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Equalities for Mixed Operations of Moore–Penrose and Group Inverses of a Matrix

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Equalities for Mixed Operations of Moore–Penrose and Group Inverses of a Matrix

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Abstract: This article shows how to establish expansion formulas for calculating the mixed operations $(A^{\dagger})^{\#}$, $(A^{\#})^{\dagger}$, $((A^{\dagger})^{\#})^{\dagger}$, $((A^{\#})^{\dagger})^{\#}$, ... of generalized inverses, where $(\cdot)^{\dagger}$ denotes the Moore–Penrose inverse of a matrix and $(\cdot)^{\#}$ denotes the group inverse of a square matrix. As applications of the formulas obtained, the author constructs and classifies some groups of matrix equalities involving the above mixed operations, and derives necessary and sufficient conditions for them to hold.

Keywords: group inverse; matrix equality; Moore–Penrose inverse; range; rank

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1. Introduction

Throughout this article, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ matrices over the field of complex numbers, A^* stands for the conjugate transpose of $A \in \mathbb{C}^{m\times n}$, $r(A)$ stands for the rank of $A \in \mathbb{C}^{m\times n}$, I_m stands for the identity matrix of order m, and $[\,A,\, B]$ stands for a columnwise partitioned matrix consisting of two submatrices A and B. We introduce the concepts of generalized inverses of a matrix. For an $A \in \mathbb{C}^{m\times n}$, the Moore–Penrose generalized inverse of A is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1.1}$$

A square matrix $A \in \mathbb{C}^{m\times m}$ is said to be group invertible if and only if there exists a matrix $X \in \mathbb{C}^{m\times m}$ that satisfies the following three matrix equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (5)\ AX = XA. \tag{1.2}$$

In such a case, the matrix X, called the group inverse of A, is unique and is denoted by $X = A^{\#}$. For more basic results and facts concerning generalized inverses of matrices and their properties, we refer the reader to the three references [1–3].

Since that the Moore–Penrose inverse and the group inverse of a matrix are defined from unique solutions of two different groups of matrix equations, the expressions of the two generalized inverses are not necessarily the same. Thus they have different performances and properties. In this case, algebraists are interested in the relationships of the two generalized inverses, as well as their possible equalities. On the other hand, algebraists are interested in mixed operations of the two generalized inverses, such as, $(A^{\dagger})^{\#}$, $(A^{\#})^{\dagger}$, $((A^{\dagger})^{\#})^{\dagger}$, $((A^{\#})^{\dagger})^{\#}$, etc. As one subject in this regard, this note considers establishing expansion formulas for calculating the mixed operations using a series of known results and facts related to ranks, ranges, and generalized inverses of matrices. As applications, the author constructs and classifies some groups of matrix equalities involving the above mixed operations, and derives necessary and sufficient conditions for them to hold.

2. Some preliminary results

In this section, we introduce a selection of existing formulas and facts related to ranks, ranges, and generalized inverses of matrices (cf. [1–3,8]) and shall use them in establishments and characterizations of matrix equalities as described in Section 1.

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Lemma 2.1. *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$(A^{\dagger})^{*} = (A^{*})^{\dagger}, \quad (A^{\dagger})^{\dagger} = A, \tag{2.1}$$
$$A^{\dagger} = A^{*}(AA^{*})^{\dagger} = (A^{*}A)^{\dagger}A^{*} = A^{*}(A^{*}AA^{*})^{\dagger}A^{*}, \tag{2.2}$$
$$(A^{*})^{\dagger}A^{*} = (AA^{\dagger})^{*} = AA^{\dagger}, \;\; A^{*}(A^{*})^{\dagger} = (A^{\dagger}A)^{*} = A^{\dagger}A, \tag{2.3}$$
$$(AA^{*})^{\dagger} = (A^{\dagger})^{*}A^{\dagger}, \;\; (A^{*}A)^{\dagger} = A^{\dagger}(A^{\dagger})^{*}, \;\; (AA^{*}A)^{\dagger} = A^{\dagger}(A^{\dagger})^{*}A^{\dagger}. \tag{2.4}$$

Lemma 2.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $P \in \mathbb{C}^{p\times m}$ and $Q \in \mathbb{C}^{q\times n}$. Then*

$$r(AA^{*}A) = r(AA^{*}) = r(A^{*}A) = r(A^{\dagger}) = r(A), \tag{2.5}$$
$$\mathscr{R}(AA^{*}A) = \mathscr{R}(AA^{*}) = \mathscr{R}(AA^{\dagger}) = \mathscr{R}((A^{\dagger})^{*}) = \mathscr{R}(A), \tag{2.6}$$
$$\mathscr{R}(A^{*}AA^{*}) = \mathscr{R}(A^{*}A) = \mathscr{R}(A^{\dagger}A) = \mathscr{R}(A^{\dagger}) = \mathscr{R}(A^{*}). \tag{2.7}$$

Lemma 2.3. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$. Then the following matrix rank and range inequalities*

$$r(A) \geq r(A^{2}) \geq \cdots \geq r(A^{k}), \tag{2.8}$$
$$\mathscr{R}(A) \supseteq \mathscr{R}(A^{2}) \supseteq \cdots \supseteq \mathscr{R}(A^{k}) \tag{2.9}$$

hold for any integer $k \geq 2$, and the following matrix rank equalities

$$r(A^{2}) = r((A^{2})^{\dagger}) = r((A^{\dagger})^{2}) \tag{2.10}$$

hold. In particular, the following facts

$$r(A) = r(A^{2}) \Leftrightarrow r(A) = r(A^{k}) \Leftrightarrow r((A^{\dagger})^{2}) = r(A) \Leftrightarrow r((A^{\dagger})^{k}) = r(A), \tag{2.11}$$
$$r(A) = r(A^{2}) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^{2}) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^{k}), \tag{2.12}$$
$$r(A) = r(A^{2}) \Leftrightarrow \mathscr{R}(A^{*}) = \mathscr{R}((A^{2})^{*}) \Leftrightarrow \mathscr{R}(A^{*}) = \mathscr{R}((A^{k})^{*}) \tag{2.13}$$

hold for any integer $k \geq 2$, and the following fact

$$r(A) = r(A^{2}) \Rightarrow r(AB) = r(A^{k}B) \tag{2.14}$$

holds for any integer $k \geq 2$.

