Article

Not peer-reviewed version

Measuremnt Problem in Statistical Signal Processing

Miloš Milovanović *

Posted Date: 19 October 2023

doi: 10.20944/preprints202310.1216.v1

Keywords: experimental mathematics; general measurement; time operator; frame wavelets; optimal decomposition.

Article

Measuremnt Problem in Statistical Signal Processing

Miloš Milovanović [1,*]

[1] Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia; milosm@mi.sanu.ac.rs
* Correspondence: milosm@turing.mi.sanu.ac.rs

Abstract: Discussing quantum theory foundations, von Neumann noted that the measurement process should not be regarded in terms of a temporal evolution. A reason for his claim is insurmountability of the gap between reversible and irreversible processes. The time operator formalism that goes beyond such a gap is adequate framework for elaboration of the measurement problem. It considers signals to be stochastic processes, whether they correspond to variables or distribution densities. Signal processing like that utilizes statistical properties to perform its tasks, which is the definition of statistical signal processing. A hierarchy of the measurement process is indicated by crossing between states and devices, which implies an evolution in the temporal domain. The concept has generalized to an open system by the use of duality in frame theory.

Keywords: experimental mathematics; general measurement; time operator; frame wavelets; optimal decomposition

---

1. Introduction

The uncertainty relation is regarded to be a fundamental principle of quantum theory. Although it has a long history starting from Sommerfeld and Heisenberg[1,2], the problem came into a focus of interest due to the discovery of wave mechanics by Schrödinger which led to its formulation in terms of mathematical physics. The concept of the wave function was utilized by Gabor in order to establish the communication theory upon decomposing signals into elementary quanta of information[3]. In that respect, uncertainty comes down to the commutator relation

$$[Q, P] = i \tag{1}$$

concerning a pair of canonically conjugate operators.

The paper is aimed to reformulate the measurement problem in the same manner. It considers signals in a relation to stochastic processes, whether corresponding to variables or distribution densities. Signal processing like that utilizes statistical properties to perform its tasks, which is the definition of statistical signal processing. The climax of such a trail should be quantum theory of information, whereat the measurement is a fundamental conception.[4]

The phrase *experimental mathematics* comes up a lot in the field of chaos, fractals and non-linear dynamics[5]. It reemerged during the last century, notwithstanding that mathematicians had always used some experiments in order to identify properties and patterns. The measurement is therefore a basic concept not only of geometry, but of mathematics overall. A link between the measurement problem and experimental mathematics has already been elaborated.[6] The paper should complement such a discussion and revise some oversights that appeared in the previous one. The multidisciplinary framework it has implied corresponds to the time operator formalism of the complex systems physics. The theory originated from the Brussels school of thermodynamics, proposing a unification of reversible and irreversible processes. A relation to the problem appears in respect to its definition that was postulated by von Neumann.[7]

Measurement is argued to be the fundamental conception of science.[8] Elaborating issues it raises is therefore significant for epistemology and methodology of the scientific research. Interrelating some aspects such as states, devices, probabilities etc., a hierarchy which is designed in that respect should coincide to the principle of psychophysical parallelism. It is indicated by crossing between states

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

and devices, which implies an evolution in the temporal domain.[6] A paradigmatic measurement corresponds to comensuration of magnitudes by the Euclidean algorithm which is an intensional procedure producing real numbers of the unit interval. Regarding that, one comes to a general definition of the process concerning a time series of binary digits.[9]

The paradigm asserts the significance of time for elaboration of the measurement problem, which has explicated a substantial relation between signals and stochastic processes. In that regard, a signal corresponds to the ensemble which is originated by a measurement. The problem is formulated in terms of mathematical physics, notwithstanding any interpretation of physical theories such as QBism or many worlds. A comparison to the uncertainty relation (1) is a picturesque instance, since it appears in statistical signal processing no matter of the interpretation imposed. The measurement problem is therefore a predominantly mathematical issue which is related to the very foundation of geometry, analysis, probability and other topics. It concerns intensionality that is the manner in which matemetatics has always applied.[10]

After the Introduction, Section 2 presents the time operator formalism of complex systems. The concept of ensamble is defined, as well as a link between reversible and irreversible processes. The measurement hierarchy is elaborated in Section 3, following a paradigm which corresponds to comensuration of magnitudes by the Euclidean algorithm. It presents a general definition of the problem in statistical signal processing, whereat an ensemble is related to the distribution density of a time series. Section 4 considers projective measurements in the hierarchical base, constituting a measurable space that is the domain of an observable. A hierarchy that has complemented the von Neumann definition arises form a temporality of the domain.

The main advancement concerns a consistent realization of psychophysical parallelism that is a principle which Bohr and von Neumann have already pointed out.[11] It is realized due to a change in representation which is the operator function of time. General measurements are considered in Section 5, whereat self-duality of the Hilbert space has been replaced by duality in frame theory.[12] In that manner, crossing between states and devices should generalize to an open system whuch is partially described by the stochastic process.[13]

2. Time and Complexity in the Physical Science

2.1. Time in Quantum Theory

Von Neumann has indicated two fundamentally diverse types of interventions in a system, the first of which corresponds to a temporal evolution that is reversible and the second one to an irreversible measurement[11]. He's been wondered by the fact that the entropy increase follows the measurement process not representing any temporal evolution, which is totally opposite to thermodynamics relating the increase of entropy to an evolution in the temporal domain. The reason for such an odd situation is the fake concept of time in quantum theory which is a classical one, considering that it is represented by linear parameterization just like in the Newtonian mechanics.[7] Von Neumann admits an essential weakness of quantum theory, which concerns the fact that it is non-relativistic whereas spatial coordinates are represented by operators and time is a mere parameter making the Poincaré symmetry impossible. The time operator, which should be a chief link between quantum and relativity theories,[14] is substantially related to the measurement problem.

