathcal{H}$ declared in (58) is well defined on the whole space $E_{(b,\rho,\beta,\mu,S_{d_1})}$ and is subjected to the next $1/2-$Lipschitz condition*

$$||\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)||_{(b,\rho,\beta,\mu,S_{d_1})} \leq \frac{1}{2}||\omega_1 - \omega_2||_{(b,\rho,\beta,\mu,S_{d_1})} \tag{190}$$

for all ω_1, ω_2 belonging to $E_{(b,\rho,\beta,\mu,S_{d_1})}$.

Proof. The proof of Proposition 9 mirrors in the very details the one of Proposition 7 and will not be presented in this work in order to avoid redundancy. □

The following proposition establish the awaited analytical connection between $\omega_{d_1,\pi}$ and $\omega_{b,\rho}$.

Proposition 10. *Let the sectorial domain S_{Q,R_D} and the radius b, ρ be prescribed as in Lemma 10. The constants $\beta,\mu > 0$ are set as in Subsection 2.2 and the unbounded sector S_{d_1} is chosen as in Lemma 4. Then, provided that the constants $\mathbf{C}_{\underline{l}} > 0$ given by (27) are taken in the vicinity of the origin for all $\underline{l} \in I$, the next identity*

$$\omega_{d_1,\pi}(\tau_1,\tau_2,m) = \omega_{b,\rho}(\tau_1,\tau_2,m) \tag{191}$$

holds for all $\tau_1 \in S_{d_1} \cap D_b$, all $\tau_2 \in D_\rho$, all $m \in \mathbb{R}$. In particular, for given $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$, the partial map $\tau_1 \mapsto \omega_{b,\rho}(\tau_1,\tau_2,m)$ is the analytic continuation of the partial map $\tau_1 \mapsto \omega_{d_1,\pi}(\tau_1,\tau_2,m)$ on the full disc D_b.

Proof. According to Proposition 2, we know that the map $\omega_{d_1,\pi}$ belongs to the Banach space $\mathrm{Exp}^{q;1}_{(k_1,\alpha,\delta,\nu,\beta,\mu,\rho)}$. According to Definition 8 it follows that the restricted map $(\tau_1,\tau_2,m) \mapsto \omega_{d_1,\pi}$, for $\tau_1 \in S_{d_1} \cap D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$ belongs to $E_{(b,\rho,\beta,\mu,S_{d_1})}$. On the other hand, we know from Proposition 8 that the map $\omega_{b,\rho}$ belongs to the space $E_{(b,\rho,\beta,\mu)}$. As a result, the restricted map $(\tau_1,\tau_2,m) \mapsto \omega_{b,\rho}(\tau_1,\tau_2,m)$ on $(S_{d_1} \cap D_b) \times D_\rho \times \mathbb{R}$ also belongs to $E_{(b,\rho,\beta,\mu,S_{d_1})}$. Furthermore, according to (98) and to (188), we observe in particular that the next two idendities

$$\mathcal{H}(\omega_{d_1,\pi}(\tau_1,\tau_2,m)) = \omega_{d_1,\pi}(\tau_1,\tau_2,m) \ , \ \mathcal{H}(\omega_{b,\rho}(\tau_1,\tau_2,m)) = \omega_{b,\rho}(\tau_1,\tau_2,m) \tag{192}$$

holds as functions provided that $\tau_1 \in S_{d_1} \cap D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. At last, if one sets $\omega_1 = \omega_{d_1,\pi}$ and $\omega_2 = \omega_{b,\rho}$ in the inequality (190), it follows from (192) that

$$||\omega_{d_1,\pi} - \omega_{b,\rho}||_{(b,\rho,\beta,\mu,S_{d_1})} \leq \frac{1}{2}||\omega_{d_1,\pi} - \omega_{b,\rho}||_{(b,\rho,\beta,\mu,S_{d_1})}.$$

It implies that $||\omega_{d_1,\pi} - \omega_{b,\rho}||_{(b,\rho,\beta,\mu,S_{d_1})} = 0$, from which the expected identity (191) follows. □

5.3. Statement of the second main result.

In this subsection, we exhibit a fine structure for the asymptotic expansion of Gevrey/$q-$Gevrey type for the solution $u(t,z)$ to the equation (139) which combines both a logarithmic scale and a power scale. The next statement represents the second deed of our work.

Theorem 2. *We consider the function $u(t,z)$ displayed in (140) which solves our main initial value problem (139) for vanishing initial data $u(0,z) \equiv 0$ built up in Theorem 1. Then, the map $u(t,z)$ can be broken up as a sum of two functions*

$$u(t,z) = u_1(t,z) + u_2(t,z) \tag{193}$$

where

- *the map $u_1(t,z)$ is bounded holomorphic on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ and possesses a generalized asymptotic expansion of so-called $q-$Gevrey type in a power scale as t tends to 0. It means that one can distinguish a formal power series*

$$\hat{u}_1(t,z) = \sum_{n\geq 0} b_n(t,z)t^n \tag{194}$$

with bounded coefficients $b_n(t,z)$ on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ which represents a generalized asymptotic expansion of $q-$Gevrey order k_1 in the scale of monomials $\{t^n\}_{n\geq 0}$ of the map $u_1(t,z)$ with respect to t on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0])$. Namely, two constants $B_1, B_2 > 0$ can be singled out for which the next error bounds

$$|u_1(t,z) - \sum_{n=0}^{N} b_n(t,z)t^n| \leq B_1(B_2)^{N+1} q^{\frac{N(N+1)}{2k_1}} |t|^{N+1} \tag{195}$$

hold for all integers $N \geq 0$, all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, provided that $z \in H_{\beta'}$.
- *the map $u_2(t,z)$ is bounded holomorphic on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ and carries the null formal series as asymptotic expansion of Gevrey order 1 in a logarithmic scale as t tends to 0. In other words, two constants $B_3, B_4 > 0$ can be identified in order that the following error bounds*

$$|u_2(t,z)| \leq B_3(B_4)^{N+1}\Gamma(N+2)|1/\log(t)|^{N+1} \tag{196}$$

hold for all integers $N \geq 0$, all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, as long as $z \in H_{\beta'}$.

