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Article

Theoretical Foundations of Macroeconomics with Impulse and Jump Characteristics

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Abstract: In recent years, the economies of many countries of the world are in a situation of intense shock processes. The current situation urgently needs to change the economic paradigm in the near future, which objectively requires the development of new conceptual models. The purpose of the paper is to develop the basic theoretical provisions of a new macroeconomics based on impulse and jump processes. Research problems are the following: 1) to describe the main types of impulse and jump functions with examples from economic theory and practice; 2) to perform an analytical representation of impulse and jump functions; 3) to select macroeconomic characteristics for the analysis of rapidly changing processes in the economy; 4) to create models and mechanisms for predicting impulsive and abrupt changes in macroeconomics. The approaches to the development of macroeconomic theory and its methods proposed in the article are not associated with the use of evolutionary continuous functions, for example, power functions, which is typical for many canonical macroeconomic models. These approaches do not include management decisions to achieve optimal values of given target functions, which is typical for recursive macroeconomic models of dynamic programming. This article is about formulating the foundations of macroeconomic theory and its methods, which, with some degree of accuracy, could give a forecast about the impending possibility of abrupt changes (impulse, shock, jump, stepwise and others) in the macroeconomic situation. The research methodology is statistical analysis, special methods for studying impulse and jump processes developed by the author. As a result of the study, the main provisions of the macroeconomic theory based on rapid impulse and spasmodic changes were formulated, approaches were outlined for constructing the tools of this theory, problems and tasks for further research were identified.

Keywords: macroeconomics; theoretical foundations; impulse and jump characteristics

MSC: 37N40

1. Introduction

There are a lot of problems in this area. But two main questions are the following: "Why are some countries rich and others poor?", and "Why does the economy in some countries grow fast and in others slowly?" [1]. To answer these questions, we need to analyze data relating to the economic performance of various countries.

Currently, there are different models of economic growth. Let's look at the most famous of them.

The foundations of modern growth theory were laid by Nobel Prize winner R. Solow, having developed the neoclassical theory of economic growth [2], in which the main role was played by the accumulation of physical capital. Solow also showed for the first time that the key factor of economic growth is technical progress, which he set exogenously [3]. The main focus of the Solow model, along with traditional issues of capital accumulation, is on the relationship between the two main factors of production - labor and capital, as well as their relationship with the exogenous source of changes in productivity - technical progress. Solow's growth theory, like all classical growth theories, is based on the law of diminishing returns of factors, which is the main condition for achieving an equilibrium state and sustainable development. According to this law, under constant technical conditions, a consistent increase in any of the production factors by an additional unit, with the others remaining

unchanged, leads to a decreasing increase in production. This occurs under conditions of perfect competition. The production function in the economic growth model is the Cobb-Douglas function, which includes a power function, the basis of which is physical capital, and the exponent varies from zero to one. The mathematical basis of Solow's economic growth model is the first-order Bernoulli differential equation.

From the point of view of this article, the disadvantage of the Solow model is that it is based on continuous functions and does not take into account impulse, shock, step, spasmodic and other types of rapidly changing processes characteristic of modern macroeconomics. In addition, this model includes a multiplier that expresses a measure of productivity. To this day, economists have been unable to quantify this multiplier with a high enough degree of realism. Solow himself called this measure "a measure of our ignorance." [4]. This drawback sharply reduces the possibility of practical use of the model for solving real problems and predicting economic events.

In the Mankiw-Romer-Weil model [5], human capital is introduced into the basic neoclassical Solow model of economic growth, which improves the theoretical provisions of the model and is more consistent with empirical data than the similar result of the Solow model without human capital. However, the main results of the basic neoclassical model remain unchanged; sustainable economic growth depends on external technological progress, while the savings rate, institutional and behavioral parameters do not affect it. Therefore, sustainable economic growth remains exogenous in nature.

The shortcomings of this model are similar to those of the Solow macroeconomic model. Just like the Solow model, the Mankiw-Romer-Weil model is based on continuous power functions, which does not allow it to be used to describe abrupt macroeconomic processes.

Recursive macroeconomic models of Stokey-Lucas-Prescott [6] are based on dynamic programming methods developed by R. Bellman [7]. Dynamic programming is associated with the ability to represent the management process in the form of a chain of sequential actions or steps, unfolded over time and leading to a goal. The control process can be divided into parts and represented as a dynamic sequence and interpreted as a step-by-step program. The dynamic programming problem is formulated as follows: it is required to determine a control that transfers the system from the initial state to the final state, in which the objective function takes an extreme value.

The goals and methods of the proposed macroeconomic theory are completely different. It is not about a dynamic management process with a specifically defined goal and an assigned target function. This article is about formulating the foundations of macroeconomic theory and its methods, which, with some degree of accuracy, could give a forecast about the impending possibility of abrupt changes (impulse, shock, stepwise and others) in the macroeconomic situation, both in a negative and positive direction. What management decisions will be made to correct the situation based on the forecasts made remains outside the scope of the proposed approach to the analysis of fast-moving processes in macroeconomics.

The economies of many countries are strongly influenced by rapidly changing processes and phenomena, crisis events, epidemics, military operations, and so on. This is especially evident in recent years [8–13]. Examples include the global pandemic, events in Ukraine, Russia, Israel, Palestine and Europe, fuel and food crises, and others. The new paradigm requires the development of new economic theories, economic and mathematical models that take into account impulse, step and spasmodic changes in economic indicators. The article describes new mathematical methods that make it possible to analyze macroeconomic processes with impulse, generalized and piecewise linear functions. Methods for approximating these functions by means of analytical expressions are considered. The possibilities of forecasting macroeconomic processes with impulse and jump characteristics are presented.

Let us note some publications in which shock situations are studied in various aspects of macroeconomic theory [14–25]. Note that these works represent only a small part of the publications on the selected topic. This once again indicates the relevance of developing a macroeconomic theory with rapidly changing characteristics.

2. Main Types of Impulse and Step Characteristics

In the scientific literature, there are discrepancies in the definitions of impulse and step characteristics depending on the areas of their use (signal transmission, electrical engineering, automatic control theory, dynamic systems, catastrophe theory, and others) [26–29]. For definiteness, in this paper, by jump characteristics we mean functions with a sharp change in values with subsequent stabilization of values at a new level (shifts). By impulse responses we mean functions with a sharp short-term change in values (shocks). The words “sharp”, “short-term” have a subjective meaning, however, the definitions can be clarified using the terms “outliers”, “standard deviation” and other statistical concepts.

Next, we present idealized models of these characteristics and their applications in the field of economic analysis. Naturally, in practice, fluctuations caused by many random causes are superimposed on trend lines.

1. Single jump function (single step function, Heaviside function, “step”).

One way to write this function is:

$$f(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

The graph of the function is shown in Figure 1a).

In practice, a sharp change in the values of the function does not occur instantly, but with a certain transitional time period x^* (Figure 1b). For example, an abrupt change in the exchange rate of the national currency tenge in Kazakhstan in February 2014 against the US dollar and the euro took place over several days (Table 1). The official website of the National Bank of the Republic of Kazakhstan is www.nationalbank.kz.

Table 1. Exchange rates in Kazakhstan.

Date	Dollar exchange rate	Euro exchange rate
2014-02-10	155,5	210,89
2014-02-11	155,56	212,25
2014-02-12	163,9	224,07
2014-02-13	184,5	251,57

The unit jump function is the antiderivative function for the delta function (Dirac function) [28]. The delta function graph is shown in Figure 1c).