The uncertainty between time and energy has been discussed frequently[15]. In the classical formulation of quantum theory, however, there is no operator that satisfies the commutator relation (1) in respect to a Hamiltonian corresponding to the energy of a system. A reason is that the Hamiltonian governs evolution by the Schrödinger equation of the wave function which is a stationary state, like orbits in the Newtonian mechanics[7]. The time operator is but definable in the Liouville-von Neumann mechanics which considers density operators of ensembles. In that regard, there is T implementing the commutator relation

$$[T, L] = i \tag{2}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

wherein L is the Liouvillian which governs evolution of density operators by the Liouville equation.

An ensemble is defined by the mapping $P \mapsto \pi(P)$ which assigns a probability $\pi(P)$ to each projector P, such that $\pi(0) = 0$ $\pi(1) = 1$ $\pi\left(\sum_{P \in \mathcal{P}} P\right) = \sum_{P \in \mathcal{P}} \pi(P)$ for orthogonal constituents of the sum[16]. According to the Gleason theorem, there is a density operator ρ that satisfies $\pi(P) = \langle \rho | P \rangle$, which should be positive semidefinite $\rho \geq 0$, Hermitian $\rho^\dagger = \rho$ and unity traced $\operatorname{Tr} \rho = 1$. It follows $\rho = FF^\dagger$ whereby F is the root operator which is unity normed, considering that $\operatorname{Tr} \rho = \|F\|^2$.

2.2. *Physics of Complex Systems*

If $\rho = |f\rangle \langle f|$ for a unity normed signal f, the density is coincident to $\rho = |f|^2$ provided that tracing the operator corresponds to integrating the function since $\operatorname{Tr} \rho = \langle \rho | 1 \rangle$. The projector $Pf = cf$, which multiplies signals by a characteristic function, has the probability $\pi(P) = \langle \rho | c \rangle$ that is an expected value of the variable c. The Koopman-von Neumann mechanics which has been postulated in that manner concers evolution of densities and variables due to the action of a one-parameter group G^t onto the measurable space that should preserve a probability measure. Variables upon the probability space evolve by the group of unitary operators $U^t : f \mapsto f \circ G^t$ and densities are governed by adjoints $U^{t\dagger} = U^{-t}$. In that instance, there is an infinitesimal generator L such that $U^{t\dagger} = e^{iLt}$ wherefrom it follows the Liouville equation $\frac{\partial \rho}{i \partial t} = L\rho$ which is governing an evolution of the density[17]. In terms of the evolutionary group, the commutator relation (2) is equivalent to $[T, U^t] = tU^t$ which comes down to

$$[T, U] = U \tag{3}$$

supposing the cyclic group generated by U.

If there is an operator T satisfying the commutator relation (3), such a system is termed to be *complex*. The time operator formalism of complex systems originated from the Brussels school of thermodynamics, that was investigating a link between reversible and irreversible processes[7]. It is realized due to a change in representation $\Lambda = \lambda(T)$ that is the operator function of time, which transfigures the Lie group $U^{t\dagger}$ into a Markov semigroup

$$W^{t\dagger} = \Lambda U^{t\dagger} \Lambda^{-1}, t \geq 0 \tag{4}$$

The semigroup (4) indicates irreversibility, since operators $W^{t\dagger}$ for $t < 0$ are not positivity preserving and therefore not related to the evolution of a density. The change in representation should preserve the positivity $\rho \geq 0 \Rightarrow \Lambda\rho \geq 0$, the trace $\operatorname{Tr} \rho = \operatorname{Tr} \Lambda\rho$, the uniform density $1 = \Lambda 1$ and it should be invertible at a dense subset. Terms of the change imply that Λ maps a density into a density without any information loss[18].

The link between reversible and irreversible processes is substantial for elucidation of the measurement problem. The evolutionary group U^t has become the semigroup W^t for $t \geq 0$ due to a change in representation (4). In that respect, irreversibile evolution corresponds to an increase of the information entropy which is a measurement characterized by.[7]

3. Wavelets and the Measurement Hierarchy

3.1. *Paradigm of the Measurement Process*

In the V book of the *Elements*, Euclid elaborates the doctrine of proportion concerning commensuration of magnitudes. Due to the Euclidean algorithm, magnitudes $a \leq b$ measure each other in terms of continued fraction

$$\frac{a}{b} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}} \tag{5}$$

which should indicate a process that takes place step by step over time [6].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

The Euclidean algorithm is regarded to be a paradigmatic measurement. The process corresponds to a sequence

$$\xi_j = \cfrac{1}{n_1 + \cfrac{1}{\ddots + \frac{1}{n_j}}}$$

whose elements $\xi_j = \frac{h_j}{k_j}$ are obtained by the recurrence equation

$$h_{j+1} = n_{j+1} h_j + h_{j-1}, \; k_{j+1} = n_{j+1} k_j + k_{j-1}$$

considering the initial condition $h_0 = 0$, $h_1 = 1$, $k_0 = 1$, $k_1 = n_1$. The difference between successive elements

$$\Delta\xi_j = \xi_{j+1} - \xi_j = \frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j} = \frac{(-1)^j}{k_j k_{j+1}}$$

expands the continued fraction (5) in the alternating series

$$\Delta\xi_0 + \cdots + \Delta\xi_j + \cdots = \frac{1}{k_0 k_1} - \cdots \frac{(-1)^j}{k_j k_{j+1}} \cdots \tag{6}$$

which is a sparse representation,[19] composing terms from the redundant dictionary $\frac{1}{1}, \frac{1}{2} \ldots$