Lemma 2.4. *$A \in \mathbb{C}^{m\times m}$ is group invertible if and only if $r(A^{2}) = r(A)$. In this case, the following equalities hold:*

$$(A^{\#})^{*} = (A^{*})^{\#}, \tag{2.15}$$
$$(A^{\#})^{\#} = A, \tag{2.16}$$
$$A^{\#} = AA^{\dagger}A^{\#} = A^{\#}A^{\dagger}A = AA^{\dagger}A^{\#}A^{\dagger}A, \tag{2.17}$$
$$A^{\#} = A(A^{3})^{\dagger}A, \tag{2.18}$$

the following rank and range equalities hold:

$$r(AA^{\#}) = r(A^{\#}A) = r(A^{\#}) = r(A), \tag{2.19}$$
$$\mathscr{R}(AA^{\#}) = \mathscr{R}(A^{\#}A) = \mathscr{R}(A^{\#}) = \mathscr{R}(A), \tag{2.20}$$
$$\mathscr{R}((AA^{\#})^{*}) = \mathscr{R}((A^{\#}A)^{*}) = \mathscr{R}((A^{\#})^{*}) = \mathscr{R}((A^{*})^{\#}) = \mathscr{R}(A^{*}). \tag{2.21}$$

Lemma 2.5. *Let $A_1 \in \mathbb{C}^{m\times n_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, $B_1 \in \mathbb{C}^{m\times p_1}$ and $B_2 \in \mathbb{C}^{m\times p_2}$. If $\mathscr{R}(A_1) = \mathscr{R}(B_1)$ and $\mathscr{R}(A_2) = \mathscr{R}(B_2)$. Then two equalities $\mathscr{R}[A_1,\, A_2] = \mathscr{R}[B_1,\, B_2]$ and $r[A_1,\, A_2] = r[B_1,\, B_2]$ hold.*

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Lemma 2.6 ([4]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then*

$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A) + r(D - CA^\dagger B). \tag{2.22}$$

In particular, if $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ and $\mathscr{R}(A^*) \supseteq \mathscr{R}(C^*)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^\dagger B). \tag{2.23}$$

Lemma 2.7 ([6]). *Let $A, B \in \mathbb{C}^{m\times n}$, and assume that $AXA = A$ and $BXB = B$ hold for an $X \in \mathbb{C}^{n\times m}$. Then the following matrix rank equality*

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] - r(A) - r(B) \tag{2.24}$$

holds. Therefore,

$$A = B \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B) \text{ and } \mathscr{R}(A^*) = \mathscr{R}(B^*). \tag{2.25}$$

Lemma 2.8 ([5,7]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following rank equalities*

$$\begin{aligned} r(AB - ABB^\dagger A^\dagger AB) &= r(A^\dagger ABB^\dagger - (A^\dagger ABB^\dagger)^2) \\ &= r[A^*,\, B] - r(A) - r(B) + r(AB), \\ r(AB - A(BB^\dagger A^\dagger A)^2 B) &= r(A^\dagger ABB^\dagger - (A^\dagger ABB^\dagger)^3) \\ &= r[A^*,\, B] - r(A) - r(B) + r(AB), \\ r((AB)^\dagger - B^\dagger A^\dagger) &= r(B(AB)^\dagger A - BB^\dagger A^\dagger A) \\ &= r((AB)^\dagger - B^\dagger A^\dagger ABB^\dagger A^\dagger) \\ &= r(B(AB)^\dagger A - (BB^\dagger A^\dagger A)^2) \\ &= r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) - r(AB) \end{aligned}$$

hold. Hence,

$$\begin{aligned} ABB^\dagger A^\dagger AB = AB &\Leftrightarrow (A^\dagger ABB^\dagger)^2 = A^\dagger ABB^\dagger \\ &\Leftrightarrow A(BB^\dagger A^\dagger A)^2 B = AB \\ &\Leftrightarrow (A^\dagger ABB^\dagger)^3 = A^\dagger ABB^\dagger \\ &\Leftrightarrow r[A^*,\, B] = r(A) + r(B) - r(AB), \end{aligned}$$

and

$$\begin{aligned} (AB)^\dagger = B^\dagger A^\dagger &\Leftrightarrow B(AB)^\dagger A = BB^\dagger A^\dagger A \\ &\Leftrightarrow (AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger \\ &\Leftrightarrow B(AB)^\dagger A = (BB^\dagger A^\dagger A)^2 \\ &\Leftrightarrow r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) = r(AB). \end{aligned}$$

Under the condition $r(AB) = r(A) = r(B)$, the following rank equalities

$$\begin{aligned} r(AB - ABB^{\dagger}A^{\dagger}AB) &= r(A^{\dagger}ABB^{\dagger} - (A^{\dagger}ABB^{\dagger})^2) \\ &= r(AB - A(BB^{\dagger}A^{\dagger}A)^2B) \\ &= r(A^{\dagger}ABB^{\dagger} - (A^{\dagger}ABB^{\dagger})^3) \\ &= r((AB)^{\dagger} - B^{\dagger}A^{\dagger}) \\ &= r(B(AB)^{\dagger}A - BB^{\dagger}A^{\dagger}A) \\ &= r((AB)^{\dagger} - B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}) \\ &= r(B(AB)^{\dagger}A - (BB^{\dagger}A^{\dagger}A)^2) \\ &= r[A^*, B] - r(A) \end{aligned}$$

hold. Hence,

$$\begin{aligned} ABB^{\dagger}A^{\dagger}AB = AB &\Leftrightarrow (A^{\dagger}ABB^{\dagger})^2 = A^{\dagger}ABB^{\dagger} \\ &\Leftrightarrow A(BB^{\dagger}A^{\dagger}A)^2B = AB \\ &\Leftrightarrow (A^{\dagger}ABB^{\dagger})^3 = A^{\dagger}ABB^{\dagger} \\ &\Leftrightarrow (AB)^{\dagger} = B^{\dagger}A^{\dagger} \\ &\Leftrightarrow B(AB)^{\dagger}A = BB^{\dagger}A^{\dagger}A \\ &\Leftrightarrow (AB)^{\dagger} = B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} \\ &\Leftrightarrow B(AB)^{\dagger}A = (BB^{\dagger}A^{\dagger}A)^2 \\ &\Leftrightarrow \mathscr{R}(A^*) = \mathscr{R}(B). \end{aligned}$$