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Proof. Our idea consists in the splitting of the triple integral representation of $u(t,z)$ given by (140) into three specific contributions

$$u(t,z) = v_1(t,z) + v_2(t,z) + v_3(t,z) \tag{197}$$

where

$$v_1(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{198}$$

and

$$v_2(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2}} \int_{L_{\pi,\rho/2,\infty}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{199}$$

in a row with

$$v_3(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2,\infty}} \int_{L_{\pi}} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{200}$$

where the integration paths are stated as follows

$$L_{d_1,b/2} = [0,b/2]e^{\sqrt{-1}d_1} \quad , \quad L_{d_1,b/2,\infty} = [b/2,+\infty)e^{\sqrt{-1}d_1}$$

along with

$$L_{\pi,\rho/2} = [0,\rho/2]e^{\sqrt{-1}\pi} \quad , \quad L_{\pi,\rho/2,\infty} = [\rho/2,+\infty)e^{\sqrt{-1}\pi},$$

where the positive real numbers $b,\rho > 0$ are prescribed in Lemma 10.

In the next first main proposition, we provide asymptotic expansions for the first piece $v_1(t,z)$ relatively to t.

Proposition 11. *There exists a sequence of maps $g_k(t,z)$, $k \geq 0$, that are well defined and bounded holomorphic relatively to (t,z) on the product $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ which are submitted to the bounds*

$$|g_k(t,z)| \leq \frac{M_1\varpi_E}{R^k} q^{\frac{k(k-1)}{2k_1}} |t|^k + \frac{M_1 k_1 \varpi_E}{\log(q) R^k} \frac{1}{C_{q,k_1}\Delta_1} \left(\frac{2\log(q)}{k_1}\right)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} |t|^k \tag{201}$$

for some well selected constants $M_1, \varpi_E > 0$ and $R > 0$, where $C_{q,k_1} > 0$ and $\Delta_1 > 0$ are the two constants arising in (15), for all integers $k \geq 0$, provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$ and $z \in H_{\beta'}$. For any given natural number $N \geq 0$, the next decomposition

$$v_1(t,z) = \sum_{k=0}^{N} g_k(t,z) + v_{1,N+1}(t,z) \tag{202}$$

holds for (t,z) on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$, where the remainder term $v_{1,N+1}(t,z)$ stands for a bounded holomorphic function on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$ and is monitored by means of the bounds

$$|v_{1,N+1}(t,z)| \leq \frac{k_1}{\log(q)} \frac{\varpi_E}{\tilde{R}^{N+1}} M_1 \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1}\Big)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}} |t|^{N+1} \tag{203}$$

for the constants $M_1, \varpi_E, C_{q,k_1}, \Delta_1$ appearing in (201) and for a suitable small radius $\tilde{R} > 0$, as long as $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$ and $z \in H_{\beta'}$.

Proof. Let $b,\rho > 0$ be fixed as in Lemma 10. Owing to Proposition 8, we know in particular that the partial map $\tau_1 \mapsto \omega_{b,\rho}(\tau_1,\tau_2,m)$ is bounded and analytic on the disc D_b for any prescribed $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. As a result, we can apply the Taylor formula with integral remainder of some fixed order $N \geq 0$ to that function and get the next expansion

$$\omega_{b,\rho}(\tau_1,\tau_2,m) = \sum_{k=0}^{N} \frac{(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)}{k!}\tau_1^k + \tau_1^{N+1}\int_0^1 \frac{(1-t)^N}{N!}(\partial_{\tau_1}^{N+1}\omega_{b,\rho})(t\tau_1,\tau_2,m)dt \tag{204}$$

provided that $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. According to Proposition 10, we know that the function $\omega_{d_1,\pi}(\tau_1,\tau_2,m)$ coincides with the map $\omega_{b,\rho}(\tau_1,\tau_2,m)$ for $\tau_1 \in S_{d_1} \cap D_b$, all $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. Hence, from the identity (204), we deduce the next development

$$\omega_{d_1,\pi}(\tau_1,\tau_2,m) = \sum_{k=0}^{N} \frac{(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)}{k!}\tau_1^k + \tau_1^{N+1}\int_0^1 \frac{(1-t)^N}{N!}(\partial_{\tau_1}^{N+1}\omega_{b,\rho})(t\tau_1,\tau_2,m)dt \tag{205}$$

for all $\tau_1 \in L_{d_1,b/2}$, all $\tau_2 \in L_{\pi,\rho/2}$ and all $m \in \mathbb{R}$. This last formula (205) enable the expansion of the map $v_1(t,z)$ in the form

$$\begin{aligned} v_1(t,z) = \sum_{k=0}^{N} g_k(t,z) + \frac{k_1}{\log(q)(2\pi)^{1/2}} \\ \times \int_{L_{d_1,b/2}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \Big[\tau_1^{N+1}\int_0^1 \frac{(1-u)^N}{N!}(\partial_{\tau_1}^{N+1}\omega_{b,\rho})(u\tau_1,\tau_2,m)du\Big] \\ \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm \end{aligned} \tag{206}$$

where

$$\begin{aligned} g_k(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \Big[\frac{(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)}{k!}\tau_1^k\Big] \\ \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2\big)e^{\sqrt{-1}zm}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}dm \end{aligned} \tag{207}$$

for $0 \leq k \leq N$.