The expansion (6) corresponds to a binary code wherein 0 is assigned to terms of the dictionary that do not participate in the series and an alternating value ± 1 to those that do participate. Such a representation of the measurement process is highly redundant, since the complete dictionary cannot be involved in a series. One should therefore eliminate excess zeros, which is realized by coding the sequence $n_1, n_2, \ldots$ The code is composed of alternative ± 1 values at positions $n_1, n_1 + n_2, \ldots$ which gives rise to the Minkowski function

$$?: \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}} \mapsto \frac{1}{2^{n_1 - 1}} - \frac{1}{2^{n_1 + n_2 - 1}} + \cdots$$

that is an automorphism transfiguring the continued fraction into the binary representation. The process is codified by binary digits of a real number

$$x = \underbrace{0.0 \ldots 0}_{n_1} \underbrace{1 \ldots 1}_{n_2} \underbrace{0 \ldots 0}_{\cdots} 1 \ldots \tag{7}$$

whose positions concern its temporality.[9]

In that regard, the measurement is conisdered to be a time series of binary digits which is a general definition of the problem. Time is related to a scale j of the binary tree whose nodes $0 < \frac{2k-1}{2^{j+1}} < 1$ correspond to both states and devices of the measurement process. A step is concerned by the Rényi map

$$R(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases} \tag{8}$$

representing a shift in terms of binary digits. It is self-similarity of the binary tree, which maps both left and right subtrees to the entire one [6].

3.2. Hierarchical Bases of the Signal Space

The binary structure reflects the hierarchy of the signal space representing the measurement process. It concerns both states Σ and devices Δ which should be considered in a dual manner [6]. If the signal space Σ has identified states, the topological dual $\widetilde{\Sigma} = \Delta$ corresponds to measurement

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

devices. Starting from devices Δ on the other hand, the topological dual $\widetilde{\Delta} = \Sigma$ concerns measurement states. These options may differ in more than a conceptual sense: taking the dual of the dual does not necessarily bring back to the departure. Even if it does, there may be some reasons to favor one of them since an aspect of the process has been obscured.[13]

A sensible solution should consider signals to be both states and devices concurrently, which leads to a source-detector interchangeability that is termed *crossing* in quantum theory [13]. It is a reason for regarding $\Sigma = \Delta$ to be self-dual, which applies to the Hilbert space L^2_μ wherein μ is the Lebesgue measure over the unit interval. The hierarchy is reflected by wavelets which concern orthonormal bases realizing the binary hierarchy.[6] The Haar base is paradigmatically designed by translation and normalized dilatation of the mother wavelet $\chi(x) = \begin{cases} -1, & 0 \leq x < \frac{1}{2} \\ +1, & \frac{1}{2} < x \leq 1 \end{cases}$ in the manner of

$$\chi_{j,k}(x) = \begin{cases} -2^{j/2}, & \frac{k}{2^j} \leq x < \frac{k+1/2}{2^j} \\ +2^{j/2}, & \frac{k+1/2}{2^j} < x \leq \frac{k+1}{2^j} \end{cases} \tag{9}$$

implying that basic elements are zero valued almost elsewhere.

Wavelets on the unit interval have arisen from those on the real line, which are orthonormal bases

$$\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$$

obtained by translation and normalized dilatation of a mother wavelet Ψ. They reappear in the signal space L^2_μ due to $\psi_{j,k}(x) = \sum_n \Psi_{j,k}(x+n)$, which gives rise to the periodization axiom

$$\psi_{j,k} = \psi_{j,k+2^j} \tag{10}$$

and the annihilation as well

$$j < 0 \Rightarrow \psi_{j,k} = 0 \tag{11}$$

In that manner, one gets the pyramid $\psi_{j,k}$ for $j \geq 0$ and $1 \leq k \leq 2^j$ which is an orthonormal base of $L^2_\mu \ominus \mathbb{1}$ representing the orthocomplement of constant signals $\mathbb{1}$. [20] Signals are decomposed in a hierarchical base due to the resolution of identity

$$1 = |1\rangle\langle 1| + \sum_{j\geq 0}\sum_{k=1}^{2^j} \left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right|$$

wherein $\langle\cdot|$ corresponds to a state and $|\cdot\rangle$ to a device of the measurement process. The translation axiom

$$\psi_{j,k}(x - \frac{m}{2^j}) = \psi_{j,k+m}(x) \tag{12}$$

is also satisfied, which means that variables are equally distributed within each scale.

The evolution of wavelets in a measurement process concerns the operator $U : f \mapsto f \circ R$ which is induced by the Rényi map (8). The evolutionary axiom holds in terms of its adjoint

$$U^\dagger\psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j-1,k} \tag{13}$$

which comes down to

$$U\psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j+1,k} + \frac{1}{\sqrt{2}}\psi_{j+1,k+2^j}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

Since R is a measure preserving transformation of the unit interval, the operator U preserves distribution of a variable. The orthogonality implies that variables are decorrelated, since $\mathrm{E}\,\psi_{j,k} = \left\langle 1|\psi_{j,k}\right\rangle = 0 = \mathrm{E}\,\overline{\psi_{j,k}}$ and

$$(j,k) \neq (l,m) \Rightarrow \mathrm{E}\,\overline{\psi_{j,k}}\psi_{l,m} = \left\langle \psi_{j,k}|\psi_{l,m}\right\rangle = 0 = \mathrm{E}\,\overline{\psi_{j,k}}\,\mathrm{E}\,\psi_{l,m}$$

The absolute square $|\psi_{j,k}|^2$ is a density function as well, which makes the base to generate both states and devices concurrently.