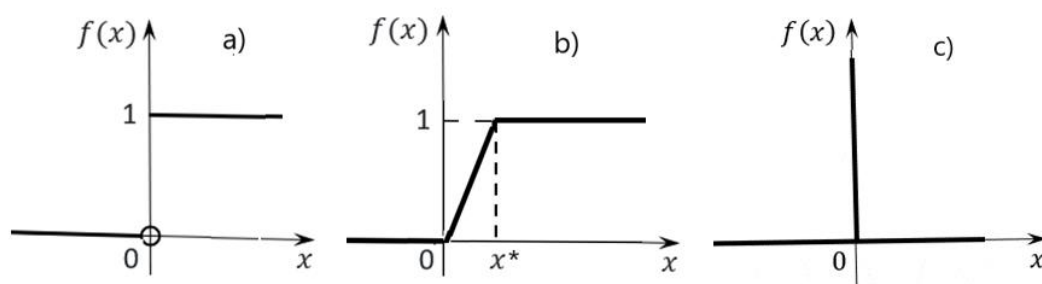


Figure 1. Graphs of the unit jump function: a) idealized; b) with a transition period and c) delta functions.

Examples of the manifestation of the unit jump function in the economy are shown in Figure 2.

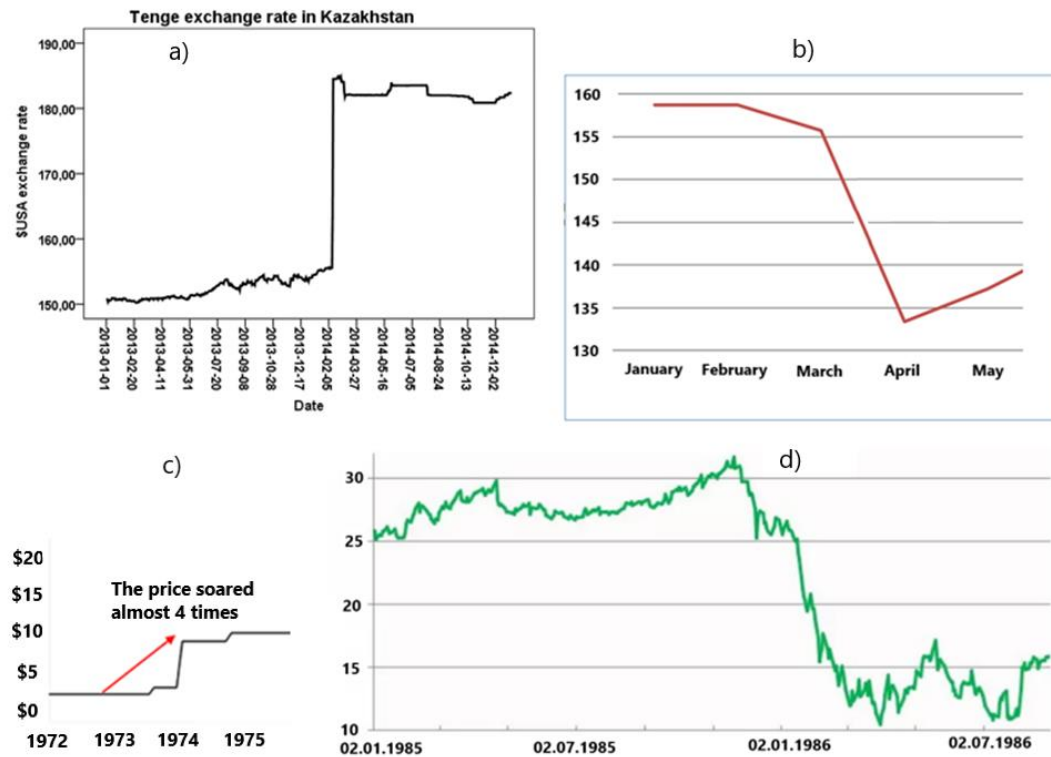


Figure 2. Examples of abrupt changes in the economy: a) change in the tenge exchange rate in Kazakhstan, the official website of the National Bank of the Republic of Kazakhstan - www.nationalbank.kz; b) the number of people employed in the USA, million people, January-May 2020, source: BLS. URL: <https://www.bls.gov> (accessed 09/21/2020); c) WTI oil price dynamics, source: FRED; d) oil price from January 1985 to July 1986 (<http://geoinform.ru/sem-s-polovinoj-tysyach-slancevyx-skvazhin-zhdut-svoego-chasa>).

In September 2020, the incidence of COVID-19 began to increase in most countries. In June 2020, the National Bureau of Economic Research (USA) announced a recession or recession in the economy since February. In absolute terms, the number of unemployed increased from 5787 thousand people. (February) to 23078 thousand people. (April). In August it amounted to 13550 thousand people. The employment rate is shown in Figure 2b).

In order to highlight the trend in more detail, it is possible to use a statistical procedure for smoothing the average value (Figure 3).

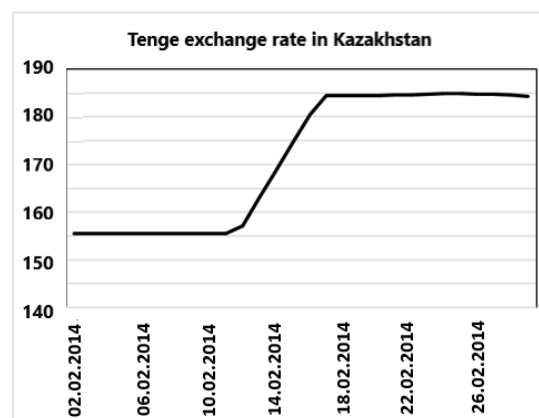


Figure 3. Graph of the smoothed average for the change in the tenge exchange rate.

2. A unit pulse (rectangular function, normalized rectangular window) is given by the following expression:

$$f(x) = \begin{cases} 0, & |x| > 0,5; \\ 0,5, & |x| = 0,5; \\ 1, & |x| < 0,5. \end{cases}$$

The graph of the function is shown in Figure 4a).

3. Triangular impulse (triangular function) is a piecewise linear function given by:

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1; \\ 0, & |x| \geq 1. \end{cases}$$

The graph of the function is shown in Figure 4b).

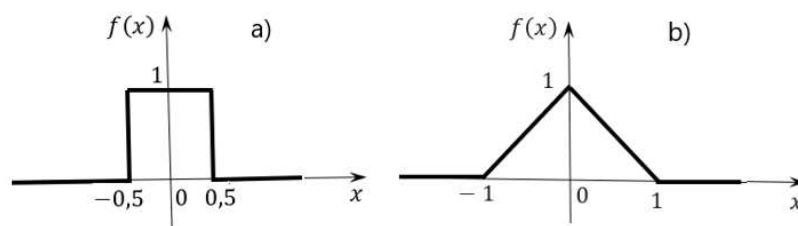


Figure 4. Graphs of impulse functions.

Examples of impulse functions in the economy are shown in Figure 5.