In the next step, we provide upper bounds for the maps $g_k(t,z)$, $0 \leq k \leq N$. We first need to remind the reader the next formula

$$\frac{k_1}{\log(q)} \int_{L_{d_1}} \frac{u^{n-1}}{\Theta_{q^{1/k_1}}(u/t)}du = q^{\frac{n(n-1)}{2k_1}} t^n$$

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for all $t \in \mathcal{R}_{d_1,\Delta_1}$ which has been applied in our recent work [10], see Lemma 3 therein, from which we deduce the splitting

$$\frac{k_1}{\log(q)} \int_{L_{d_1,b/2}} \frac{\tau_1^{k-1}}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 = q^{\frac{k(k-1)}{2k_1}} t^k - \frac{k_1}{\log(q)} \int_{L_{d_1,b/2,\infty}} \frac{\tau_1^{k-1}}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 \tag{208}$$

for all $t \in \mathcal{R}_{d_1,\Delta_1}$. As a result, one can further break up the term g_k as follows

$$g_k(t,z) = a_k(t,z) q^{\frac{k(k-1)}{2k_1}} t^k - \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2,\infty}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \Big[\frac{(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)}{k!}\tau_1^k\Big] \\ \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2\big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \tag{209}$$

where

$$a_k(t,z) = \frac{1}{(2\pi)^{1/2}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \Big[\frac{(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)}{k!}\Big] \exp\big(-(\log(t))\tau_2\big) e^{\sqrt{-1}zm} \frac{d\tau_2}{\tau_2} dm \tag{210}$$

In the next lemma, we focus on bounds for the function $a_k(t,z)$.

Lemma 13. *For all $0 \le k \le N$, the map $a_k(t,z)$ is well defined and bounded holomorphic with respect to (t,z) on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$. Furthermore, there exists two constants $M_1 > 0$ and $0 < R < b$ such that*

$$|a_k(t,z)| \le \frac{M_1 \varpi_E}{R^k} \tag{211}$$

for all (t,z) on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$, provided that $0 < \beta' < \beta$.

Proof. We remind from Proposition 8 that the map $\omega_{b,\rho}$ belongs to the space $E_{(b,\rho,\beta,\mu)}$ and that a constant $\varpi_E > 0$ can be pinpointed with the bounds

$$|\omega_{b,\rho}(\tau_1,\tau_2,m)| \le \varpi_E (1+|m|)^{-\mu} e^{-\beta|m|} |\tau_2| \tag{212}$$

provided that $\tau_1 \in D_b$, $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$. Besides, from the classical Cauchy's formula, we know the next integral representation

$$(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m) = \frac{k!}{2\sqrt{-1}\pi} \int_{C_R} \frac{\omega_{b,\rho}(\xi,\tau_2,m)}{\xi^{k+1}} d\xi \tag{213}$$

to hold for $\tau_2 \in D_\rho$ and $m \in \mathbb{R}$, where the integration is realized along any positively oriented circle C_R centered at 0 with radius R subjected to $0 < R < b$. On account of (213) and the bounds (212), we reach the estimates

$$|(\partial_{\tau_1}^k \omega_{b,\rho})(0,\tau_2,m)| \le \frac{k!}{R^k} \varpi_E (1+|m|)^{-\mu} e^{-\beta|m|} |\tau_2| \tag{214}$$

for all $\tau_2 \in D_\rho$, all $m \in \mathbb{R}$. As a result of (214), we arrive at

$$|a_k(t,z)| \le \frac{1}{(2\pi)^{1/2}} \int_0^{\rho/2} \int_{-\infty}^{+\infty} \frac{1}{R^k} |\exp((\log(t))s)| \\ \times \varpi_E (1+|m|)^{-\mu} e^{-\beta|m|} e^{|\mathrm{Im}(z)||m|} ds dm \le \frac{M_1 \varpi_E}{R^k} \tag{215}$$

where

$$M_1 = \frac{1}{(2\pi)^{1/2}} \int_0^{\rho/2} e^{\log(R_1)s} ds \times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \tag{216}$$

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for all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$, all $z \in H_{\beta'}$ with $0 < \beta' < \beta$. $\square$

In the next lemma, bounds for the second piece of (209) are determined.

Lemma 14. *For all $0 \leq k \leq N$, the map*

$$g_{k,1}(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2,\infty}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \Big[\frac{(\partial_{\tau_1}^{k} \omega_{b,\rho})(0,\tau_2,m)}{k!} \tau_1^k \Big] \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\big(-(\log(t))\tau_2 \big) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1} \frac{d\tau_2}{\tau_2} dm \quad (217)$$

is well defined and stand for a bounded holomorphic function relatively to (t,z) on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$. In addition, the next upper bounds

$$|g_{k,1}(t,z)| \leq \frac{M_1 k_1 \varpi_E}{\log(q) R^k} \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} |t|^k \quad (218)$$

hold for all $t \in ((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0])$, $z \in H_{\beta'}$, where the constants $M_1 > 0$ and $R > 0$ are prescribed in Lemma 13 and where $C_{q,k_1} > 0$ and $\Delta_1 > 0$ are the two constants appearing in (15).

Proof. The technical estimates displayed in the next lemma are crucial.

Lemma 15. *The next inequality*

$$\Big| \int_{L_{d_1,b/2,\infty}} \tau_1^{k-1} \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 \Big| \leq \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} |t|^k \quad (219)$$

holds for all $t \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$, all integers $k \geq 0$, where $C_{q,k_1} > 0$ and $\Delta_1 > 0$ are the two constants appearing in (15).