A step of the measurement hierarchy has corresponded to a scale, which implies the time operator

$$T = \sum_{j\geq 0}\sum_{k=1}^{2^j} j\left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right| \tag{14}$$

that is defined on a dense subset of $L^2_\mu \ominus \mathbb{1}$ [21]. The commutator relation (3) follows immediately from the evolutionary axiom, considering that

$$[U^\dagger, T]\,\psi_{j,k} = U^\dagger j\psi_{j,k} - (j-1)U^\dagger\psi_{j,k} = U^\dagger\psi_{j,k}$$

3.3. Space of Ensembles

A trouble might occur concerning the generation of an evolutionary group, since the operator U is not invertible. However, it extends naturally to an invertible operator $U_\chi : F \mapsto F\circ B$ which is induced by the baker map

$$B(x,y) = \begin{cases} (2x, \frac{y}{2}), & 0 \leq x < \frac{1}{2} \\ (2x-1, \frac{y+1}{2}), & \frac{1}{2} < x \leq 1 \end{cases} \tag{15}$$

that is a measure preserving transformation of the unit square [22]. It is a reason to embed the signal space L^2_μ into an extended one $L^2_{\mu^2} = L^2_\mu \otimes L^2_\mu$ whereat $\mu^2 = \mu\otimes\mu$ is the product measure.[6]

The space of ensembles $L^2_{\mu^2} = \Delta\otimes\Sigma$ is regarded to be a tensor product of devices and states. The resolution of identity gives rise to a decomposition

$$F = |1\rangle\langle A| + \sum_{j\geq 0}\sum_{k=1}^{2^j}\left|\psi_{j,k}\right\rangle\left\langle D_{j,k}\right|$$

wherein $\langle A| = \langle 1|F$ is the approximation coefficient and $\left\langle D_{j,k}\right| = \left\langle\psi_{j,k}|F\right.$ are detail coefficients at a certain scale of the measurement hierarchy. One implies the matrix multiplication

$$F_1F_2(x,y) = \int F_1(x,t)F_2(t,y)dt$$

The time operator T_χ of the system evolving by U_χ has been explicitly constructed [16]. Its projection onto the signal space L^2_μ concerns the hierarchy of the Haar base (9). The time operator of any wavelets (14) is obtained through conjugation $T = CT_\chi C^\dagger$ by $C : \left|\chi_{j,k}\right\rangle\langle\chi_{l,m}| \mapsto \left|\psi_{j,k}\right\rangle\langle\psi_{l,m}|$ which transforms the Haar base to the other one. It corresponds to the system whose evolution is governed by $U = CU_\chi C^\dagger$ that is also an extension of the evolutionary operator U, which is a reason to be denoted in the same manner.

One defines the density opeartor of an ensemble $\rho = FF^\dagger$, whereupon the root F should be unity normed. The density evolves by an adjoint of $\mathfrak{U}\rho = (UF)(UF)^\dagger$, which is the superoperator

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

$\mathfrak{U}^\dagger\rho = U^\dagger\rho U$. The time operator T that concerns the evolution by U is relevant to $\mathfrak{U}$ as well, considering that the commutator relation (3) is satisfied

$$[T,\mathfrak{U}]\rho = [T,U]\rho U^\dagger = \mathfrak{U}\rho$$

It induces a change in representation $\Lambda = \lambda(T)$ which should transfigure the evolutionary group generated by $\mathfrak{U}^\dagger$ to a semigroup (4) generated by

$$\mathfrak{W}^\dagger = \Lambda\mathfrak{U}^\dagger\Lambda^{-1} \tag{16}$$

4. Orthonormal Wavelets and Projective Measurements

4.1. Measurement in the Hierarchical Base

The von Neumann measurement corresponds to a complete set of orthogonal projectors in the Hilbert space. Considering the paradigmatic measurement, one should suppose a hierarchy that is realized by the time series of binary digits. It is a reason to represent orthonormal wavelets $\psi_{j,k}$ in terms of projectors $P_{j,k} = \left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right|$, which concerns an embedment of $L^2_\mu \ominus \mathbb{1}$ into $(L^2_\mu \ominus \mathbb{1})^2$.

Projectors constitute the Boolean algebra which is isomorphic to an algebra of sets due to the Stone representation theorem. It is the measurable space corresponding to devices which an observable has been defined upon.[10] A measurement state on the other hand corresponds to a density $\rho = FF^\dagger$ which is defined upon the same domain. One concludes that it should commute with each of projectors which comes down to the requirement $\rho = \sum_{j,k} P_{j,k}\rho P_{j,k}$. In that manner, a density reduces to the subspace of commutative operators

$$\mathfrak{M}\rho = \sum_{j,k} P_{j,k}\rho P_{j,k} = \sum_{j,k} \|D_{j,k}\|^2 P_{j,k}$$

and the measurement problem concerns the issue of how such a reduction has taken place.

It is obvious that the problem occurs only if the measurable space has not accorded to the state. If one measures a density itself, there is no reduction since devices are generated by eigenprojectors $P^o_{j,k}$ of the density operator. Such a measurement

$$\mathfrak{M}^o\rho = \sum_{j,k} P^o_{j,k}\rho P^o_{j,k} = \sum_{j,k} \|D^o_{j,k}\|^2 P^o_{j,k}$$

is termed to be *optimal*, considering that the density operator $\rho = \mathfrak{M}^o\rho$ is an invariance of the process.