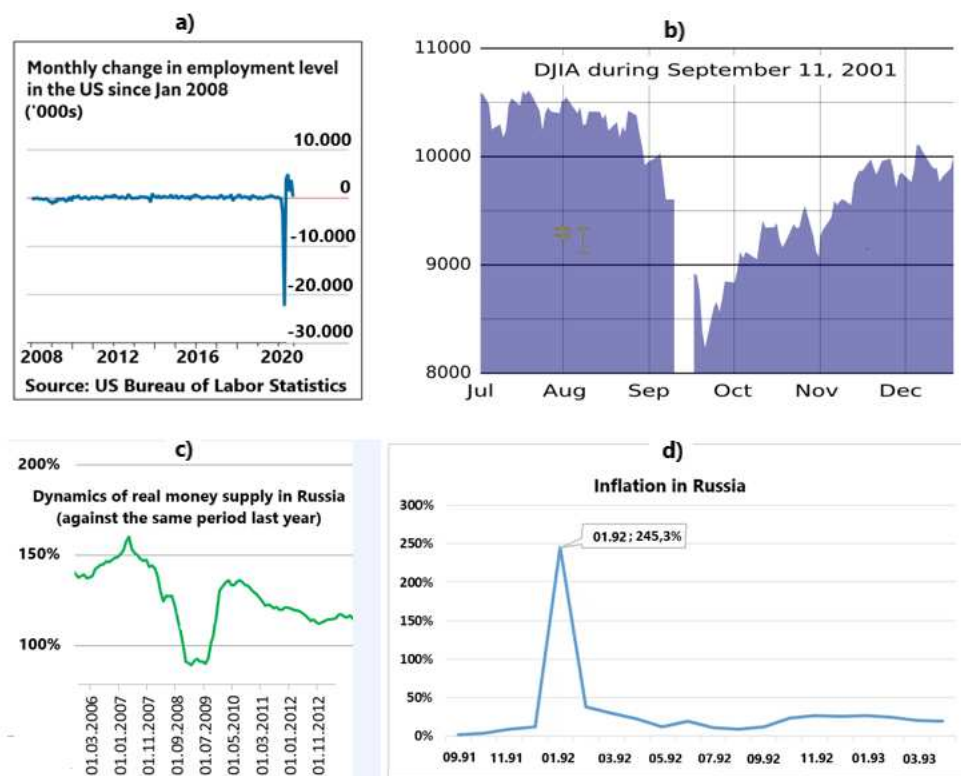


Figure 5. Examples of impulse changes in the economy: a) monthly change in employment level in the USA, source: US Bureau of Labor Statistics; b) Dow Jones industrial index, source: <https://www.economicportal.ru/ponyatiya-all/makroekonomicheskij-shok.html>; c) dynamics of real money supply in Russia, source: Central Bank of Russia, Rosstat; d) inflation in Russia, source: Rosstat data.

The graph in Figure 5a largely corresponds to the delta function.

On Figure 5b shows the macroeconomic shock caused by the September 11, 2001 terrorist attacks and reflected in the Dow Jones Industrial Average. After the initial panic, the DJIA rose rapidly, falling only marginally from its pre-attack levels.

4. To describe rapidly changing macroeconomic processes, various combinations of impulse and jump models are possible. For example, Figure 6 shows the stepwise transformation of the Russian ruble exchange rate from one stable state to another through an impulse transition process. This shows the effect of the interaction of the step and impulse functions. The trend line is highlighted in red. Figure 7 shows the dynamics of the world cycle: industrial production, 2019–2022, seasonally adjusted quarterly growth rates. In the figure we see a combination of two oppositely directed impulses. It is also possible to display various variants of piecewise linear functions.

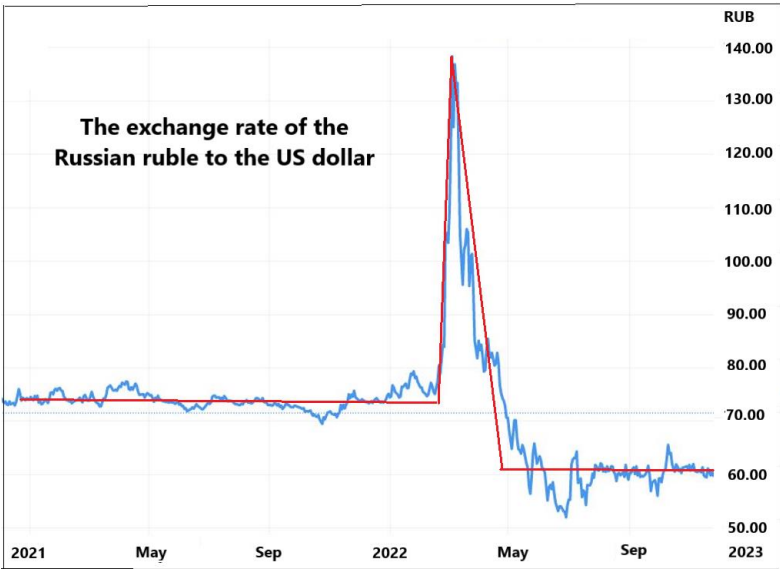


Figure 6. The exchange rate of the Russian ruble against the US dollar.

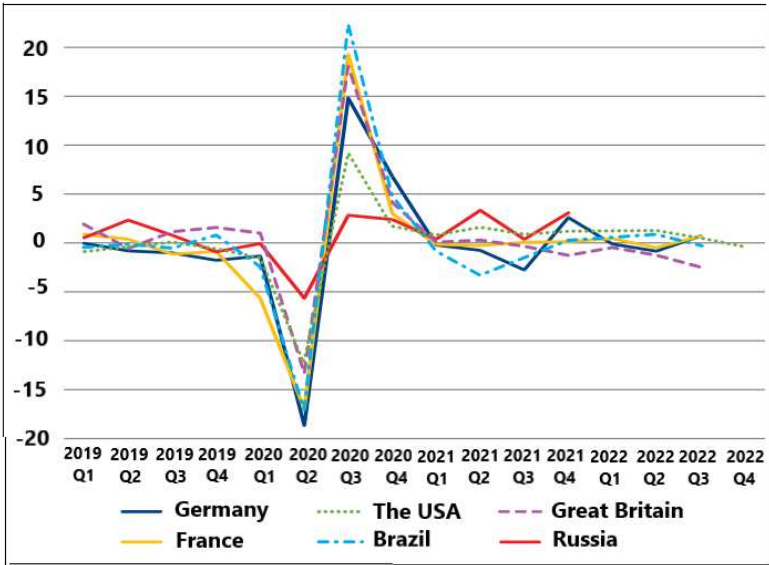


Figure 7. Dynamics of the world cycle: industrial production, 2019–2022, seasonally adjusted quarterly growth rates. Source: Federal Reserve Economic Data.

3. Approximations of Step, Impulse and Generalized Functions

3.1. Description of the Methods

Approximations of step, impulse and generalized functions makes it possible to more accurately identify trends in rapidly changing processes and describe existing processes in macroeconomics. These functional forms may become useful here as predictive tools.

Currently, there are many methods for approximating piecewise linear functions using analytic expressions. This article describes only the methods proposed by the author and having a number of advantages over known approximations. These methods, like Fourier series, are based on the use of trigonometric expressions. The difference lies in the fact that trigonometric expressions in the developed methods have the form of nested, recursive functions [30–33]. The methods were further developed in [34,35] and many other scientific papers.

For example, let's take a step function, which will explain the main idea of the approximation:

$$f_0(x) = \text{sign}(\sin x) \quad (1)$$

The step function is often used to describe Fourier series. Therefore, the choice of a step function is convenient for comparing and identifying the advantages and disadvantages of expansion into Fourier series and the methods developed by the author.

It is known that the Fourier expansion of the proposed function (1) is characterized by certain disadvantages [36,37]. The Gibbs effect can be noted. In the developed methods, the initial step function is approximated by a recursive sequence of periodic trigonometric functions, which eliminates the shortcomings of the Fourier series expansion:

$$\{f_n(x) \mid f_n(x) = \sin((\pi/2) \cdot f_{n-1}(x)), f_1(x) = \sin x; n-1 \in N\} \subset C^\infty[-\pi, \pi] \quad (2)$$

In Figure 8, the thickened line marks the initial step function. The thin lines show the graphs of five successive approximations by the proposed method.

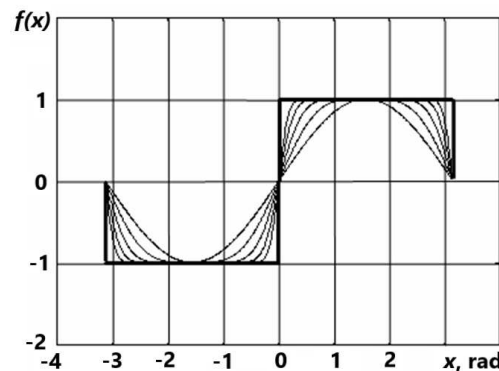


Figure 8. Graphs of the step function (the thick line) and five successive approximations of this function (the thin lines).

Figure 8 shows that even the first approximations of the proposed iterative sequence quickly converge to the step function (1). At the same time, the shortcomings of the expansion into Fourier series, in particular, the Gibbs effect [36], are completely leveled.

The developed methods of trigonometric approximation using an iterative recursive procedure differ in some of their properties. Let's note some of them.