Proof. Owing to (15), we first observe that

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|} \leq \frac{1}{C_{q,k_1}\Delta_1} \exp\Big(-\frac{k_1}{2} \frac{\log^2(|\tau_1|/|t|)}{\log(q)} \Big) \frac{1}{|\tau_1/t|^{1/2}} \quad (220)$$

for all $\tau_1 \in L_{d_1,b/2,\infty}$ and $t \in \mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}$. Based on (220), we deduce that

$$\Big| \int_{L_{d_1,b/2,\infty}} \tau_1^{k-1} \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 \Big| \leq \frac{1}{C_{q,k_1}\Delta_1} \int_{b/2}^{+\infty} r_1^{k-3/2} \exp\Big(-\frac{k_1}{2} \frac{\log^2(r_1/|t|)}{\log(q)} \Big) dr_1 \times |t|^{1/2} = \frac{1}{C_{q,k_1}\Delta_1} I_{k,|t|,b} |t|^k \quad (221)$$

where the quantity $I_{k,|t|,b}$ is derived by performing the change of variable $s_1 = r_1/|t|$ in the integral along the segment $[b/2, +\infty)$ above and stands for

$$I_{k,|t|,b} = \int_{\frac{b}{2|t|}}^{+\infty} s_1^{k-3/2} \exp\Big(-\frac{k_1}{2} \frac{\log^2(s_1)}{\log(q)} \Big) ds_1. \quad (222)$$

In the next step, we reach upper bounds for $I_{k,|t|,b}$. By coarse upper estimates, we first get

$$I_{k,|t|,b} \leq I_{k,0} = \int_{0}^{+\infty} s_1^{k-3/2} \exp\Big(-\frac{k_1}{2} \frac{\log^2(s_1)}{\log(q)} \Big) ds_1. \quad (223)$$

Then, at last, we show that the constant $I_{k,0}$ can be computed in an exact manner. Indeed, we make the change of variable

$$t_1 = \left(\frac{k_1}{2\log(q)}\right)^{1/2} \log(s_1)$$

in the integral $I_{k,0}$, which gives rise to

$$I_{k,0} = \int_{-\infty}^{+\infty} \exp\left((k - \frac{1}{2}) (\frac{2\log(q)}{k_1})^{1/2} t_1 \right) \exp(-t_1^2) dt_1 \times (\frac{2\log(q)}{k_1})^{1/2}. \quad (224)$$

On the other hand, we recall the Gaussian identity

$$e^{A^2/4}\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2 - Ax} dx$$

which is valid for any given real number $A \in \mathbb{R}$, that has been already used in our former paper [11] and stems from the book [1], Chapter 10, p. 498. This last identity enables the straight computation of (224) as follows

$$I_{k,0} = \left(\frac{2\log(q)}{k_1}\right)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} \quad (225)$$

for any integer $k \geq 0$.

Eventually, we gather all the above bounds (221), (223) and (225) and arrive at (219). □

With the help of the above lemma, we achieve bounds for the map $g_{k,1}(t,z)$. Indeed, from (219) in a row with (214), we get

$$\begin{aligned} |g_{k,1}(t,z)| &\leq \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{1}{C_{q,k_1}\Delta_1} \left(\frac{2\log(q)}{k_1}\right)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} |t|^k \\ &\times \int_0^{\rho/2} \int_{-\infty}^{+\infty} \frac{1}{R^k} \varpi_E |e^{s\log(t)}| (1+|m|)^{-\mu} e^{-\beta|m|} |e^{\sqrt{-1}zm}| ds dm \\ &\leq \frac{M_1 k_1 \varpi_E}{\log(q) R^k} \frac{1}{C_{q,k_1}\Delta_1} \left(\frac{2\log(q)}{k_1}\right)^{1/2} \sqrt{\pi} q^{\frac{(k-\frac{1}{2})^2}{2k_1}} |t|^k \quad (226) \end{aligned}$$

for all $t \in ((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0])$, $z \in H_{\beta'}$ for the constant $M_1 > 0$ defined in (216). □

In the next lemma, we address bounds for the remainder part of the expansion (206) for $v_1(t,z)$.

Lemma 16. *Let us denote*

$$\begin{aligned} v_{1,N+1}(t,z) &= \frac{k_1}{\log(q)(2\pi)^{1/2}} \\ &\times \int_{L_{d_1,b/2}} \int_{L_{\pi,\rho/2}} \int_{-\infty}^{+\infty} \left[\tau_1^{N+1} \int_0^1 \frac{(1-u)^N}{N!} (\partial_{\tau_1}^{N+1} \omega_{b,\rho})(u\tau_1, \tau_2, m) du \right] \\ &\times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2 \right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1} \frac{d\tau_2}{\tau_2} dm \quad (227) \end{aligned}$$

the tail piece of (206). The map $v_{1,N+1}(t,z)$ is well defined and represents a bounded holomorphic function on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta'}$. Moreover, the next estimates

$$|v_{1,N+1}(t,z)| \leq \frac{k_1}{\log(q)} \frac{\varpi_E}{\tilde{R}^{N+1}} M_1 \frac{1}{C_{q,k_1}\Delta_1} \left(\frac{2\log(q)}{k_1}\right)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}} |t|^{N+1} \quad (228)$$

hold whenever $t \in ((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0])$ *and* $z \in H_{\beta'}$*, for the constant* $M_1 > 0$ *defined in (216) and where* $\tilde{R} > 0$ *is some fixed small radius.*