Lemma 2.9 ([5]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $B \in \mathbb{C}^{p\times q}$, and assume that $r(ABC) = r(B)$. Then the following equality holds:*

$$(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}.$$

3. Equalities composed of the mixed operations of the Moore–Penrose inverses and group inverses of a matrix

We first give a family of range equalities for mixed operations of the Moore–Penrose inverses and group inverses of a matrix.

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times m}$ with $r(A^2) = r(A)$ and define*

$$X_1 = A,\ X_2 = A^{\#},\ X_3 = (A^{\dagger})^*,\ X_4 = ((A^{\dagger})^{\#})^*, \tag{3.1}$$

$$X_5 = ((A^{\#})^{\dagger})^*,\ X_6 = (((A^{\dagger})^{\#})^{\dagger},\ X_7 = (((A^{\#})^{\dagger})^{\#})^*, \tag{3.2}$$

$$Y_1 = A^*,\ Y_2 = A^{\dagger},\ Y_3 = (A^{\#})^*,\ Y_4 = (A^{\dagger})^{\#}, \tag{3.3}$$

$$Y_5 = (A^{\#})^{\dagger},\ Y_6 = ((A^{\#})^{\dagger})^{\#},\ Y_7 = (((A^{\dagger})^{\#})^{\dagger})^*. \tag{3.4}$$

Then the following range equalities

$$\mathscr{R}(X_i) = \mathscr{R}(A), \qquad \mathscr{R}(X_i^*) = \mathscr{R}(A^*), \tag{3.5}$$
$$\mathscr{R}(Y_i) = \mathscr{R}(A^*), \qquad \mathscr{R}(Y_i^*) = \mathscr{R}(A), \tag{3.6}$$
$$\mathscr{R}(X_iX_j) = \mathscr{R}(A), \qquad \mathscr{R}((X_iX_j)^*) = \mathscr{R}(A^*), \tag{3.7}$$
$$\mathscr{R}(Y_iY_j) = \mathscr{R}(A^*), \qquad \mathscr{R}((Y_iY_j)^*) = \mathscr{R}(A), \tag{3.8}$$
$$\mathscr{R}(X_iY_j) = \mathscr{R}((X_iY_j)^*) = \mathscr{R}(A), \qquad \mathscr{R}(Y_iX_j) = \mathscr{R}((Y_iX_j)^*) = \mathscr{R}(A^*), \tag{3.9}$$
$$\mathscr{R}(X_iY_jX_k) = \mathscr{R}(A), \qquad \mathscr{R}((X_iY_jX_k)^*) = \mathscr{R}(A^*), \tag{3.10}$$
$$\mathscr{R}(Y_iX_jY_k) = \mathscr{R}(A^*), \qquad \mathscr{R}((Y_iX_jY_k)^*) = \mathscr{R}(A), \tag{3.11}$$
$$\mathscr{R}(X_{i_1}Y_{j_1}X_{i_2}Y_{j_2}) = \mathscr{R}((X_{i_1}Y_{j_1}X_{i_2}Y_{j_2})^*) = \mathscr{R}(A), \quad \mathscr{R}(Y_{i_1}X_{j_1}Y_{i_2}X_{j_2}) = \mathscr{R}((Y_{i_1}X_{j_1}Y_{i_2}X_{j_2})^*) = \mathscr{R}(A^*), \tag{3.12}$$
$$\mathscr{R}(X_{i_1}X_{i_2}Y_{j_1}Y_{j_2}) = \mathscr{R}(A), \qquad \mathscr{R}((X_{i_1}X_{i_2}Y_{j_1}Y_{j_2})^*) = \mathscr{R}(A^*), \tag{3.13}$$
$$\mathscr{R}(Y_{i_1}Y_{i_2}X_{i_1}X_{i_2}) = \mathscr{R}(A^*), \qquad \mathscr{R}((Y_{i_1}Y_{i_2}X_{i_1}X_{i_2})^*) = \mathscr{R}(A) \tag{3.14}$$

hold for $i, j, k, i_1, i_2, j_1, j_2 = 1, 2, \ldots, 7$.

Proof. Follows from (2.6), (2.7), (2.20), and (2.21). □

Let $A = B$ in Lemmas 2.8 and 2.9, we first obtain the following results.