Starting from the decomposition $F = \sum_{j,k}\left|\psi^o_{j,k}\right\rangle\left\langle D^o_{j,k}\right|$ of an ensemble from $(L^2_\mu \ominus \mathbb{1})^2$, one obtains $\rho = \sum_{j,k,l,m}\left|\psi^o_{j,k}\right\rangle\left\langle D^o_{j,k}|D^o_{l,m}\right\rangle\left\langle\psi^o_{l,m}\right|$ as well as $\mathfrak{M}^o\rho = \sum_{j,k}\|D^o_{j,k}\|^2\left|\psi^o_{j,k}\right\rangle\left\langle\psi^o_{j,k}\right|$. It follows that $(j,k) \neq (l.m) \Rightarrow \left\langle D^o_{j,k}|D^o_{l,m}\right\rangle = 0$, meaning that detail coefficients are decorrelated in the optimal base.[9,10] In respect to another base $\psi_{l,m}$ that is suboptimal, the same ensemble is decomposed by coefficients

$$\langle D_{l,m}| = \sum_{j,k}\left\langle\psi_{l,m}|\psi^o_{j,k}\right\rangle\left\langle D^o_{j,k}\right|$$

Since basic elements $\psi^o_{j,k}$ and $\psi_{l,m}$ are almost entirely supported by domains $[\frac{k-1}{2^j},\frac{k}{2^j}]$ and $[\frac{l-1}{2^m},\frac{l}{2^m}]$ respectively, values $\left\langle\psi_{l,m}\middle|\psi^o_{j,k}\right\rangle$ are negligible if these segments do not intersect. It follows an approximate decorrelation of the ensemble, which implies that correlation between detail coefficients predominantly concerns inheritance along branches of the binary tree.

The wavelet domain hidden Markov model which is obtained like that has been proven tremendously useful in a variety of applications, including speech recognition and artificial intelligence

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

[26]. The model is based upon approximate decorrelation of detail coefficients $\mathbf{D} = \left(D_{j,k}\right)$. Correlation is transmitted only through the Markovian tree of hidden variables $\mathbf{S} = \left(S_{j,k}\right)$ which have attributed to each node and out of such an interdeptitide the ensemble is considered decorrelated. The conditional distibution $\mathbf{D}|\mathbf{S}$ is supposed to be normal, which implies that $D_{j,k}|S_{j,k}$ are independent variables.

4.2. Psychophysical Parallelism

The projective measurement $\mathfrak{M} = \sum_j \mathfrak{M}_j$ has temporally decomposed into the sum of superprojectors $\mathfrak{M}_j = \sum_k \mathfrak{P}_{j,k}$, whereby each $\mathfrak{P}_{j,k}\rho = P_{j,k}\rho P_{j,k}$ is superprojection onto the ensemble $P_{j,k} = \left|\psi_{j,k}\right\rangle \left\langle \psi_{j,k}\right|$. If one defines $\mathcal{U}\mathfrak{P} = \mathfrak{U}\mathfrak{P}\mathfrak{U}^\dagger$, it holds for $j \geq 1$

$$\mathcal{U}^\dagger \mathfrak{P}_{j,k}\rho = \begin{cases} 2\left|\psi_{j-1,k}\right\rangle \left\langle \psi_{j-1,k}(x)\right| \int_0^{1/2} F(x,t)F^\dagger(t,y)dt \left|\psi_{j-1,k}(y)\right\rangle \left\langle \psi_{j-1,k}\right|, & k \leq 2^{j-1} \\ 2\left|\psi_{j-1,k-2^{j-1}}\right\rangle \left\langle \psi_{j-1,k-2^{j-1}}(x)\right| \int_{1/2}^1 F(x,t)F^\dagger(t,y)dt \left|\psi_{j-1,k-2^{j-1}}(y)\right\rangle \left\langle \psi_{j-1,k-2^{j-1}}\right|, & k > 2^{j-1} \end{cases}$$

In that respect,

$$\mathcal{U}^\dagger \mathfrak{M}_j = \mathcal{U}^\dagger \sum_{k \leq 2^{j-1}} \mathfrak{P}_{j,k} + \mathcal{U}^\dagger \sum_{k > 2^{j-1}} \mathfrak{P}_{j,k} = \sum_k 2\mathfrak{P}_{j-1,k} = 2\mathfrak{M}_{j-1}$$

and since $\mathcal{U}$ is unitary

$$\mathfrak{M}_j = 2\mathcal{U}\mathfrak{M}_{j-1} = 2^j \mathcal{U}^j \mathfrak{M}_0 = 2^j \mathfrak{U}^j \mathfrak{M}_0 \mathfrak{U}^{j\dagger} \tag{17}$$

which relates all superprojectors to the primary measurement $\mathfrak{M}_0\rho = P_0\rho P_0$.

The evolutionary operator U that maps a scale of the measurement hierarchy to the next one is extended to the space of ensembles $\Delta \otimes \Sigma$ due to the baker map (15). It crosses information between coordinates of the domain in such a manner that the first binary digit of one, which has been lost by the Rényi map, becomes the first digit of another. The induced operator should cross spaial components, which is evident in the relation $U_\chi |\chi\rangle \langle 1| = |1\rangle \langle \chi|$ and likewise for other wavelets. Considering identifications $|\cdot\rangle \hookrightarrow |\cdot\rangle \langle 1|$ and $\langle \cdot| \hookrightarrow |1\rangle \langle \cdot|$, the operator has crossed a measurement device into a state.[6]

The superprojector (17) is factorized into measurement operators $\mathfrak{M}_j\rho = M_j\rho M_j^\dagger$ whereat $M_jF = 2^{j/2}U^jP_0U^{j\dagger}F$. First of all, it concerns the evolution by $U^\dagger$ crossing states into devices. Thereafter P_0 projects the ensemble onto a primary device, which annihilates all devices out of the measurement display. Finally, the evolution by U crosses devices into states. Supposing the measurement hierarchy that is reflected by the Haar base, the primary device corresponds to the ensemble $\chi_0 = |\chi\rangle \langle 1|$ which produces by the evolution $\chi_j = U_\chi^j \chi_0$ the base of ensembles $\prod_{j\in(j_1<\cdots<j_n)} \chi_j$ [16]. Each element $\chi_{\vec{j}}$ is specified by an increasing sequence of integers $\vec{j} = (j_1 < \cdots < j_n)$ and it evolves by $U_\chi \chi_{\vec{j}} = \chi_{\vec{j}+1}$ wherein $\vec{j}+1 = (j_1+1 < \cdots < j_n+1)$. The measurement operator $M_j = 2^{j/2}U_\chi^j P_0 U_\chi^{j\dagger}$ implies the process $U_\chi^{j\dagger}\chi_{\vec{j}} = \chi_{\vec{j}-j}$ due to which some states have become devices. The projector P_0 should fix an element $\chi_{\vec{j}} = \chi_{j_1}\cdots\chi_{j_n}$ if it is started by the primary device $\chi_{j_1} = \chi_0$ and annihilate it if not, which means that all devices out of the measurement display come to be annihilated. The terminal step concerns the evolution $U_\chi^j \chi_{\vec{j}} = \chi_{\vec{j}+j}$ whereat some devices have become states. In that respect, crossing between them due to an evolution in the temporal domain is substantial for a hierarchy [6].