Proof. We first need to upper bound the next quantity

$$J_N = \Big| \int_{L_{d_1,b/2}} \tau_1^N \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 \Big| \tag{229}$$

relatively to t and N. Indeed, owing to (220), we deduce

$$J_N \leq \frac{1}{C_{q,k_1}\Delta_1} \int_0^{b/2} r_1^{N-\frac{1}{2}} \exp\Big(-\frac{k_1}{2} \frac{\log^2(r_1/|t|)}{\log(q)} \Big) dr_1 \times |t|^{1/2} = \frac{1}{C_{q,k_1}\Delta_1} \tilde{I}_{N,|t|,b} |t|^{N+1} \tag{230}$$

where the element $\tilde{I}_{N,|t|,b}$ is obtained by applying the change of variable $s_1 = r_1/|t|$ in the integral along the segment $[0, b/2]$ overhead and stands for

$$\tilde{I}_{N,|t|,b} = \int_0^{\frac{b}{2|t|}} s_1^{N-\frac{1}{2}} \exp\Big(-\frac{k_1}{2} \frac{\log^2(s_1)}{\log(q)} \Big) ds_1. \tag{231}$$

In the next step, we merely observe that

$$\tilde{I}_{N,|t|,b} \leq I_{N+1,0} = \int_0^{+\infty} s_1^{N-\frac{1}{2}} \exp\Big(-\frac{k_1}{2} \frac{\log^2(s_1)}{\log(q)} \Big) ds_1, \tag{232}$$

where $I_{N+1,0}$ is given in the inequality (223). According to the computation made in (225), we notice that

$$I_{N+1,0} = \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}}. \tag{233}$$

At last, with the combination of (230), (231), (232) and (233) we arrive at

$$J_N \leq \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}} |t|^{N+1} \tag{234}$$

provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$, for any given integer $N \geq 0$.

Besides, owing to the classical Cauchy's formula, the next integral representation

$$(\partial_{\tau_1}^{N+1} \omega_{b,\rho})(u\tau_1, \tau_2, m) = \frac{(N+1)!}{2\sqrt{-1}\pi} \int_{C_{\tilde{R}}(u\tau_1)} \frac{\omega_{b,\rho}(\xi, \tau_2, m)}{(\xi - u\tau_1)^{N+2}} d\xi \tag{235}$$

holds for all $\tau_1 \in D_{b/2}$, $\tau_2 \in D_\rho$, $u \in [0,1]$ and $m \in \mathbb{R}$, where the integration is performed along a positively oriented circle $C_{\tilde{R}}(u\tau_1)$ centered at $u\tau_1$ with small radius $\tilde{R} > 0$ chosen in a way that $C_{\tilde{R}}(u\tau_1) \subset D_b$. From (235) together with (212), we deduce the useful bounds

$$|(\partial_{\tau_1}^{N+1} \omega_{b,\rho})(u\tau_1, \tau_2, m)| \leq \frac{(N+1)!}{\tilde{R}^{N+1}} \varpi_E (1+|m|)^{-\mu} e^{-\beta|m|} |\tau_2| \tag{236}$$

for all $\tau_1 \in D_{b/2}$, $\tau_2 \in D_\rho$, $u \in [0,1]$ and $m \in \mathbb{R}$. Eventually, the gathering of (234) and (236) gives rise to

$$|v_{1,N+1}(t,z)| \leq \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{(N+1)!}{\tilde{R}^{N+1}} \varpi_E \times \big(\int_0^1 \frac{(1-u)^N}{N!} du \big)$$
$$\times \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}} |t|^{N+1} \times \int_0^{\rho/2} e^{s \log|t|} ds \times \int_{-\infty}^{+\infty} (1+|m|)^{-\mu} e^{-\beta|m|} e^{|\mathrm{Im}(z)||m|} dm$$
$$\leq \frac{k_1}{\log(q)} \frac{\varpi_E}{\tilde{R}^{N+1}} M_1 \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{(N+\frac{1}{2})^2}{2k_1}} |t|^{N+1} \quad (237)$$

for all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$ and $z \in H_{\beta'}$ where the constant $M_1 > 0$ is defined in (216). □

In conclusion, the proposition 11 ensues from the decompositions (206), (209) and the collection of Lemma 13, 14 and 16. □

In the second main proposition, we show that the second piece $v_2(t,z)$ has the null formal series as Gevrey asymptotic expansion of order 1 in a logarithmic scale with respect to t.

Proposition 12. *The map $v_2(t,z)$ is well defined and bounded holomorphic relatively to (t,z) on the product $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]) \times H_{\beta'}$. Furthermore, for some well chosen constants $M_{\varpi,b}, \tilde{M}_1, Q_1 > 0$ and any given integer $N \geq 0$, the next error bounds*

$$|v_2(t,z)| \leq \frac{k_1}{\log(q)(2\pi)^{1/2}} M_{\varpi,b} \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{1}{8k_1}} |t|$$
$$\times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \times \frac{Q_1}{\tilde{M}_1} \big(\frac{1}{\tilde{M}_1 \rho/2} \big)^N N^{1/2} \Gamma(N) \Big(-\frac{1}{\log|t|} \Big)^{N+1} \quad (238)$$

hold provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$ and $z \in H_{\beta'}$, where $C_{q,k_1} > 0$ and $\Delta_1 > 0$ are the two constants stemming from (15).