Periodic functions $f_n(x)$ and $f_0(x)$ with period 2π are odd. The periodic functions $f_n(x + \pi/2)$ and $f_0(x + \pi/2)$ are even. It follows from this fact that one can only study the approximating sequence of functions (2) on the interval on the interval $[0, \pi/2]$. It would be enough.

Let $\{f_n(x)\} \subset L_2[0, \pi/2]$ and $f_0(x) \in L_2[0, \pi/2]$. Because $\sup_{n \in N} \sup_{x \in [0, \pi/2]} |f_n(x)| = 1 < \infty$ (because the function is limited) $f_n(x)$ and $\sup_{n \in N} \sup_{x \in [0, \pi/2]} f_n(x) = 1 < \infty$ (since the function $f_n(x)$ is monotonic on the segment $[0, \pi/2]$), then by Helly's theorem we can extract a subsequence from the given sequence $\{f_n(x)\}$. This subsequence converges to some function f at each point from $[0, \pi/2]$, and, additionally, $\sup_{x \in [0, \pi/2]} f \leq \lim_{n \rightarrow \infty} \sup_{x \in [0, \pi/2]} f_n$. The original function f may be such a function $f_0(x)$. Let's prove this statement.

Theorem 1. The sequence $f_n(x)$ converges pointwise to the function $f_0(x)$, but not uniformly.

Proof.

In $x = 0$ and $x = \pi/2$ there is $f_n(x) - f_0(x) = 0, \forall n \in N$. Therefore $f_n(x) \xrightarrow{n \rightarrow \infty} f_0(x)$ since $\forall \varepsilon > 0 \exists n^* \in N \forall n : n > n^* \Rightarrow |f_n(x) - f_0(x)| = 0 < \varepsilon$. For instance, $n^* = 1$.

$f_n(x) = \sin((\pi/2) \cdot f_{n-1}(x)) > f_{n-1}(x) > \dots > f_1(x) > 0$ for any $x \in (0, \pi/2)$, since $\sin x > (2/\pi) \cdot x, \forall x \in (0, \pi/2)$. $f_n(x), \forall x \in (0, \pi/2)$ has a finite limit $\lim_{n \rightarrow \infty} f_n(x) = A \in R$.

$A = \lim_{n \rightarrow \infty} \sin((\pi/2) \cdot f_{n-1}(x)) = \sin((\pi/2) \cdot \lim_{n \rightarrow \infty} f_{n-1}(x)) = \sin((\pi/2) \cdot A)$, $A = 0$ or $A = 1$. $A = 1 = f_0(x)$, since $f_n(x)$ is increasing and positive. For $x \in (0, \pi/2)$, $f_n(x) \xrightarrow{n \rightarrow \infty} f_0(x)$. $f_n(x)$ converges at $x = 0$, $x = \pi/2$. Therefore, $f_n(x) \xrightarrow{n \rightarrow \infty} f_0(x), \forall x \in [0, \pi/2]$. $f_0(x)$ is not continuous on $[0, \pi/2]$. Therefore, the convergence is pointwise, but not uniform. ■

Theorem 2. $f_n(x)$ converges in norm to $f_0(x)$ in the spaces $L_1[0, \pi/2]$ and $L_2[0, \pi/2]$.

Proof.

We take a sequence of minorant functions for $f_n(x)$:

$$\{\eta_n(x) \mid \eta_n(x) = (2/\pi) \cdot \arctg(n\pi); n \in N\} \subset C^\infty[0, \pi/2]$$

$f_n(x) \geq \eta_n(x), \forall n \in N, \forall x \in [0, \pi/2]$. The measure for the set of discontinuity points of $f_0(x)$ is zero. $f_n(x)$ and $\eta_n(x)$ are non-negative and limited on the segment $[0, \pi/2]$ in $L_1[0, \pi/2]$:

$$\|f_0(x) - f_n(x)\| = \int_0^{\pi/2} (1 - f_n(x)) dx \leq \int_0^{\pi/2} (1 - \eta_n(x)) dx = \frac{\pi}{2} - \arctg \frac{\pi n}{2} + \frac{1}{\pi n} \cdot \ln(1 + (\pi n)^2/4)$$

$$\lim_{n \rightarrow \infty} \left(\frac{\pi}{2} - \arctg \frac{\pi n}{2} + \frac{1}{\pi n} \cdot \ln(1 + (\pi n)^2/4) \right) = 0, \text{ therefore, } \|f_0(x) - f_n(x)\| \xrightarrow{n \rightarrow \infty} 0.$$

The same is for $f_n(x)$ in $L_2[0, \pi/2]$. ■

$f_n(x)$ is fundamental in $L_1[-\pi, \pi]$ and $L_2[-\pi, \pi]$, but not in $C[-\pi, \pi]$.

The angular function $f_1(x)$ may not necessarily be the sine, but another one, including non-periodic ones. For $|f_1(x)| < 2$, $\lim_{n \rightarrow \infty} f_n(x) = \text{sign}(f_1(x))$. With (2) it is possible to approximate an arbitrary step function. For example, consider

$$f(x) = \begin{cases} h, & x \in (x_1, x_2), \\ 0, & x \notin (x_1, x_2) \end{cases} \quad (3)$$

and take the function $f_1(x) = \exp(1 - (ax + b)^2) - 1$ as the angular one. $f_1(x_1) = f_1(x_2) = 0$, $a = 2/(x_1 - x_2)$; $b = (x_1 + x_2)/(x_2 - x_1)$, so

$$\left\{ f_n(x) \mid f_n(x) = (h/2) \cdot (1 + \sin \varphi_n(x)), \varphi_n(x) = (\pi/2) \cdot \sin \varphi_{n-1}, \varphi_1(x) = (\pi/2) \cdot f_1(x), n-1 \in \mathbb{N} \right\}$$

converges to $f(x)$. It is shown in Figure 9.

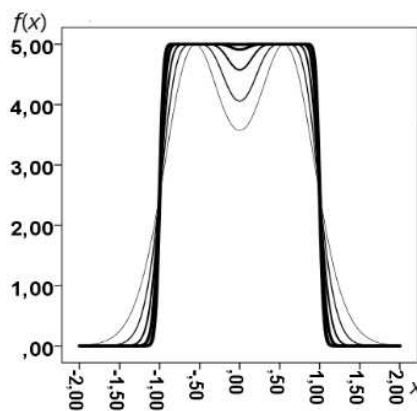


Figure 9. Graphs of successive approximations of the bar function.

Any step function with h_i on (x_{1i}, x_{2i}) can be approximated by the sum $\sum_{i=1}^k \{f_n(x)\}_i$.

$f(x) = \sum_{i=1}^k h_i \cdot f_{0i}(x)$, $h_i \in \mathbb{R}$, where $f_{0i}(x) = \text{sign}(\sin(l_i x - x_i))$, $l_i, x_i \in \mathbb{R}$, can approximate any periodic step function. The convergence of $f_n(x) = \sum_{i=1}^k h_i \cdot f_{ni}(x)$ to $f(x)$ in the norm is ensured by the proved theorem, since $\|f_{0i}(x) - f_{ni}(x)\| \xrightarrow[n \rightarrow \infty]{} 0$ in $L_1[-\pi, \pi]$ and $L_2[-\pi, \pi]$, and

$$\|f(x) - f_n(x)\| = \left\| \sum_{i=1}^k h_i \cdot f_{0i}(x) - \sum_{i=1}^k h_i \cdot f_{ni}(x) \right\| \leq \sum_{i=1}^k |h_i| \cdot \|f_{0i}(x) - f_{ni}(x)\| \xrightarrow[n \rightarrow \infty]{} 0.$$

3.2. Generalized Functions. Approximation

Generalized functions [38] began to be used in the 20th century to solve problems in the field of quantum physics. The solution of such tasks required supplementing the generally recognized mathematical concept of a function. This concept suggested that by a function we understand a certain statement, according to which for each value of an independent variable belonging to a certain set, one value of the dependent variable is indicated. To solve the problems of quantum physics, it was necessary to introduce functions that go beyond the usual definitions. At present, generalized functions are used not only for solving problems of quantum physics, but also for solving other problems in various fields of science and engineering.