The measurement display defines boundary between states and devices, which is arbitrary to a very large extent. Self-duality of the signal space representing both states and devices concerns the principle of psychophysical parallelism, as has been noticed by von Neumann [11]. The problem occurs in that the principle is violated so long as it is not demonstrated that the display has been placed in an arbitrary manner, which is achieved by crossing due to the evolutionary operator. In that regard, the evolution of measurement operators corresponds to its displacement by designating another χ_j to be a primary device. Devices of the measurement process are continually crossing into states and the term *psychophysics* is used in order to transcend any separation between the two.[6]

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

Von Neumann made a reference to Bohr, who was the first to have pointed out that the dual description of quantum theory relates to the principle of psychophysical parallelism [11]. Although Bohr never mentioned it in the print, he had adopted Fechner's psychophysics as taught to him by Høffding [23]. The most significant source for phychophysical parallelism is the foreword and the introduction from the *Elements of Psychophysics* [24]. Fechner's attitude is termed the *identity view*, since the observer is not to be considered a conglomeration of two substances but one single entity. The *outer psychophysics*, which is a link between sensation and stimulation, is realized through the neuroesthetical computation that relates sensation to neural activity, which is termed by Fechner to be the *inner psychophysics* [25].

An important repercussion of von Neumann's solution to the measurement problem is that the irreversibility takes place in the presence of the observer's mind, which seems to play an active role in the process. The only manner to make such an unpleasant situation compatible to psychophysical parallelism concerns switching into the inner psychophysics by a change in representation [6]. In that manner, the inner psychophysics should corresponds to a Markovian tree of the wavelet domain hidden Markov model.[25]

The irreversibility is actually manifested by the fact that a state before the measurement process results in the sum of diverse states thereafter. The primary measurement designed by an operator $M_0 = P_0$ is not irreversible in that respect, since it corresponds to the projector onto a single state. The problem occurs considering that the measurement operator $M_j = 2^{j/2}\mathfrak{U}^j P_0 = \sum_k 2^{j/2} P_{j,k}$ evolves into a combination of diverse projectors. It concerns the evolution represented by $\mathfrak{U}$ whose irreversibility comes to prominence due to a change in representation (16). The evolution $M_{j+1} = \sqrt{2}\mathfrak{U}M_j$ in terms of the Markov process $\mathfrak{W}$ becomes $M_{j+1} = \sqrt{2}\mathfrak{U}\sum_k 2^{j/2}P_{j,k} = \sqrt{2}\sum_k \Lambda^{\dagger}\mathfrak{W}\Lambda^{-1\dagger}2^{j/2}P_{j,k}$ and one denotes $S_{j,k} = \Lambda^{-1\dagger}2^{j/2}P_{j,k}F = 2^{j/2}\Lambda^{-1\dagger}\left|\psi_{j,k}\right\rangle\left\langle D_{j,k}\right|$ which has indicated an irreversible evolution of hidden variables $\sqrt{2}\mathfrak{W}\sum_k S_{j,k} = \sum_k S_{j+1,k}$. In that manner, the change of representation should transfigure detail coeffieicnets $\mathbf{D} = (D_{j,k})$ into a Markovian tree $\mathbf{S} = (S_{j,k})$.

The outer psychophysical information of an ensemble is independent of orthonormal wavelets, considering that $\mathrm{H}(C\mathbf{D}) = \mathrm{H}(\mathbf{D}) + \log|\det C| = \mathrm{H}(\mathbf{D})$ for any operator C which should be unitary since it represents a base substitution. The canonical relation $\mathrm{H}(\mathbf{D}) = \mathrm{H}(\mathbf{S}) + \mathrm{H}(\mathbf{D}|\mathbf{S})$ separates the inner psychophysical information $\mathrm{H}(\mathbf{S})$ from an irreducible randomness $\mathrm{H}(\mathbf{D}|\mathbf{S})$ [25]. The global entropy $\mathrm{H}(\mathbf{S})$ is related to the increase of the local one $\mathrm{H}(S_{j,k})$ in the temporal domain corresponding to the scale of the measurement hierarchy [27]. The optimal decomposition concerns the most significant increase of the information entropy, which is the measurement process characterized by [11].

5. Frame Wavelets and General Measurements

5.1. Duality in Frame Theory

The concept of *frame* refers to elements $\psi_{j,k}$ such that

$$A \leq \sum_{j\geq 0}\sum_{k=1}^{2^j}\left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right| \leq B$$

for positive numbers A and B which are termed *frame bounds* [12]. If $A = B = 1$, i.e., $1 = \sum_{j,k}\left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right|$, such a frame is the *Parseval* one. It is termed to be *frame wavelets* on the unit interval if axioms (10) – (13) hold.