Theorem 3.2. *Let* $A \in \mathbb{C}^{m\times m}$. *Then we have the following results.*

(a) *The following rank equalities hold*:

$$\begin{aligned} r(A^2 - A^2(A^\dagger)^2A^2) &= r(A^\dagger A^2A^\dagger - (A^\dagger A^2A^\dagger)^2) \\ &= r(A^2 - A^2(A^\dagger)^2A^2(A^\dagger)^2A^2) \\ &= r(A^\dagger A^2A^\dagger - (A^\dagger A^2A^\dagger)^3) \\ &= r[A,\ A^*] - 2r(A) + r(A^2), \end{aligned} \tag{3.15}$$

$$\begin{aligned} r((A^2)^\dagger - (A^\dagger)^2) &= r(A(A^2)^\dagger A - A(A^\dagger)^2A) \\ &= r((A^2)^\dagger - (A^\dagger)^2A^2(A^\dagger)^2) \\ &= r(A(A^2)^\dagger A - A(A^\dagger)^3A) \\ &= r\left(\begin{bmatrix} A^2A^* \\ A \end{bmatrix}[A^*A^2,\ A]\right) - r(A^2). \end{aligned} \tag{3.16}$$

Consequently, the following implications hold:

$$\begin{aligned} A^2(A^\dagger)^2A^2 = A^2 &\Leftrightarrow (A^\dagger A^2A^\dagger)^2 = A^\dagger A^2A^\dagger \\ &\Leftrightarrow A^2(A^\dagger)^2A^2(A^\dagger)^2A^2 = A^2 \\ &\Leftrightarrow (A^\dagger A^2A^\dagger)^3 = A^\dagger A^2A^\dagger \\ &\Leftrightarrow r[A,\ A^*] = 2r(A) - r(A^2), \end{aligned} \tag{3.17}$$

$$\begin{aligned} (A^2)^\dagger = (A^\dagger)^2 &\Leftrightarrow A(A^2)^\dagger A = A(A^\dagger)^2A \\ &\Leftrightarrow (A^2)^\dagger = (A^\dagger)^2A^2(A^\dagger)^2 \\ &\Leftrightarrow A(A^2)^\dagger A = (A(A^\dagger)^2A)^2 \\ &\Leftrightarrow r\left(\begin{bmatrix} A^2A^* \\ A \end{bmatrix}[A^*A^2,\ A]\right) = r(A^2). \end{aligned} \tag{3.18}$$

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(b) *Under the condition $r(A^2) = r(A)$, the following rank equalities*

$$\begin{aligned} r(A^2 - A^2(A^\dagger)^2A^2) &= r(A^\dagger A^2A^\dagger - (A^\dagger A^2A^\dagger)^2) \\ &= r(A^2 - A^2(A^\dagger)^2A^2(A^\dagger)^2A^2) \\ &= r(A^\dagger A^2A^\dagger - (A^\dagger A^2A^\dagger)^3) \\ &= r((A^2)^\dagger - (A^\dagger)^2) \\ &= r(A(A^2)^\dagger A - A(A^\dagger)^2A) \\ &= r((A^2)^\dagger - (A^\dagger)^2A^2(A^\dagger)^2) \\ &= r(A(A^2)^\dagger A - A(A^\dagger)^2A^2(A^\dagger)^2A) \\ &= r[A^*, A] - r(A), \end{aligned} \tag{3.19}$$

$$\begin{aligned} r((A^\dagger)^2 - (A^\dagger)^2A^2(A^\dagger)^2) &= r(A(A^\dagger)^2A - (A(A^\dagger)^2A)^2) \\ &= r((A^\dagger)^2 - (A^\dagger)^2A^2(A^\dagger)^2A^2(A^\dagger)^2) \\ &= r(A(A^\dagger)^2A - (A(A^\dagger)^2A)^3) \\ &= r(((A^\dagger)^2)^\dagger - A^2) \\ &= r(A^\dagger((A^\dagger)^2)^\dagger A^\dagger - A^\dagger A^2A^\dagger) \\ &= r(((A^\dagger)^2)^\dagger - A^2(A^\dagger)^2A^2) \\ &= r(A^\dagger((A^\dagger)^2)^\dagger A^\dagger - A^\dagger A^2(A^\dagger)^2A^2A^\dagger) \\ &= r[A^*, A] - r(A) \end{aligned} \tag{3.20}$$

hold. Hence,

$$\begin{aligned} A^2(A^\dagger)^2A^2 = A^2 &\Leftrightarrow (A^\dagger A^2A^\dagger)^2 = A^\dagger A^2A^\dagger \\ &\Leftrightarrow A^2(A^\dagger)^2A^2(A^\dagger)^2A^2 = A^2 \\ &\Leftrightarrow (A^\dagger A^2A^\dagger)^3 = A^\dagger A^2A^\dagger \\ &\Leftrightarrow (A^2)^\dagger = (A^\dagger)^2 \\ &\Leftrightarrow A(A^2)^\dagger A = A(A^\dagger)^2A \\ &\Leftrightarrow (A^2)^\dagger = (A^\dagger)^2A^2(A^\dagger)^2 \\ &\Leftrightarrow A(A^2)^\dagger A = A(A^\dagger)^2A^2(A^\dagger)^2A \\ &\Leftrightarrow (A^\dagger)^2A^2(A^\dagger)^2 = (A^\dagger)^2 \\ &\Leftrightarrow (A^\dagger)^2A^2(A^\dagger)^2A^2(A^\dagger)^2 = (A^\dagger)^2 \\ &\Leftrightarrow ((A^\dagger)^2)^\dagger = A^2(A^\dagger)^2A^2 \\ &\Leftrightarrow A^\dagger((A^\dagger)^2)^\dagger A^\dagger = A^\dagger A^2(A^\dagger)^2A^2A^\dagger \\ &\Leftrightarrow \mathscr{R}(A^*) = \mathscr{R}(A). \end{aligned} \tag{3.21}$$

(c) *Under the condition $r(A^2) = r(A)$, the following equalities hold:*

$$(A^3)^\dagger = (A^2)^\dagger A(A^2)^\dagger, \tag{3.22}$$

$$A^\# = A(A^2)^\dagger A(A^2)^\dagger A. \tag{3.23}$$

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times m}$, Then the following rank equality holds:*

$$r(AA^\dagger - A^\dagger A) = 2r[A, A^*] - 2r(A). \tag{3.24}$$

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If $r(A^2) = r(A)$, then the following rank equalities hold:

$$r(A^\dagger - A^\#) = r((A^2)^\dagger - (A^2)^\#) = 2r[A,\ A^*] - 2r(A). \tag{3.25}$$