Let us consider the concept of a generalized function in more detail.

Let's take a linear space Y . Functions in the traditional mathematical sense are points in this space. A functional is given on Y , if for each point $y \in Y$ from the space under consideration we can specify a certain number. The defined functional will be written $I: Y \rightarrow \mathbb{R}$, or $I(y)$.

The functional is linear under the assumption $I(\alpha y_1 + \beta y_2) = \alpha I(y_1) + \beta I(y_2), \forall y_1, y_2 \in Y, \forall \alpha, \beta \in R$, and continuous when $y_n \Rightarrow y^*$ follows $I(y_n) \rightarrow I(y^*), \forall y_n, y^* \in Y$. We consider our functions on the set of real numbers R .

A function $\varphi(x)$ is finite, if it is equal to zero outside $[a, b]$, and the boundaries of this segment are determined by $\varphi(x)$. Any continuous function with compact support is basic. C_0 is a set of basic functions.

Next, consider a continuous function $f(x)$, which can have a finite number of discontinuity points. The function is bounded on any finite interval.

A functional is defined by the integral $I(\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx$. Any basic function $\varphi(x)$ will be finite. In this case we have a regular functional.

Definition [32]. Any linear continuous functional $I(\varphi)$ on C_0 is called a generalized function, if

1. $I(\alpha\varphi_1 + \beta\varphi_2) = \alpha I(\varphi_1) + \beta I(\varphi_2), \forall \varphi_1, \varphi_2 \in C_0, \forall \alpha, \beta \in R;$
2. $I(\varphi_n) \rightarrow I(\varphi),$ if $\varphi_n \Rightarrow \varphi$ in $C_0.$

A generalized function that cannot be represented by the integral $I(\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx$, is called singular. For example, the δ -function or the Dirac function is a singular generalized function $I(\varphi) = \varphi(0)$.

The basic idea of singular generalized functions can be visualized if we consider a generalized function from the point of view of the limit to which a sequence of approximating functions with traditional definitions tends. In this case, generalized functions are perceived through the prism of their approximating procedures. For example, suppose

$$\delta_n(x) = \begin{cases} n/2, & \forall x \in [-1/n, 1/n], \\ 0, & \forall x \notin [-1/n, 1/n]. \end{cases}$$

The functions of this sequence have graphs corresponding to the graph of the step function shown in Figure 10.

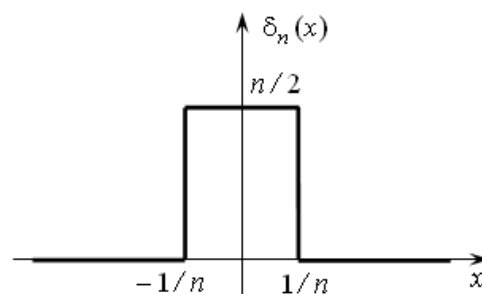


Figure 10. Graph of step function.

It is easy to see that for any area of the figure under the graph of such a step function is equal to one.

The limit of the sequence of the functions, the graph of one of them is schematically shown in Figure 10 is the delta function. However, step functions have discontinuity points of the 1st kind. At these points the step functions are not differentiable. Therefore, this approach prevents the analytical representation of the approximation dependences for the derivatives of the delta function. Note that the derivatives of the delta function also belong to the class of generalized functions. To enable the analytical representation of derivatives, the author proposed a procedure using an approximating sequence that allows one to find derivatives of delta functions of any order. Let us write an expression that implements this approach

$$f(x) = \cos\left(A\left(A(\dots A(x))\right)\right), \text{ where } A(x) = \frac{\pi}{2}\sin(x).$$

For example, the graph of the function

[illegible]

from the proposed sequence is shown in Figure 11.

The approximation error of the δ -function by the proposed methods is much smaller compared to the Fourier series. Moreover, the approximation error can be arbitrarily small with an increase in the number of nested functions in the proposed expression. Using the integral condition written in the definition of the δ -function, one can find the amplitude of the approximation dependence.

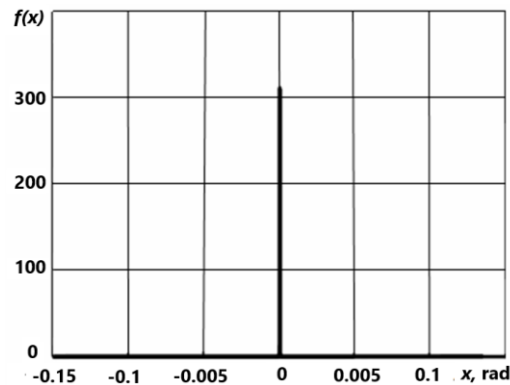


Figure 11. Graph one of the approximations for δ - functions.

Let us show how to find the height of the approximation peak. We know that the δ -function is the derivative of the unit jump function

$$H(x) = \begin{cases} 1, & \forall x > 0; \\ 0, & \forall x < 0. \end{cases}$$

We can approximate the unit jump function (Heaviside function) by a sequence of analytic functions $H_n(x) = 0,5(1 + f_n(x))$. The sequence $f_n(x)$ is defined by (2) on $[-\pi/2, \pi/2]$.

For instance, we have (Figure 12)

$$H_9(x) = 0,5(1 + \sin(A(A(A(A(A(A(A(A(x))))))))))$$

$$H_{10}(x) = 0,5(1 + \sin(A(A(A(A(A(A(A(A(A(x))))))))))$$

$$H_{11}(x) = 0,5(1 + \sin(A(A(A(A(A(A(A(A(A(A(x))))))))))$$

where $A(x) = \frac{\pi}{2} \sin x$.

The larger the approximation number, the greater the graph thickness (Figure 12).

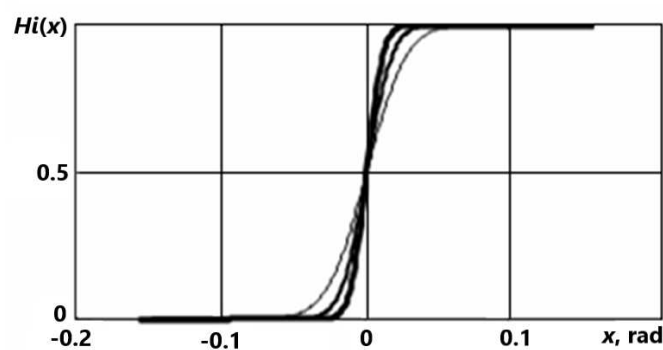


Figure 12. Graphs of approximations of the Heaviside function.

Graphs of the first derivatives of successive functions $H_9(x), H_{10}(x), H_{11}(x)$ are shown in Figure 13. They characterize successive approximations of the delta function.

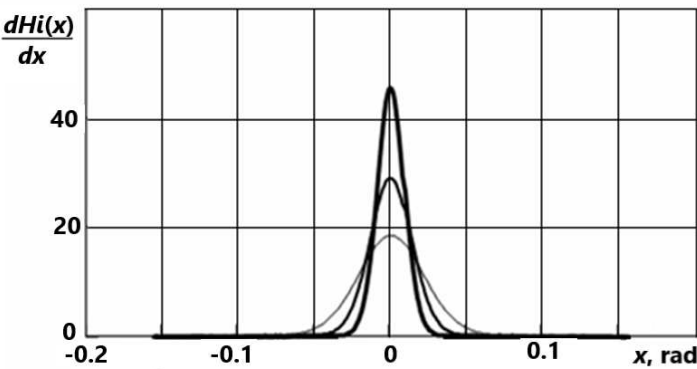


Figure 13. Graphs of approximations for δ - function.