Proof. According to (97) in Proposition 2, one can find a constant $M_{\varpi,b} > 0$ for which the map $\omega_{d_1,\pi}$ is subjected to the next upper bounds

$$|\omega_{d_1,\pi}(\tau_1,\tau_2,m)| \leq \varpi(1+|m|)^{-\mu} e^{-\beta|m|} |\tau_1| \exp\Big(\frac{k_1}{2} \frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha \log(|\tau_1|+\delta) \Big)$$
$$\times |\tau_2| e^{\nu|\tau_2|} \leq M_{\varpi,b}(1+|m|)^{-\mu} e^{-\beta|m|} |\tau_1||\tau_2| e^{\nu|\tau_2|} \quad (239)$$

provided that $\tau_1 \in S_{d_1} \cap D_{b/2}$, $\tau_2 \in L_{\pi,\rho/2,\infty}$ and $m \in \mathbb{R}$. Besides, the bounds (234) for the quantity (229) in the special case $N = 0$ yields the next estimates

$$\Big| \int_{L_{d_1,b/2}} \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} d\tau_1 \Big| \leq \frac{1}{C_{q,k_1}\Delta_1} \Big(\frac{2\log(q)}{k_1} \Big)^{1/2} \sqrt{\pi} q^{\frac{1}{8k_1}} |t| \quad (240)$$

for all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$. Furthermore, a constant $\tilde{M}_1 \in (0,1)$ can be singled out with

$$H(t) = \int_{\rho/2}^{+\infty} e^{\nu r_2} |\exp(\log(t) r_2)| dr_2 \leq \int_{\rho/2}^{+\infty} e^{\nu r_2} e^{r_2 \log|t|} dr_2 \leq \int_{\rho/2}^{+\infty} e^{(\tilde{M}_1 \log|t|) r_2} dr_2$$
$$= -\frac{1}{\tilde{M}_1 \log|t|} e^{(\tilde{M}_1 \log|t|)\rho/2} \quad (241)$$

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as long as $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$, for $R_1 > 0$ chosen small enough. In the next step, we remind the reader the following technical estimates that are taken from Lemma 14 of [10]. Namely, for any given real number $M > 0$, one can select a constant $Q_1 > 0$ such that

$$(\frac{1}{r})^N \exp\left(-\frac{M}{r}\right) \leq Q_1(1/M)^N N^{1/2}\Gamma(N) \tag{242}$$

for all integers $N \geq 1$, all real numbers $r > 0$. Based on (242) for the constant $M = \tilde{M}_1\rho/2$ and specific value $r = -1/\log|t|$, we deduce from (241) that

$$H(t) \leq \frac{Q_1}{\tilde{M}_1}(\frac{1}{\tilde{M}_1\rho/2})^N N^{1/2}\Gamma(N)\left(-\frac{1}{\log|t|}\right)^{N+1} \tag{243}$$

for all integers $N \geq 1$, provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$.

At last, the collection of the bounds (239), (240) and (243) triggers the next error bounds for the piece $v_2(t,z)$. Namely,

$$|v_2(t,z)| \leq \frac{k_1}{\log(q)(2\pi)^{1/2}} M_{\varpi,b} \frac{1}{C_{q,k_1}\Delta_1}\left(\frac{2\log(q)}{k_1}\right)^{1/2}\sqrt{\pi}q^{\frac{1}{8k_1}}|t| \\ \times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}dm \times \frac{Q_1}{\tilde{M}_1}(\frac{1}{\tilde{M}_1\rho/2})^N N^{1/2}\Gamma(N)\left(-\frac{1}{\log|t|}\right)^{N+1}$$

for all integers $N \geq 0$, whenever $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$ and $z \in H_{\beta'}$. □

In the last principal proposition, the third piece $v_3(t,z)$ is shown to have the null formal series as asymptotic expansion of $q-$Gevrey order k_1 in the scale of monomials relatively to t.

Proposition 13. *The map $v_3(t,z)$ is well defined and bounded holomorphic relatively to (t,z) on the product $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta'}$. In addition, for some suitable constants $\varpi > 0$, $M_{\delta,1,b}, M_{\delta,2,b} > 0$, $M_3 > 0$ and any given integer $N \geq 0$, the next error bounds*

$$|v_3(t,z)| \leq \frac{k_1}{\log(q)(2\pi)^{1/2}}\frac{\varpi}{C_{q,k_1}\Delta_1}\exp\left(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\right)q^{\frac{(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))^2}{2k_1}} \\ \times \int_{b/2}^{1} r_1^{\alpha-\frac{1}{2}}dr_1 \times M_3(q^{-(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))/k_1})^{N+1}q^{\frac{(N+1)^2}{2k_1}}|t|^{N+1} \\ + \frac{k_1}{\log(q)(2\pi)^{1/2}}\frac{\varpi}{C_{q,k_1}\Delta_1}\exp\left(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\right)q^{\frac{1}{8k_1}} \\ \times \int_{1}^{+\infty}\frac{1}{r_1^{3/2}}dr_1 \times M_3(q^{-\frac{1}{2k_1}})^{N+1}q^{\frac{(N+1)^2}{2k_1}}|t|^{N+1} \tag{244}$$

hold provided that $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$ and $z \in H_{\beta'}$, where $C_{q,k_1} > 0$ and $\Delta_1 > 0$ are the two constants appearing in (15).

Proof. We further break up the integral $v_3(t,z)$ in two parts

$$v_3(t,z) = v_{3.1}(t,z) + v_{3.2}(t,z) \tag{245}$$

where

$$v_{3.1}(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,b/2,1}} \int_{L_\pi} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (246)$$

with $L_{d_1,b/2,1} = [b/2,1]e^{\sqrt{-1}d_1}$ and for

$$v_{3.2}(t,z) = \frac{k_1}{\log(q)(2\pi)^{1/2}} \int_{L_{d_1,1,\infty}} \int_{L_\pi} \int_{-\infty}^{+\infty} \omega_{d_1,\pi}(\tau_1,\tau_2,m) \times \frac{1}{\Theta_{q^{1/k_1}}(\tau_1/t)} \exp\left(-(\log(t))\tau_2\right) e^{\sqrt{-1}zm} \frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2} dm \quad (247)$$

along the segment $L_{d_1,1,\infty} = [1,+\infty)e^{\sqrt{-1}d_1}$.