Hence,

$$AA^\dagger = A^\dagger A \Leftrightarrow A^\dagger = A^\# \Leftrightarrow (A^2)^\dagger = (A^2)^\# \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^*). \tag{3.26}$$

Proof. Note that $AA^\dagger$ and $A^\dagger A$ are idempotent matrices by definition, and $A^\dagger AA^\dagger = A^\dagger\ A^\# AA^\# = A^\#$, $(A^2)^\dagger A^2 (A^2)^\dagger = (A^2)^\dagger$, $(A^2)^\# A^2 (A^2)^\# = (A^2)^\#$ hold by definition. In these cases, applying (2.24) to the differences $AA^\dagger - A^\dagger A$, $(A^\dagger - A^\#$, and $(A^2)^\dagger - (A^2)^\#$ and simplifying by Lemma 2.5, (2.6), (2.7), (2.20) and (2.21). □

In what follows, we show how to establish expansion formulas for calculating the mixed operations of the Moore–Penrose and group inverses of a matrix.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times m}$ with $r(A^2) = r(A)$. Then we have the following results.*

(a) *The following equalities always hold*:

$$(A^\dagger)^\# = A^\dagger((A^\dagger)^3)^\dagger A^\dagger = A^\dagger((A^\dagger)^2)^\dagger A^\dagger((A^\dagger)^2)^\dagger A^\dagger, \tag{3.27}$$
$$A(A^\dagger)^\# A = A((A^\dagger)^3)^\dagger A = ((A^\dagger)^2)^\dagger A^\dagger((A^\dagger)^2)^\dagger, \tag{3.28}$$
$$(A^\dagger)^2 A(A^\dagger)^\# A(A^\dagger)^2 = (A^\dagger)^2 A((A^\dagger)^3)^\dagger A(A^\dagger)^2 = A^\dagger. \tag{3.29}$$

(b) *The following equalities always hold*:

$$(A^\#)^\dagger = A^\dagger A^3 A^\dagger, \tag{3.30}$$
$$((A^2)^\#)^\dagger = A^\dagger A^4 A^\dagger, \tag{3.31}$$
$$((A^3)^\#)^\dagger = A^\dagger A^5 A^\dagger, \tag{3.32}$$
$$A(A^\#)^\dagger A = A^3, \tag{3.33}$$
$$A((A^2)^\#)^\dagger A = A^4, \tag{3.34}$$
$$A((A^3)^\#)^\dagger A = A^5. \tag{3.35}$$

(c) *The following equalities always hold*:

$$((A^\dagger)^\#)^\dagger = A(A^\dagger)^3 A, \tag{3.36}$$
$$(((A^2)^\dagger)^\#)^\dagger = AA^\dagger(A^2)^\dagger A^\dagger A, \tag{3.37}$$
$$(((A^3)^\dagger)^\#)^\dagger = AA^\dagger(A^3)^\dagger A^\dagger A, \tag{3.38}$$
$$A^\dagger((A^\dagger)^\#)^\dagger A^\dagger = (A^\dagger)^3, \tag{3.39}$$
$$A^\dagger(((A^2)^\dagger)^\#)^\dagger A^\dagger = A^\dagger(A^2)^\dagger A^\dagger, \tag{3.40}$$
$$A^\dagger(((A^3)^\dagger)^\#)^\dagger A^\dagger = A^\dagger(A^3)^\dagger A^\dagger. \tag{3.41}$$

(d) *The following equalities always hold*:

$$(((A^\#)^\dagger)^\#)^\dagger = AA^\dagger(A^\#)^\dagger A^\dagger A = A(A^\dagger)^2 A^3 (A^\dagger)^2 A, \tag{3.42}$$
$$((((A^\dagger)^\#)^\dagger)^\#)^\dagger = A^\dagger A((A^\dagger)^\#)^\dagger AA^\dagger = A^\dagger A^2 (A^\dagger)^3 A^2 A^\dagger, \tag{3.43}$$
$$(((((A^\#)^\dagger)^\#)^\dagger)^\#)^\dagger = A^\dagger A(((A^\#)^\dagger)^\#)^\dagger AA^\dagger = A^\dagger A^2 (A^\dagger)^2 A^3 (A^\dagger)^2 A^2 A^\dagger, \tag{3.44}$$
$$((((((A^\dagger)^\#)^\dagger)^\#)^\dagger)^\#)^\dagger = AA^\dagger((((A^\dagger)^\#)^\dagger)^\#)^\dagger A^\dagger A = A(A^\dagger)^2 A^2 (A^\dagger)^3 A^2 (A^\dagger)^2 A, \tag{3.45}$$
$$\begin{aligned}(((((((A^\#)^\dagger)^\#)^\dagger)^\#)^\dagger)^\#)^\dagger &= AA^\dagger(((((A^\#)^\dagger)^\#)^\dagger)^\#)^\dagger A^\dagger A \\ &= A(A^\dagger)^2 A^2 (A^\dagger)^2 A^3 (A^\dagger)^2 A^2 (A^\dagger)^2 A.\end{aligned} \tag{3.46}$$

(e) *The following equalities always hold*:

$$((((A^2)^{\#})^{\dagger})^{\#})^{\dagger} = AA^{\dagger}((A^2)^{\#})^{\dagger}A^{\dagger}A = A(A^{\dagger})^2A^4(A^{\dagger})^2A, \tag{3.47}$$

$$(((((A^2)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} = A^{\dagger}A(((A^2)^{\dagger})^{\#})^{\dagger}AA^{\dagger} = A^{\dagger}A^2A^{\dagger}(A^2)^{\dagger}A^{\dagger}A^2A^{\dagger}, \tag{3.48}$$

$$((((((A^2)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} = A^{\dagger}A((((A^2)^{\#})^{\dagger})^{\#})^{\dagger}AA^{\dagger} = A^{\dagger}A^2(A^{\dagger})^2A^4(A^{\dagger})^2A^2A^{\dagger}, \tag{3.49}$$

$$\begin{aligned}(((((((A^2)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} &= AA^{\dagger}(((((A^2)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger}A^{\dagger}A \\ &= A(A^{\dagger})^2A^2A^{\dagger}(A^2)^{\dagger}A^{\dagger}A^2(A^{\dagger})^2A,\end{aligned} \tag{3.50}$$

$$\begin{aligned}((((((((A^2)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} &= AA^{\dagger}((((((A^2)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger}A^{\dagger}A \\ &= A(A^{\dagger})^2A^2(A^{\dagger})^2A^4(A^{\dagger})^2A^2(A^{\dagger})^2A.\end{aligned} \tag{3.51}$$