For the sequence $H_n(x) = 0,5(1 + f_n(x))$, we get

$$\frac{dH_n(x)}{dx} = \frac{\pi^{n-1}}{2^n} \prod_{k=1}^{n-1} \cos\left(\frac{\pi}{2} f_k(x)\right) \cdot \cos x$$

Therefore, we have

$$A_n = \frac{\pi^{n-1}}{2^n}$$

The performed analytical approximations of step, impulse and generalized functions make it possible to use the methods of mathematical analysis in the analysis and forecasting of rapid changes in the macroeconomic situation.

The developed methods are universal and have found application in various fields of scientific research and their practical applications [39–44].

4. Choice of Macroeconomic Indicators for the Analysis of Impulse and Spasmodic Processes

The totality of macroeconomic parameters is controversial. Nevertheless, it is possible to single out macroeconomic parameters with the importance of which the overwhelming majority of macroeconomists agree. We will divide the parameters into two groups (Table 2) [8]. Note that macroeconomic indicators are largely interconnected.

Table 2. The macroeconomic parameters.

Main Indicators	Additional Indicators
1. GDP per capita PPP, USD	1. GDP, USD Billion
2. GDP Annual Growth Rate, Last, %	2. Population, million
3. GDP Annual Growth Rate, Previous, %	3. Retirement Age Men, years
4. Unemployment Rate, %	4. Retirement Age Women, years
5. Consumer price index	5. Males Life Expectancy at birth, years
6. Trade Balance, % of GDP	6. Females Life Expectancy at birth, years
7. Corporate Tax Rate, %	7. Current Account to GDP, %
8. Personal Income Tax Rate, %	
9. Sales Tax Rate, %	
10. Exchange rate	
11. Net External Debt, % of GDP	
12. Interest Rate, %	
13. Inflation Rate, %	

14. Government Debt to GDP, %
15. Saving Rate (National savings/GNDI), %

As a rule, macroeconomic indicators have annual values, in some cases, quarterly and monthly. This is the difficulty of applying macroeconomic indicators for the analysis of fast-moving (impulsive and spasmodic) economic processes. As practice shows, in some cases the macroeconomic situation can change dramatically in a matter of days. Examples of macroeconomic indicators with daily values are the exchange rate, energy prices (and even by hours and minutes) on stock exchanges, and others. Table 3, for example, shows prices for Brent crude oil.

Table 3. Brent oil price.

Date	07.07.2023	06.07.2023	05.07.2023	04.07.2023	03.07.2023
Price, \$US per barrel	78,47	76,52	76,65	76,25	74,65

The daily change in the situation on some macroeconomic indicators can be estimated indirectly. For example, the daily unemployment rate can be estimated by the number of welfare claims filed in a country during the day. Modern computer technology makes it quite easy to do this.

To improve the accuracy of forecasting crisis situations, economic analysis should be carried out on a set of macroeconomic indicators, which provides a more holistic view of pre-crisis realities. First of all, macroeconomic indicators with greater inertia to changes are of interest. Note that macroeconomic shifts can also occur in a positive direction, which can be determined, for example, by investments and a high level of innovative development.

5. Approaches to Methods for Forecasting Macroeconomic Events

5.1. Control Rules

In quality control systems [45–47], control rules have been developed (Table 4), which should be paid attention to if violations of specified standards are suspected in the near future. The probability of manifestation of each of these rules within the framework of a random process is small, therefore, each of these manifestations indicates a probable violation of the course of a normal process. Therefore, with some probability, a loss of quality should be expected.

Table 4. Control rules.

Values above + 3 sigma	Values below are 3 sigma
2 of last 3 above + 2 sigma	2 of the last 3 below are 2 sigma
4 of last 5 above + 1 sigma	4 of the last 5 below are 1 sigma
8 points in a row above the center line	8 points in a row below the center line
6 points in a row ascending	6 points in a row descending
14 points in a row alternately	

The problem of applying similar rules in the analysis of macroeconomic processes lies in the fact that, in practice, changes in macroeconomic processes can occur not only for random reasons, but also as a result of purposeful managerial actions. Therefore, for application in macroeconomic theory with impulsive and abrupt changes, these rules must be formulated with some “margin”, for example, not one value above +3 sigma, but two, not one value above + 3 sigma, but one value above + 4 sigma, etc. At the same time, the probability of an error of the first kind decreases (false prediction of an impulse or abrupt development of an event series), but the probability of an error of the second kind increases (ignoring such a development). For example, Figure 14 shows graphs of the same data set in the $\mp 3\sigma$ range (Figure 14a) and in the $\mp 4\sigma$ range (Figure 14b). The range boundaries are marked

with dotted lines. Some points that are critical in the range $\mp 3\sigma$ are not critical in the range $\mp 4\sigma$. Therefore, the standard quality control rules for predicting possible crisis phenomena in the framework of the new macroeconomic theory need to be reformulated.

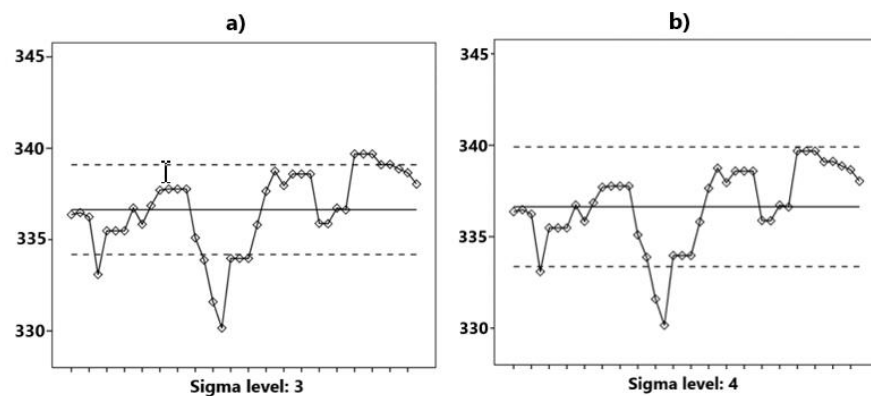


Figure 14. Graphs of the data set in the range: a) $\pm 3\sigma$ and b) $\pm 4\sigma$.

Some of the rules can be left unchanged, for example, 6 points in a row in ascending (descending) order. This rule indicates the existence of a trend, which can lead to abrupt changes in the macroeconomic process.

The creation of a new system of rules for predicting rapid changes in macroeconomic indicators is possible on the basis of statistical processing of information and heuristic methods. Pay attention to weak signals. Note that predictions made on the basis of a system of attention-grabbing criteria can only be realized with a certain probability, which is common when using statistical methods.

In the context of the frequent lack of daily data, we will give examples of forecasts for a rapid change in macroeconomic indicators by months.

Figure 15a shows a graph of the Brent oil price (\$US per barrel) for several consecutive months in 2007-08. Source of information: 1. <https://investfunds.ru>, 2. <https://ru.investing.com>. It follows from the graph that 3 points break the rules. Of particular interest is point number 3, since at this point 3 rules are violated at once (Table 5).

Figure 15b shows the graph of inflation in Russia by months as a percentage of the previous period. Source of information: Rosstat data, <https://rosstat.gov.ru/folder/10705>. Of critical points 1 and 2, point 2 is of particular interest, since in the subsequent period (in March 2022), the inflation rate jumped sharply to a value of 7.61.