As stated in (97) in Proposition 2, the map $\omega_{d_1,\pi}$ is subjected to the next upper bounds

$$|\omega_{d_1,\pi}(\tau_1,\tau_2,m)| \leq \varpi(1+|m|)^{-\mu}e^{-\beta|m|}|\tau_1|\exp\left(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\right) \times |\tau_2|e^{\nu|\tau_2|} \quad (248)$$

provided that $\tau_1 \in L_{d_1,b/2,\infty} = [b/2,+\infty)e^{\sqrt{-1}d_1}$, $\tau_2 \in L_\pi = [0,+\infty)e^{\sqrt{-1}\pi}$ and $m \in \mathbb{R}$. On the other hand, we need the next technical upper bounds.

Lemma 17. *One can single out two constants $M_{\delta,1,b}, M_{\delta,2,b} > 0$ such that*

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|}\exp\left(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\right) \leq \frac{1}{C_{q,k_1}\Delta_1}\exp\left(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\right) \times \exp\left(-\frac{k_1}{2\log(q)}\log^2|t|\right)|t|^{1/2} \times \exp\left(\alpha\log|\tau_1| + \frac{k_1}{\log(q)}\log|\tau_1|\log|t|\right)\frac{1}{|\tau_1|^{1/2}} \quad (249)$$

provided that $\tau_1 \in L_{d_1,b/2,\infty}$ and $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$.

Proof. For $\tau_1 \in L_{d_1,b/2,\infty}$ and $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, we first expand

$$\log^2(|\tau_1|/|t|) = \log^2|\tau_1| - 2\log|\tau_1|\log|t| + \log^2|t| \quad (250)$$

along with

$$\log^2(|\tau_1|+\delta) = \log^2|\tau_1| + 2\log|\tau_1|\log(1+\frac{\delta}{|\tau_1|}) + \log^2(1+\frac{\delta}{|\tau_1|}) \quad (251)$$

and

$$\log(|\tau_1|+\delta) = \log|\tau_1| + \log(1+\frac{\delta}{|\tau_1|}). \quad (252)$$

Since $\log(1+x) \sim x$ holds as x tends to 0 and bearing in mind the classical growth comparison $\lim_{x\to+\infty}\log(x)/x = 0$, we get from (251) and (252) two constants $M_{\delta,1,b}, M_{\delta,2,b} > 0$ with

$$\log^2(|\tau_1|+\delta) \leq \log^2|\tau_1| + M_{\delta,1,b} \ , \ \log(|\tau_1|+\delta) \leq \log|\tau_1| + M_{\delta,2,b} \quad (253)$$

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for all $|\tau_1| \geq b/2$. Eventually, the gathering of (220) and (253) with the expansion (250) yields the awaited estimates (249). □

In the next lemma, we exhibit $q-$Gevrey type estimates on both segments $L_{d_1,b/2,1}$ and $L_{d_1,1,\infty}$.

Lemma 18. *The next two $q-$Gevrey type estimates hold.*

- *On the segment $L_{d_1,b/2,1}$, we get that*

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|}\exp\Big(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\Big)$$
$$\leq \frac{1}{C_{q,k_1}\Delta_1}\exp\Big(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\Big) \times q^{\frac{(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))^2}{2k_1}}$$
$$\times (q^{-(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))/k_1})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1}|\tau_1|^{\alpha-\frac{1}{2}} \quad (254)$$

holds for all $\tau_1 \in L_{d_1,b/2,1}$, all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, for all integers $N \geq 0$.

- *On the segment $L_{d_1,1,\infty}$, we arrive at*

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|}\exp\Big(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\Big)$$
$$\leq \frac{1}{C_{q,k_1}\Delta_1}\exp\Big(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\Big) \times q^{\frac{1}{8k_1}}(q^{-\frac{1}{2k_1}})^{N+1}q^{\frac{(N+1)^2}{2k_1}}|t|^{N+1}\frac{1}{|\tau_1|^{1+\frac{1}{2}}}. \quad (255)$$

provided that $\tau_1 \in L_{d_1,1,\infty}$, all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$, for all integers $N \geq 0$.

Proof. 1) Consider $\tau_1 \in L_{d_1,b/2,1}$ and $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$. In particular, we notice that $b/2 \leq |\tau_1| \leq 1$ and $|t| < R_1 < 1$. It follows that

$$\frac{k_1}{\log(q)}\log|\tau_1|\log|t| \leq \frac{k_1}{\log(q)}\log(b/2)\log|t| = \log\Big(|t|^{\frac{k_1}{\log(q)}\log(b/2)}\Big). \quad (256)$$

As a result, the inequality (249) becomes

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|}\exp\Big(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\Big)$$
$$\leq \frac{1}{C_{q,k_1}\Delta_1}\exp\Big(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\Big) \times \exp\Big(-\frac{k_1}{2\log(q)}\log^2|t|\Big)|t|^{\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2)}|\tau_1|^{\alpha-\frac{1}{2}}. \quad (257)$$

2) Let us take $\tau_1 \in L_{d_1,1,\infty}$. In particular $|\tau_1| \geq 1$. We select $R_1 > 0$ small enough and fulfilling (100) in a way that

$$\frac{k_1}{\log(q)}\log|t|\log|\tau_1| \leq -(\alpha+1)\log|\tau_1| \quad (258)$$

for all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty,0]$. The inequality (249) is then changed into

$$\frac{1}{|\Theta_{q^{1/k_1}}(\tau_1/t)|}\exp\Big(\frac{k_1}{2}\frac{\log^2(|\tau_1|+\delta)}{\log(q)} + \alpha\log(|\tau_1|+\delta)\Big)$$
$$\leq \frac{1}{C_{q,k_1}\Delta_1}\exp\Big(\frac{k_1}{2\log(q)}M_{\delta,1,b} + \alpha M_{\delta,2,b}\Big) \times \exp\Big(-\frac{k_1}{2\log(q)}\log^2|t|\Big)|t|^{1/2}\frac{1}{|\tau_1|^{1+\frac{1}{2}}}. \quad (259)$$