(f) *The following equalities always hold*:

$$((((A^3)^{\#})^{\dagger})^{\#})^{\dagger} = AA^{\dagger}((A^3)^{\#})^{\dagger}A^{\dagger}A = A(A^{\dagger})^2A^5(A^{\dagger})^2A, \tag{3.52}$$

$$(((((A^3)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} = A^{\dagger}A(((A^3)^{\dagger})^{\#})^{\dagger}AA^{\dagger} = A^{\dagger}A^2A^{\dagger}(A^3)^{\dagger}A^{\dagger}A^2A^{\dagger}, \tag{3.53}$$

$$((((((A^3)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} = A^{\dagger}A((((A^3)^{\#})^{\dagger})^{\#})^{\dagger}AA^{\dagger} = A^{\dagger}A^2(A^{\dagger})^2A^5(A^{\dagger})^2A^2A^{\dagger}, \tag{3.54}$$

$$\begin{aligned}(((((((A^3)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} &= AA^{\dagger}(((((A^3)^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger}A^{\dagger}A \\ &= A(A^{\dagger})^2A^2A^{\dagger}(A^3)^{\dagger}A^{\dagger}A^2(A^{\dagger})^2A,\end{aligned} \tag{3.55}$$

$$\begin{aligned}((((((((A^3)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger} &= AA^{\dagger}((((((A^3)^{\#})^{\dagger})^{\#})^{\dagger})^{\#})^{\dagger}A^{\dagger}A \\ &= A(A^{\dagger})^2A^2(A^{\dagger})^2A^5(A^{\dagger})^2A^2(A^{\dagger})^2A.\end{aligned} \tag{3.56}$$

Proof. The two equalities in (3.27) follow from (2.18) and (3.23) by replacing A with $A^{\dagger}$. Pre- and post-multiplying the two equalities with A and simplifying leads to (3.28). Pre- and post-multiplying the two equalities with $((A^{\dagger})^2)^{\dagger}$ and simplifying leads to (3.29).

By (2.23),

$$\begin{aligned}r(A(A^{\#})^{\dagger}A - A^3) = r\begin{bmatrix} A^{\#} & A \\ A & A^3 \end{bmatrix} - r(A^{\#}) &= r\begin{bmatrix} AA^{\#}A & A^2 \\ A^2 & A^3 \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} - r(A) = 0,\end{aligned}$$

which is equivalent to (3.33) by the basic fact: $r(M) = 0$ if and only if $M = 0$. Pre- and post-multiplying both sides of (3.33) with A and simplifying leads to (3.30). Replacing A with A^2 and A^3 in (3.30), respectively, and noting that $A^2(A^2)^{\dagger} = AA^{\dagger}$, $(A^2)^{\dagger}A^2 = A^{\dagger}A$, $A^3(A^3)^{\dagger} = AA^{\dagger}$, and $(A^3)^{\dagger}A^3 = A^{\dagger}A$ under $r(A^2) = r(A)$, we obtain (3.31) and (3.32). Pre- and post-multiplying both sides of (3.31) and (3.32) with A and simplifying leads to (3.34) and (3.35).

Replacing A with $A^{\dagger}$ in (3.30)–(3.32) and simplifying results in (3.36)–(3.38). Pre- and post-multiplying both sides of (3.36)–(3.38) with $A^{\dagger}$ and simplifying leads to (3.39)–(3.41).

Replacing A with $A^{\#}$ in (3.36) and simplifying by $A^{\#}(A^{\#})^{\dagger} = AA^{\dagger}$ and $(A^{\#})^{\dagger}A^{\#} = A^{\dagger}A$ yields the first equality in (3.42). Substituting (3.30) in the second term in (3.42) leads to the second equality in (3.42). Replacing A with $A^{\dagger}$ in (3.42) and simplifying yields the first equality in (3.43). Substituting (3.36) in the second term in (3.43) leads to the second equality in (3.43). Eqs. (3.44)–(3.46) can be inductively established by similar steps.

Replacing A with A^2 in (3.42)–(3.46) and simplifying by $A^2(A^2)^{\dagger} = AA^{\dagger}$ and $(A^2)^{\dagger}A^2 = A^{\dagger}A$ under $r(A^2) = r(A)$ leads to (3.47)–(3.51).

Replacing A with A^3 in (3.47)–(3.51) and simplifying by $A^3(A^3)^{\dagger} = AA^{\dagger}$ and $(A^3)^{\dagger}A^3 = A^{\dagger}A$ under $r(A^2) = r(A)$ leads to (3.52)–(3.56). □

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As applications, we give the following two groups of results on equalities composed of mixed operations of the Moore–Penrose and group inverses of a matrix.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times m}$ with $r(A^2) = r(A)$. Then the following 8 conditions are equivalent*:

(a) $(A^{\#})^{\dagger} = (A^{\dagger})^{\#}$.
(b) $((A^{\#})^{\dagger})^{\#} = A^{\dagger}$.
(c) $((A^{\dagger})^{\#})^{\dagger} = A^{\#}$.
(d) $(((A^{\#})^{\dagger})^{\#})^{\dagger} = A$.
(e) $(((A^{\dagger})^{\#})^{\dagger})^{\#} = A$.
(f) $(A^3)^{\dagger} = (A^{\dagger})^3$.
(g) $A^*A(A^3)^{\dagger}AA^* = A^*A^{\dagger}A^*$.
(h) $r\begin{bmatrix} (A^3)^* & 0 & A^*A \\ 0 & -AA^*A & A^2 \\ AA^* & A^2 & 0 \end{bmatrix} = 2r(A)$.