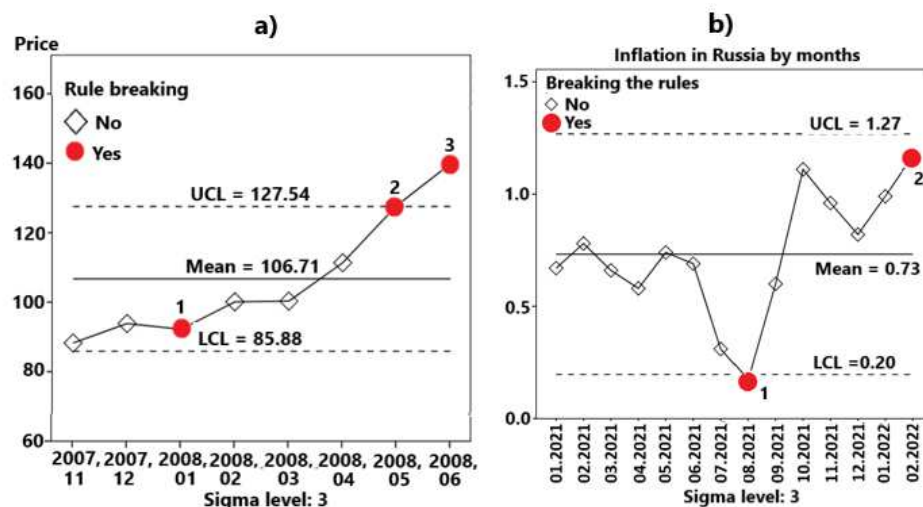


Figure 15. Examples of completed forecasts: a) Brent oil price; b) Inflation in Russia by months.

Table 5. Dynamics of oil prices.

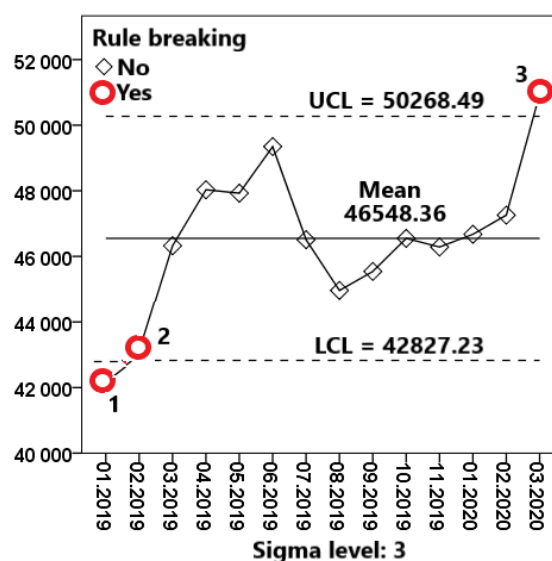
Date	Violations for points
January 2008	2 points from the last 3 below - 2 sigma
May 2008	Greater than + 3 sigma
June 2008	Greater than + 3 sigma
June 2008	2 points from last 3 above + 2 sigma
June 2008	6 points in a row ascending

Let's take another example. Table 6 contains the values of the average monthly nominal accrued wages in rubles of employees in the whole economy of the Russian Federation. Source: Rosstat, <https://rosstat.gov.ru>. The December 2019 salary figure is omitted as the annual bonus is usually paid in the month of December, which greatly distorts the overall picture.

Table 6. Average monthly salary in rubles.

01. 2019	02. 2019	03. 2019	04. 2019	05. 2019	06. 2019	07. 2019	08. 2019	09. 2019	10. 2019	11. 2019	01. 2020	02. 2020	03. 2020.
42263	43062	46324	48030	47926	49348	46509	44961	45541	46549	46285	46674	47257	50948

The graph of the average monthly salary with critical points 1,2 and 3 is shown in Figure 16.

**Figure 16.** Schedule of the average monthly salary.

Violation of control rules at critical points is explained in Table 7.

Table 7. Violation of control rules.

Date	Violations for points
01.2019	Less than -3 sigma
02.2019	2 points from the last 3 below - 2 sigma
03.2020	Greater than + 3 sigma

It should be noted that in the considered examples (Figure 15, 16) the control rules of the theory of quality were applied. Within the framework of the new macroeconomic paradigm, it is necessary to revise and adjust the system of control rules based on the accumulated statistics and heuristic considerations.

We note the main issues that require further development within the framework of the new macroeconomic theory:

Creation of a system of macroeconomic indicators with fixing daily values.

Detailing the set of statistical rules for predicting crisis, fast-moving, impulsive and spasmodic changes in macroeconomic processes.

5.1. Theoretical Foundations of the Technique

For each macroeconomic indicator X we define:

n – number of consecutive days with daily observations $X_1, X_2, \dots, X_n$;

$\bar{X}$ – sample mean, $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$;

σ – sample standard (mean square) deviation, $\sigma = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}}$.

We construct the central line Mean corresponding to the sample mean $\bar{X}$, and the lines $\pm\sigma$; $\pm2\sigma$; $\pm3\sigma$; $\pm4\sigma$;

UCL – upper control limit;

LCL – lower control limit.

The upper UCL and lower LCL reference lines can correspond to values of $\pm3\sigma$ or $\pm4\sigma$ depending on our choice, as explained in the previous Section 5.1. Next, to forecast possible abrupt changes in the macroeconomic situation, we use the control rules from Section 5.1.

As noted earlier, in order to improve the accuracy of forecasts based on the provisions of the new macroeconomic theory, macroeconomic indicators should be considered not separately, but in their totality. In this case, it is required to apply methods of multivariate analysis, such as: multivariate scaling, factor analysis, cluster analysis, and others.

The new macroeconomic theory requires the accumulation of statistical data on the daily values of macroeconomic indicators due to their possible rapid change. This work is only at the beginning of the journey and requires additional efforts. However, the development of computer technology does not raise doubts about the possibility of successfully solving this problem. At present, however, the overwhelming majority of macroeconomic indicators are measured by years, and only at best by quarters and months. Therefore, it is still difficult to give specific examples of forecasts based on the daily values of a set of macroeconomic indicators. It is only possible to outline some analogies of such approaches based on the annual values of macroeconomic indicators. As such analogies, let us consider the dynamics of a certain set of macroeconomic indicators of the Russian Federation for the period 2012-22 (Table 8). Source: compiled by the author based on data from the Federal State Statistics Service: <https://rosstat.gov.ru/folder/10705>.

Table 8. Dynamics of macroeconomic indicators of Russia for the period 2012-22.

Year	GDP growth at current prices, %	Unemploye nt rate, %	Industrial production index in % of the previous year	Consumer price indices for services at the end of the period, in % to December of the previous year	Labor productivity index in the economy in % of the previous year
2012	13,3	5,5	103,4	107,28	103,8
2013	7,2	5,5	100,4	108,01	102,1
2014	8,3	5,2	101,7	110,45	100,8
2015	5,1	5,6	100,2	110,20	98,7
2016	3,0	5,5	101,8	104,89	100,1
2017	7,3	5,2	103,7	104,35	102,1
2018	13,1	4,8	103,5	103,94	103,1
2019	5,5	4,6	103,4	103,75	102,4
2020	-1,8	5,8	97,9	102,70	99,6

2021	25,7	4,8	105,3	104,98	103,7
2022	13,4	3,9	99,4	113,19	102,8

Using the ALSCAL multidimensional scaling procedure, you can get a visual representation of multidimensional tabular information (Figure 17a). The figure shows that 2020, 2021 and 2022 stand out sharply in terms of the totality of their indicators. Further analysis shows the economic crisis in 2020 and a sharp improvement in the situation in 2021 and 2022.

The conclusions made are confirmed by cluster analysis. On Figure 17b is a dendrogram showing the possibility of splitting the variable "Year" into 5 clusters (section along the dotted line).

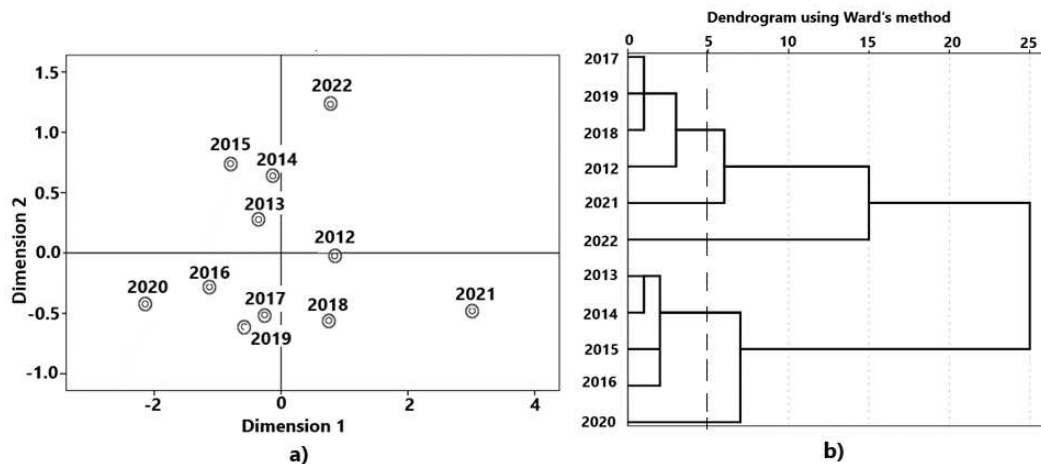


Figure 17. Visual representation of multidimensional information: a) multidimensional scaling; b) cluster analysis dendrogram.