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The next estimates have been presented in Lemma 12 of our recent work [10]. Namely, for any prescribed real number $h \in \mathbb{R}$, the next inequality

$$x^{h} \exp\Big(-\frac{k_1}{2}\frac{\log^{2}(x)}{\log(q)} \Big) \leq q^{\frac{h^2}{2k_1}} (q^{-h/k_1})^{N} q^{\frac{N^2}{2k_1}} x^{N} \quad (260)$$

occurs for all integers $N \geq 1$, all positive real numbers $x > 0$. In particular, the next upper bounds

$$\exp\Big(-\frac{k_1}{2\log(q)}\log^{2}|t| \Big) |t|^{\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2)} \leq q^{\frac{(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))^2}{2k_1}} \times (q^{-(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))/k_1})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \quad (261)$$

along with

$$\exp\Big(-\frac{k_1}{2\log(q)}\log^{2}|t| \Big) |t|^{1/2} \leq q^{\frac{1}{8k_1}} (q^{-\frac{1}{2k_1}})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \quad (262)$$

hold for all $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$, for all integers $N \geq 0$.

At last, the $q-$Gevrey type bounds (254) result from (257) together with (261) and the combination of (259) with (262) yields (255). □

In the last part of the proof, we can now provide upper bounds for each piece $v_{3.1}(t,z)$ and $v_{3.2}(t,z)$. Namely, based on (248), (254) and (255) we get

$$\begin{aligned} |v_{3.1}(t,z)| \leq \; & \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{\varpi}{C_{q,k_1}\Delta_1} \exp\Big(\frac{k_1}{2\log(q)} M_{\delta,1,b} + \alpha M_{\delta,2,b} \Big) q^{\frac{(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))^2}{2k_1}} \\ & \times \int_{b/2}^{1} r_1^{\alpha-\frac{1}{2}} dr_1 \times \int_{0}^{+\infty} e^{\nu r_2} e^{r_2 \log|t|} dr_2 \times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \\ & \times (q^{-(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))/k_1})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \\ \leq \; & \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{\varpi}{C_{q,k_1}\Delta_1} \exp\Big(\frac{k_1}{2\log(q)} M_{\delta,1,b} + \alpha M_{\delta,2,b} \Big) q^{\frac{(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))^2}{2k_1}} \\ & \times \int_{b/2}^{1} r_1^{\alpha-\frac{1}{2}} dr_1 \times M_3 (q^{-(\frac{1}{2}+\frac{k_1}{\log(q)}\log(b/2))/k_1})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \end{aligned} \quad (263)$$

along with

$$\begin{aligned} |v_{3.2}(t,z)| \leq \; & \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{\varpi}{C_{q,k_1}\Delta_1} \exp\Big(\frac{k_1}{2\log(q)} M_{\delta,1,b} + \alpha M_{\delta,2,b} \Big) q^{\frac{1}{8k_1}} \\ & \times \int_{1}^{+\infty} \frac{1}{r_1^{3/2}} dr_1 \times \int_{0}^{+\infty} e^{\nu r_2} e^{r_2 \log|t|} dr_2 \times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \\ \times (q^{-\frac{1}{2k_1}})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \leq \; & \frac{k_1}{\log(q)(2\pi)^{1/2}} \frac{\varpi}{C_{q,k_1}\Delta_1} \exp\Big(\frac{k_1}{2\log(q)} M_{\delta,1,b} + \alpha M_{\delta,2,b} \Big) q^{\frac{1}{8k_1}} \\ & \times \int_{1}^{+\infty} \frac{1}{r_1^{3/2}} dr_1 \times M_3 (q^{-\frac{1}{2k_1}})^{N+1} q^{\frac{(N+1)^2}{2k_1}} |t|^{N+1} \end{aligned} \quad (264)$$

for the constant

$$M_3 = \int_{0}^{+\infty} e^{\nu r_2} e^{r_2 \log R_1} dr_2 \times \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \quad (265)$$

for all integers $N \geq 0$, whenever $t \in (\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]$ and $z \in H_{\beta'}$.

Eventually, the splitting (245) together with the above upper estimates (263) and (264) promotes the expected bounds (244). □

We return to the proof of Theorem 2. On the ground of the decomposition (197), we set

$$u_1(t,z) = v_1(t,z) + v_3(t,z).$$

According to Proposition 11 and Proposition 13, we observe that $u_1(t,z)$ represents a bounded holomorphic map on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta'}$. Moreover, u_1 is submitted to error bounds of the form (195) for the sequence of functions $b_n(t,z)$, $n \geq 0$ given by $b_n(t,z) = g_n(t,z)/t^n$, which represent bounded holomorphic maps on the domain $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta'}$, owing to the upper bounds (201).

On the other hand, we assign

$$u_2(t,z) = v_2(t,z).$$

As claimed by Proposition 12, we check that $u_2(t,z)$ stands for a bounded holomorphic function on $((\mathcal{R}_{d_1,\Delta_1} \cap D_{R_1}) \setminus (-\infty, 0]) \times H_{\beta'}$. Furthermore, u_2 is subjected to error bounds shaped in (238). Theorem 2 is established. □

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