Proof. The equivalences of (a)–(e) follow from (2.1) and (2.16). Substituting (3.27) and (3.30) into $(A^{\#})^{\dagger} = (A^{\dagger})^{\#}$ leads to $A^{\dagger}((A^{\dagger})^3)^{\dagger}A^{\dagger} = A^{\dagger}A^3A^{\dagger}$, which is also equivalent to $((A^{\dagger})^3)^{\dagger} = A^3$, thus establishing the equivalence of (a) and (f).

Pre- and post-multiplying both sides of (f) with A^*A and AA^* respectively, and simplifying leads to (g).

Note that $\mathscr{R}(AA^*) = \mathscr{R}(A^3)$ and $\mathscr{R}(A^*A) = \mathscr{R}((A^3)^*)$ under $r(A^2) = r(A)$. In this case, we find from (2.23) that

$$\begin{aligned} r(A^*A(A^3)^{\dagger}AA^* - A^*A^{\dagger}A^*) &= r\begin{bmatrix} A^3 & AA^* \\ A^*A & A^*A^{\dagger}A^* \end{bmatrix} - r(A^3) \\ &= r\left(\begin{bmatrix} A^3 & AA^* \\ A^*A & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A^* \end{bmatrix} A^{\dagger}[0,\ A^*] \right) - r(A) \\ &= r\begin{bmatrix} A^3 & 0 & AA^* \\ 0 & -A^*AA^* & (A^2)^* \\ AA^* & (A^2)^* & 0 \end{bmatrix} - 2r(A). \end{aligned}$$

Setting the left hand side of the equalities equal to zero results in the equivalence of (g) and (h). □

Theorem 3.6. *Let $A \in \mathbb{C}^{m\times m}$ with $r(A^2) = r(A)$. Then the following 10 conditions are equivalent*:

(a) $(A^2)^{\dagger} = (A^{\dagger})^2$.
(b) $(A^2)^{\dagger} = (A^{\#})^2$.
(c) $((A^2)^{\#})^{\dagger} = ((A^{\#})^{\dagger})^2$.
(d) $((A^2)^{\dagger})^{\#} = ((A^{\dagger})^{\#})^2$.
(e) $(((A^2)^{\#})^{\dagger})^{\#} = (((A^{\#})^{\dagger})^{\#})^2$.
(f) $(((A^2)^{\dagger})^{\#})^{\dagger} = (((A^{\dagger})^{\#})^{\dagger})^2$.
(g) $(((A^2)^{\dagger})^{\#})^{\dagger} = (((A^{\#})^{\dagger})^{\#})^2$.
(h) $((((A^2)^{\#})^{\dagger})^{\#})^{\dagger} = ((((A^{\#})^{\dagger})^{\#})^{\dagger})^2$.
(i) $((((A^2)^{\dagger})^{\#})^{\dagger})^{\#} = ((((A^{\dagger})^{\#})^{\dagger})^{\#})^2$.
(j) $\mathscr{R}(A) = \mathscr{R}(A^*)$.

Proof. The equivalence of (a) and (j) follows from (3.21).

The equivalence of (b) and (j) follows from (3.26) and the rule $(A^{\#})^2 = (A^2)^{\#}$.

Replacing A with $A^{\#}$ in (a) and noting $(A^{\#})^2 = (A^2)^{\#}$ and (2.20) and (2.21) leads to the equivalence of (c) and (j).

By the rule $(A^{\#})^2 = (A^2)^{\#}$, we rewrite the matrix equality in (d) as $((A^2)^{\dagger})^{\#} = ((A^{\dagger})^2)^{\#}$, which is also equivalent to $(A^2)^{\dagger} = (A^{\dagger})^2$ by (2.16), thus establishing the equivalence of (a) and (d).

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Taking the group inverses of both sides of the equality in (c) and applying the rule $(A^{\#})^2 = (A^2)^{\#}$ leads to the equality in (e).

By (3.37) and (3.38), the equality in (f) is equivalent to $AA^{\dagger}(A^2)^{\dagger}A^{\dagger}A = A(A^{\dagger})^3A^2(A^{\dagger})^3A$. Pre- and post-multiplying both sides of the equality with $(A^{\dagger})^{\#}A^{\dagger}$ and $A^{\dagger}(A^{\dagger})^{\#}$, respectively, and simplifying leads to $(A^2)^{\dagger} = (A^{\dagger})^2A^2(A^{\dagger})^2$, thus establishing the equivalence of (a) and (f) through the the fifth and the seventh equalities in (3.21).

Notice from (3.27) and (3.28) that $\mathscr{R}((((A^2)^{\dagger})^{\#})^{\dagger}) = \mathscr{R}(A^*)$ and $\mathscr{R}((((A^{\#})^{\dagger})^{\#})^2) = \mathscr{R}(A^*)$ hold. Then, (g) is equivalent to (j) by (2.1), (2.16), and (3.26).

By (3.42) and (3.47), the equality in (h) is equivalent to

$$A(A^{\dagger})^2A^4(A^{\dagger})^2A = A(A^{\dagger})^2A^3(A^{\dagger})^2A^2(A^{\dagger})^2A^3(A^{\dagger})^2A.$$

Pre- and post-multiplying both sides of the equality with $A^{\#}(A(A^{\dagger})^2A)^{\dagger}$ and $(A(A^{\dagger})^2A)^{\dagger}A^{\#}$, respectively, and simplifying leads to $A^2 = A^2(A^{\dagger})^2A^2(A^{\dagger})^2A^2$, thus establishing the equivalence of (a) and (h) through the the third and fifth equalities in (3.21).