Clustering is presented in Table 9. Average values of variables by clusters are presented in Table 10.

Table 9. Clustering.

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Cluster	1	2	2	2	2	1	1	1	3	4	5

Table 10. Mean values of variables by clusters.

Ward Method		GDP growth at current prices, %	Unemployment rate, %	Industrial production index in % of the previous year	Consumer price indices for services at the end of the period, in % to December of the previous year	Labor productivity index in the economy in % of the previous year
1	Mean	9,8000	5,0250	103,5000	104,8300	102,8500
	N	4	4	4	4	4
2	Mean	5,9000	5,4500	101,0250	108,3875	100,4250
	N	4	4	4	4	4
3	Mean	-1,8000	5,8000	97,9000	102,7000	99,6000
	N	1	1	1	1	1
4	Mean	25,7000	4,8000	105,3000	104,9800	103,7000
	N	1	1	1	1	1
5	Mean	13,4000	3,9000	99,4000	113,1900	102,8000
	N	1	1	1	1	1
Total	Mean	9,1000	5,1273	101,8818	106,7036	101,7455
	N	11	11	11	11	11

Multidimensional scaling methods have a great advantage, they allow you to process multidimensional information and present the results of processing visually. This allows the use of macroeconomic variables in their totality to identify possible pre-crisis and forecast other rapidly changing macroeconomic situations. In addition to the previous examples, let's consider a more complete set of Russia's macroeconomic indicators in the dynamics of their changes over the period 2004-2011 (Table 11).

Table 11. Dynamics of the main macroeconomic indicators of Russia in 2004-2011 (growth rate, %).

Source: Rosstat and Bank of Russia.

Years	2004	2005	2006	2007	2008	2009	2010	2011
Unemployment	7.9	7.6	6.9	6.1	6.4	8.6	7.6	7.3
Inflation, December to December	11.7	10.9	9.0	9.3	13.0	8.8	7.4	6.3
Gross domestic product	7.2	6.4	6.7	8.1	5.6	-7.9	4.0	4.1
Industrial products	8.3	4.0	4.9	6.1	2.1	-10.8	8.3	4.1
Agricultural products	3.0	2.4	3.6	3.3	1.8	-5.5	-2.3	7.4
Real disposable income of the population	10.4	11.1	10.0	10.7	2.9	1.1	3.5	4.2
Retail turnover	13.3	12.8	13.0	15.9	13.2	-5.0	4.5	4.8
Gold and foreign exchange reserves of the Central Bank (billion US dollars)	130.0	175.0	300.0	479.4	426.3	439.0	500.2	585.8
Volume of the stabilization fund (billion rubles)	489.0	522.3	2180.0	3859.0	4027.6	1830.5	1279.9	1300.0
Investments in fixed assets	11.7	10.5	13.5	20.3	9.8	-11.0	5.9	9.0
World oil price (Urals)	34.4	50.6	61.1	69.3	94.4	61.1	78.2	90.0
Export of goods, billion US dollars	183.2	243.6	304.5	355.5	471.6	304.0	400.4	380.4
Import of goods, billion US dollars	97.4	125.3	163.9	223.4	291.9	192.0	248.7	250.1

A visual representation of the results of applying multidimensional scaling is given in Figure 18.

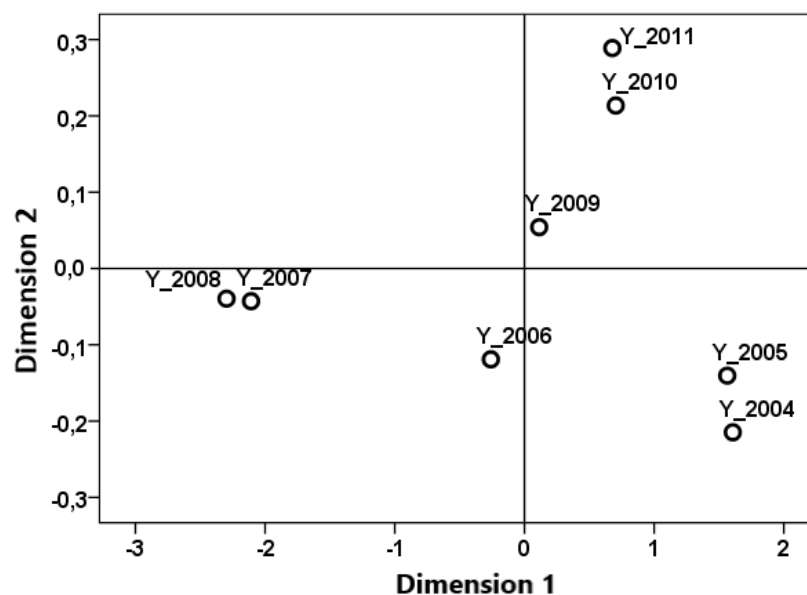


Figure 18. Graphical representation of the results of multivariate scaling.

It can be seen from the figure that the years 2005 and 2005, 2007 and 2008, 2010 and 2011 are pairwise close in terms of the totality of initial macroeconomic indicators. The years 2006 and 2009 occupy rather isolated positions and, therefore, differ from the rest and from each other.

A similar processing of information for predicting rapid macroeconomic changes is assumed in the framework of the new macroeconomic theory in the presence of a set of variables with daily values and daily monitoring.

Currently, work is also underway to improve the mechanism of foresight controlling [48–50].

6. Conclusions

1. This article describes only the main approaches to the creation of a macroeconomic theory with rapidly changing characteristics of an impulse and jump type, and formulates its main provisions. These provisions need to be further developed and improved on the basis of statistical analysis.
2. The current macroeconomic paradigm is characterized by rapid impulsive and spasmodic changes, in some cases within days, so a new macroeconomic theory should be based on the analysis of macroeconomic characteristics with daily values.
3. For daily monitoring of the macroeconomic situation within the framework of the new macroeconomic theory, it is required to form a system for the daily collection and processing of macroeconomic information, which can be done on the basis of automated systems with the widespread use of computer technology.
4. Daily values of macroeconomic indicators are best expressed in relative terms, in the form of the dynamics of their changes for comparative analysis and identification of critical points.
5. It is necessary to clarify the system of criteria for predicting possible rapid changes in the macroeconomics and identifying critical points. This may require extended ranges of acceptable changes compared to conventional statistical methods and control rules, for example, a point going outside the $\mp 4\sigma$ range, not one but several points going beyond the $\mp 3\sigma$ range. The system of criteria may include such rules as several (6 or 8) points in a row in ascending (or descending) order to identify an emerging trend, and other control rules. Refinement of the system should be carried out using the accumulated statistics on rapidly changing macroeconomic processes and using heuristic methods and rules.
6. To improve the accuracy of forecasts using the methods of the new macroeconomic theory, macroeconomic indicators should be considered not separately, but in their totality. At the same time, to identify pre-crisis conditions and possible rapid positive changes, multidimensional information processing methods, for example, multidimensional scaling, cluster analysis, factor analysis, and others, may be useful.
7. The paper describes the methods developed by the author for analytical approximation of stepwise and generalized functions, which makes it possible to describe impulsive and jump functions by conventional methods of mathematical analysis.

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