$\widetilde{\psi_{j,k}}$ is a *dual frame* of $\psi_{j,k}$ if the resolution of identity applies

$$1 = \sum_{j,k}\left|\widetilde{\psi_{j,k}}\right\rangle\left\langle\psi_{j,k}\right|$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2023 doi:10.20944/preprints202310.1216.v1

If there is an operator [] such that $\widetilde{\psi_{j,k}} = []\psi_{j,k}$, the frame is *canonical dual*. Let] be an invertible operator such that $]\psi_{j,k}$ is the Parseval frame and [its ajoint. In that regard,

$$1 = \sum_{j,k} [\Big] \psi_{j,k} \Big\rangle \Big\langle \psi_{j,k} \Big| = [] \sum_{j,k}]^{-1} \Big] \psi_{j,k} \Big\rangle \Big\langle \psi_{j,k} \Big[[^{-1} = []^{-1} \sum_{j,k} \Big] \psi_{j,k} \Big\rangle \Big\langle \psi_{j,k} \Big[[^{-1} = []^{-1} [^{-1}$$

wherefrom it follows that [] is factorized into the product of operators [and].

The general measurement $\mathfrak{M}\rho = \sum_{j,k} M_{j,k} \rho M_{j,k}^{\dagger}$ is characterized by operators $M_{j,k}$ satisfying $1 = \sum_{j,k} M_{j,k}^{\dagger} M_{j,k}$, which means

$$\operatorname{Tr} \mathfrak{M}\rho = \sum_{j,k} \|M_{j,k} F\|^2 = \left\langle F \middle| \sum_{j,k} M_{j,k}^{\dagger} M_{j,k} F \right\rangle = \|F\|^2 = \operatorname{Tr} \rho$$

that a density is mapped into a density. In order to elucidate how it relates to the frame concept, one should consider operators $Q_{j,k} = \Big| \widetilde{\psi_{j,k}} \Big\rangle \Big\langle \psi_{j,k} \Big|$ that meet the resolution of identity $1 = \sum_{j,k} Q_{j,k}$. Under the term $\|\widetilde{\psi_{j,k}}\| = 1$, it follows $1 = \sum_{j,k} \Big] \psi_{j,k} \Big\rangle \Big\langle \psi_{j,k} \Big[= \sum_{j,k}] Q_{j,k}^{\dagger} Q_{j,k} [$ which has implied that $Q_{j,k} [= \Big| \widetilde{\psi_{j,k}} \Big\rangle \Big\langle \psi_{j,k} \Big[$ is the measurement operator $M_{j,k}$. Its evolution requires the Parseval frame $]\psi_{j,k}$ and the dual one $\widetilde{\psi_{j,k}}$ to be wavelets satisfying (10) – (13).

The evolutionary operator $U = C U_{\chi} C^{\dagger}$ on the space of ensembles is obtained through conjugation of the natural extension U_{χ} by $C : \Big| \chi_{j,k} \Big\rangle \Big\langle \chi_{j,k} \Big| \mapsto \Big| \widetilde{\psi_{j,k}} \Big\rangle \Big\langle \psi_{j,k} \Big[$ which transforms the Haar base to the Parseval frame and the dual one. Crossing devices into states due to the evolution by U concerns a duality relation $\Sigma = \widetilde{\Delta}$. [13] The signal space of the general measurement might not be self-dual, but it separates into dual spaces generated by $\psi_{j,k}$ and $\widetilde{\psi_{j,k}}$ respectively of which the first one should correspond to states and the second one to devices [12].

5.2. Measuring an Open System

According to the Naimark theorem, states of the measurement extend to a direct sum $\Sigma^* = \Sigma \oplus \Sigma'$ wherein the Parseval frame $\Big\langle \psi_{j,k} \Big[$ corresponds to the projection of an orthonormal base $\Big\{ \psi_{j,k} \Big[$ onto the subspace Σ. Likewise, the dual frame concerns measurement devices which are extended to $\Delta^* = \Delta \oplus \Delta'$. The measurement operator $M_{j,k}$ is restriction of the projector $M_{j,k}^* = [^{-1} \widetilde{\psi_{j,k}} \Big\} \Big\{ \psi_{j,k} \Big[$ onto $\Delta \otimes \Sigma$ and in that manner, the projective measurement $\mathfrak{M}^*$ restricts to the general one $\mathfrak{M}$ by neglecting an environment which has remained out of the scope [28]. The general measurement is therefore related to an open system that has been partially described by the stochastic process. Devices and states might be some subspaces of signals, respecting the duality between them. In that regard, frames $\Big\langle \psi_{j,k} \Big|$ and $\Big| \widetilde{\psi_{j,k}} \Big\rangle$ are projection of the Riesz base $\Big\{ \psi_{j,k} \Big|$ and its dual $\Big| \widetilde{\psi_{j,k}} \Big\} = \Big[\Big] \psi_{j,k} \Big\}$ which are biorthonormal [12].

A practical realization of the Naimark theorem implies a method analogous to heterodyne detection in communication engineering: the ensemble to be observed combines with another one, which is termed *ancilla* [29]. Thereafter, the von Neumann measurement corresponding to projectors $M_{j,k}^*$ has been performed on the combined space $\Delta^* \otimes \Sigma^*$ that is the tensor product of states and devices which are extended by the environment. The amount of information which is obtained in that manner might be larger than if the observer is restricted to the von Neumann measurements without ancilla. Optimal measurements are therefore not even close to be just projective ones which correspond to orthonormal wavelets in statistical signal processing [28].