Taking the group inverses of both sides of the equality in (f) and applying the rule $(A^{\#})^2 = (A^2)^{\#}$ leads to the equality in (i), and the vas versa, thus establishing the equivalence of (f) and (i). □

Theorem 3.7. *Let $A \in \mathbb{C}^{m\times m}$ with $r(A^2) = r(A)$. Then the following equalities always hold:*

$$(AA^{\#})^{\dagger} = (A^{\#}A)^{\dagger} = A^{\dagger}A^2A^{\dagger}, \tag{3.57}$$
$$(A^{\dagger}(A^{\dagger})^{\#})^{\dagger} = ((A^{\dagger})^{\#}A^{\dagger})^{\dagger} = A(A^{\dagger})^2A, \tag{3.58}$$
$$A(AA^{\#})^{\dagger}A = A(A^{\#}A)^{\dagger}A = A^2, \tag{3.59}$$
$$A^{\dagger}(A^{\dagger}(A^{\dagger})^{\#})^{\dagger}A^{\dagger} = A^{\dagger}((A^{\dagger})^{\#}A^{\dagger})^{\dagger}A^{\dagger} = (A^{\dagger})^2. \tag{3.60}$$

Proof. It is easy to verify that $AA^{\#}$ and $(A^{\dagger}A^2A^{\dagger})^{\dagger}$ are two idempotent matrices. In this case, applying (2.24) to the difference $AA^{\#} - (A^{\dagger}A^2A^{\dagger})^{\dagger}$, we obtain

$$r(AA^{\#} - (A^{\dagger}A^2A^{\dagger})^{\dagger}) = r\begin{bmatrix} AA^{\#} \\ (A^{\dagger}A^2A^{\dagger})^{\dagger} \end{bmatrix} + r[AA^{\#},\ (A^{\dagger}A^2A^{\dagger})^{\dagger}] - r(AA^{\#}) - r((A^{\dagger}A^2A^{\dagger})^{\dagger}), \tag{3.61}$$

where by Lemma 2.5, (2.5)–(2.7), and (2.19)–(2.21),

$$r\begin{bmatrix} AA^{\#} \\ (A^{\dagger}A^2A^{\dagger})^{\dagger} \end{bmatrix} = r\begin{bmatrix} A \\ (A^{\dagger}A^2A^{\dagger})^{*} \end{bmatrix} = r\begin{bmatrix} A \\ A(A^{\dagger})^2A \end{bmatrix} = r(A),$$
$$r[AA^{\#},\ (A^{\dagger}A^2A^{\dagger})^{\dagger}] = r[A,\ (A^{\dagger}A^2A^{\dagger})^{*}] = r[A,\ A(A^{\dagger})^2A] = r(A),$$
$$r((A^{\dagger}A^2A^{\dagger})^{\dagger}) = r(A^{\dagger}A^2A^{\dagger}) = r(A^2) = r(A).$$

Substituting these rank equalities into (3.61) results in $r(AA^{\#} - (A^{\dagger}A^2A^{\dagger})^{\dagger}) = 0$, i.e., $AA^{\#} = (A^{\dagger}A^2A^{\dagger})^{\dagger}$, which is further equivalent to (3.57) by the second equality in (2.1). Replacing A with $A^{\dagger}$ in (3.57) and applying the second equality in (2.1) leads to (3.58). Pre- and post-multiplying both sides of two equalities in (3.57) and (3.58) with A and $A^{\dagger}$, respectively, leads to (3.59) and (3.60). □

Finally, the author points out that numerous matrix expansion equalities and equivalent facts analogous to these in Theorem 3.4 can be constructed and classified by induction. As applications of these expansion equalities, it is possible to establish and characterize a wide range of equivalent facts related to matrix equalities that are composed of a matrix and its generalized inverses. As some concrete examples in this regard, the author proposes the following equivalent facts:

$$((A^2)^{\#})^{\dagger} = ((A^{\dagger})^{\#})^2 \Leftrightarrow (A^4)^{\dagger} = (A^{\dagger})^4,$$

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$$((A^3)^{\#})^{\dagger} = ((A^{\dagger})^{\#})^3 \Leftrightarrow (A^5)^{\dagger} = (A^{\dagger})^5,$$
$$(A^{\dagger})^{\#}(A^{\#})^{\dagger} = (A^{\#})^{\dagger}(A^{\dagger})^{\#} \Leftrightarrow A^2((A^{\dagger})^2)^{\dagger} = ((A^{\dagger})^2)^{\dagger}A^2,$$
$$(((A^{\#})^{\dagger})^{\#})^{\dagger} = (((A^{\dagger})^{\#})^{\dagger})^{\#} \Leftrightarrow (A^3)^{\dagger} = (A^{\dagger})^2A^2(A^{\dagger})^3A^2(A^{\dagger})^2.$$

and leaves their verifications for the reader.

1. A. Ben–Israel, T.N.E. Greville. Generalized Inverses: Theory and Applications. 2nd ed., Springer, New York, 2003.
2. D.S. Bernstein. Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas–Revised and Expanded Edition. Princeton University Press, Princeton, NJ, USA; Oxford, UK, 2018.
3. S.L. Campbell, C.D. Meyer. Generalized Inverses of Linear Transformations. Pitman Publishing Limited, 1979; Dover Publications, 1991; SIAM, Philadelphia, 2009.
4. G. Marsaglia, G.P.H. Styan. Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra 2(1974), 269–292.
5. Y. Tian. Reverse order laws for the generalized inverses of multiple matrix products. Linear Algebra Appl. 211(1994), 85–100.
6. Y. Tian. Rank equalities related to outer inverses of matrices and applications. Linear Multilinear Algebra 49(2002), 269–288.
7. Y. Tian. Equivalence analysis of different reverse order laws for generalized inverses of a matrix product. Indian J. Pure Appl. Math. 53(2022), 939–947.
8. Y. Tian. Characterizations of the group invertibility of a matrix revisited. Demonstratio Math. 55(2022), 866–890.