A frame $\psi_{j,k}$ should be optimal for the ensemble from $\Delta \otimes \Sigma$ if the orthonormal base $\Big] \psi_{j,k} \Big\}$ is optimal for an ensemble in the combined space $\Delta^* \otimes \Sigma^*$. One assumes $F = \sum_{j,k} \Big| \widetilde{\psi_{j,k}^{o}} \Big\rangle \Big\langle D_{j,k}^{o} \Big|$ wherein $\psi_{j,k}^{o}$ is the optimal frame. Detail coefficients correspond to those of $F^* = \sum_{j,k} \Big[^{-1} \widetilde{\psi_{j,k}^{o}} \Big\} \Big\langle D_{j,k}^{o} \Big|$ in the base

$\left[{}^{-1}\widetilde{\psi^{o}_{j,k}}\right\} = \left]\psi^{o}_{j,k}\right\}$ which is orthonormal, though it might not imply any hierarchy (10) – (13). In that respect, general measurements spread the optimal decomposition to some ensembles which cannot be decorrelated in a hierarchical base but which has restricted to the frame providing a hierarchy of devices and states.

6. Conclusion

The measurement problem is formulated in terms of mathematical physics, notwithstanding any interpretation of physical theories. The significance of time for its elaboration has explicated a substantial relation between signals and stochastic processes, which is the definition of statistical signal processing. A paradigmatic measurement concerns commensuration of magnitudes by the Euclidean algoritm producing a time series of binary digits. It constitutes the hierarchy of the binary tree whose nodes correspond to both states amd devices of the measurement process.

The time operator formalism of complex systems which has proposed a unification of reversible and irreversible processes relates the problem in respect to its definition that was postulated by von Neumann. He's indicated two fundamentally diverse types of interventions in a system, the first of which corresponds to a temporal evolution that is reversible and the second one to an irreversible measurement. The main advancement concerns a formulation of the measurement process in terms of a temporal evolution, whereat irreversibility has occured due to the change in representation switching from outer to inner psychophysics. The principle of psychophysical parallelism that was pointed out by Bohr and von Neumann should consistently realize in that manner.

The optimal measurement corresponds to the most significant increase of the information entropy in the temporal domain. It has implied decorrelation of the ensemble, which is a consequence of its invariance under the process. Generalization to an open system is performed by the use of duality in frame theory, spreading the optimal decomposition to some ensembles which cannot be decorrelated in a hierarchical base.

Acknowledgments: The author acknowledges support by the Ministry of Education, Science and Technological Development of the Republic of Serbia through Mathmatical Institute of the Serbian Academy of Sciences and Arts.

References

1. A. Sommerfeld, *Annalen der Physik* **44**, 177 (1914).
2. W. Heisenberg, *Zeitschrift für Physik* **43**, 172–198 (1927).
3. D. Gabor, *Journal of the Institution of Electrical Engineers* **93**, 429–457 (1946).
4. M. A. Nielson and I. L. Chung, *Quantum Computation and Quantum Information*, 2nd edn. (Cambridge University Press, Cambridge, 2010).
5. G. J. Chaitin, *Chaos, Solitons & Fractals* **5**, 143–159 (1995).
6. M. Milovanović, S. Vukmirović and N. Saulig, *Mathematics* **9**, 1452 (2021).
7. I. Prigogine, *From Being to Becoming: Time and Complexity in the Physical Science* (W.H. Freeman & Co., New York, 1980), pp. 65–67, 241.
8. T. S. Kuhn, *Isis* **52**, 161–193 (1961).
9. M. Miilovanović, B. M. Tomić and N. Saulig, *Chaos, Solitons & Fractals* **173**, 113724 (2023).
10. M. Miilovanović and N. Saulig, *Mathematics* **10**, 4294 (2022).
11. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955), pp. 207, 351–354, 398–420.
12. D. Han and D. R. Larson, *Frames, Bases and Group Representations* (American Mathematical Society, Providence, 2010), pp. 21–45.
13. L. Florack, *Image Structure*, (Springer, Heidelberg, 2010), pp. 16–23.
14. I. E. Antoniou and B. Misra, *International Journal of Theoretical Physics* **31**, 119–136 (1992).
15. W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover Publications, Mineola, 1949), pp. 39–46.

doi:10.20944/preprints202310.1216.v1

16. I. Antoniou, B. Misra and Z. Suchanecki, *Time Operator, Innovation and Complexity* (John Wiley & Sons, Hoboken, 2003), pp. 29–32, 47–60, 131–134.
17. B. O. Koopman, *Proceedings of the National Academy of Sciences of the United States of America* **17**, 315–318 (1931).
18. B. Misra, I. Prigogine and M. Courbage, *Physica* **98A**, 1–26 (1979).
19. S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way* (Elsevier, Amsterdam, 2009), p. 611.
20. I. Daubechies, *Ten Lectures on Wavelets* (Society for Industrial and Applied Mathematics, Philadelphia, 1992), pp. 304–307.
21. I. E. Antoniou and K. E. Gustafson, *Chaos, Solitons & Fractals* **11**, 443–452 (2000).
22. I. Antoniou and K. Gustafson, *Mathematics and Computers in Simulation* **49**, 81–104 (1999).
23. M. Heidelberger, The mind-body problem in the origin of logical empiricism: Herbert Feigel and psychophysical parallelism, in *Logical Empiricism: Historical and Contemporary Perspectives*, eds. P. Parrini, W. C. Salmon and M. H. Salmon (University of Pittsburgh Press, Pittsburg, 2003), pp. 233–262.
24. G. T. Fechner, *Elemente der Psychophysik, Erster Theil* (Breitkopf und Härtel, Wesbaden, 1860), pp. vii–xiii, 1–20.
25. M. Milovanović and G. Medić-Simić, *Neural Computing and Applications* **33**, 2137–2156 (2021).
26. M. C. Crouse, R. D. Novak and R. G. Baraniuk, *IEEE Transactions on Signal Processing* **46**, 886–902 (1998).
27. M. Milovanović and M. Rajković, *Europhysics Letters* **102**, 40004 (2013).
28. A. Peres and W. K. Wootters, *Physical Review Letters* **66**, 1119–1122 (1991).
29. C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976), pp. 74–83.