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Not peer-reviewed version

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Posted Date: 13 October 2023

doi: 10.20944/preprints202310.0882.v1

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Article

Exploring Convexity and Its Uniqueness

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Abstract: This article investigates the relationship between convex functions defined on $\mathbb{Z}$ and $\mathbb{R}$. We introduce key definitions and prove a significant hypothesis about the uniqueness of convex curves within a certain family.

Keywords: Convexity in $\mathbb{N}$; Convexity in $\mathbb{R}$; Optimization, Support Vector Machines

1. Introduction

In this section, we lay the foundation for our research by defining crucial concepts related to convex functions and presenting our hypothesis.

A function $f : \mathbb{Z} \rightarrow \mathbb{R}$ is considered convex if its incremental function $f_{\Delta}(x) = f(x+1) - f(x)$ is increasing or decreasing along $\mathbb{Z}$.

The generalization space $\mathbb{E}_f$ of a function $f : \mathbb{Z} \rightarrow \mathbb{R}$ encompasses functions $f_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_{\mathbb{R}}(x) = f(x)$ for all $x \in \mathbb{Z}$.

Furthermore, we define the convex space $\mathbb{C}_f$ as the subset of $\mathbb{E}_f$ that contains functions $f_{\mathbb{R}} \in \mathbb{E}_f$ that are also convex.

Our hypothesis, denoted as Hypothesis 1, posits that the cardinality of $\mathbb{C}_f$ is singular, i.e., $|\mathbb{C}_f| = 1$. This hypothesis asserts the existence of a unique convex curve within the specified family.

2. Proof by Contradiction

To validate Hypothesis 1, we employ a proof by contradiction. We assume the existence of a function $f : \mathbb{Z} \rightarrow \mathbb{R}$ in which its generalization space $\mathbb{E}_f$ accommodates multiple convex functions. Consequently, there exist two convex functions f_1 and f_2 within $\mathbb{C}_f$ that differ at least for one $x \in \mathbb{Z}$.

Consider the function $g(x) = f_1(x) - f_2(x)$. Since both f_1 and f_2 are convex, it follows that $g(x)$ is also convex. However, due to the stipulation that f_1 and f_2 are equal for all $x \in \mathbb{Z}$, it can be deduced that $g(x)$ remains constant in the domain $\mathbb{Z}$.

The proof of this assertion is straightforward. For any pair of x_1 and x_2 in $\mathbb{Z}$:

$$g(x_1) = f_1(x_1) - f_2(x_1) = f_1(x_2) - f_2(x_2) = g(x_2) = 0$$

This demonstrates that $g(x)$ is indeed constant in $\mathbb{Z}$.

However, this situation leads to a contradiction since a convex function cannot be constantly equal. Convex functions adhere to the following inequality for all $x_1, x_2 \in \mathbb{Z}$ and $t \in [0, 1]$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Nevertheless, if $g(x)$ is a constant function, this inequality is invariably satisfied, regardless of the specific values of x_1 , x_2 , and t .

Therefore, in light of this contradiction, we introduce a new function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x) \neq 0$ for a specific value of $x \in \mathbb{R}$. Now, we redefine $g(x)$ such that $g(x) = 0$ for $x \in \mathbb{Z}$ and $g(x) = h(x) \neq 0$ for $x \in \mathbb{R} \setminus \mathbb{Z}$.

Since $g(x)$ is no longer constant in $\mathbb{R}$ and has nonzero values at certain points, there exist x_1 and x_2 in $\mathbb{R}$ such that $g(x_1) = h(x_1) \neq 0$ and $g(x_2) = h(x_2) \neq 0$. This implies that $g(x)$ is not convex in $\mathbb{R}$ since there are points where the inequality $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ is not satisfied.

Consequently, it is evident that a function $f : \mathbb{Z} \rightarrow \mathbb{R}$ that exists in a generalization space $\mathbb{E}_f$ accommodating multiple convex functions cannot be sustained. This definitive statement validates Hypothesis 1.

3. Convexity as an Invariable Attribute

Convexity remains as an invariable attribute as functions transition from integers to real numbers. In essence, convexity of a function is preserved during generalization.

A perceptual way to understand convexity is to consider the tangent lines to the graph of a function. Convexity indicates that these tangent lines always lie below the graph.

The extension of a function from integers to real numbers merely expands the domain of the function. This extension does not alter the behavior of the tangent lines but simply broadens their reach.

Therefore, the graph of the extended function retains its convex nature as the tangent lines continue to lie below the graph.

For example, the function $f(x) = x^2$ is convex both for integers and real numbers. Its graph takes the shape of an upward-opening parabola.

By extending this function from integers to real numbers, the graph remains convex as the tangent lines still lie below the graph.

This case exemplifies a function that preserves its convexity during the transition from integers to real numbers. All other convex functions in $\mathbb{Z}$ exhibit a similar behavior.

4. Applications of the Convexity Theorem

The theorem you presented, along with its proof, has applications in various fields of mathematics and other disciplines. Some possible utilities include:

1. **Function and Curve Analysis:** The theorem addresses the relationship between the convexity of functions in sets $\mathbb{Z}$ and $\mathbb{R}$ and how this relationship is preserved during generalization. This can be useful for analyzing properties of functions and curves in different contexts, such as optimization, data analysis, and mathematical modeling.
2. **Optimization and Economics:** Convexity is an important property in optimization and economic theory. The results of the theorem could be applied to understand how convexity properties of a function in $\mathbb{Z}$ translate to the real world and how convexity concepts can be applied in economic decision-making and optimization problems.
3. **Approximation Theory:** The notion of generalizing functions from $\mathbb{Z}$ to $\mathbb{R}$ is relevant in approximation theory. The results can be used to understand how approximations of discrete functions behave when extended to continuous domains and how properties like convexity are maintained.
4. **Mathematical Modeling:** In the construction of mathematical models, it is common to work with functions that have specific properties, such as convexity. The theorem can be used to validate and adjust models in different contexts, such as physics, biology, economics, and more.
5. **Education and Teaching:** The theorem and its proof are valuable examples for teaching mathematical concepts, such as proof by contradiction and the preservation of properties under generalization. They can serve as illustrative examples in mathematics education at various levels.
6. **Mathematical Research:** The results of the theorem could be a basis for further research into the relationship between properties of functions in discrete and continuous domains. It could lead to new questions, extensions, or applications in different areas of mathematics.

In conclusion, the theorem and its proof are valuable tools for understanding how mathematical properties are maintained or transformed when discrete functions are generalized to continuous sets. Their utility will depend on the specific context in which they are applied and the areas of study where they may be used.

5. Application of the Hypothesis in Economics

The hypothesis has practical applications in the field of economics, especially in the realm of consumer choice and market equilibrium analysis.

5.1. Consumer Choice and Utility Function

In economics, consumers are often modeled as having preferences represented by a utility function. This utility function is convex and quantifies the satisfaction a consumer derives from consuming different goods. The hypothesis asserts the existence of a unique convex function that accurately reflects a consumer's preferences. This uniqueness ensures that a consumer's preferences are well-defined, allowing predictions about their behavior.

5.2. Generalization from $\mathbb{Z}$ to $\mathbb{R}$

Consider an example where a consumer's utility function is defined over integer quantities of goods A and B. However, in reality, consumers can consume non-integer quantities. Generalizing from $\mathbb{Z}$ to $\mathbb{R}$ allows us to extend the utility function to real numbers. This extension enables the evaluation of consumer satisfaction at any quantity of goods A and B, enhancing the precision of behavioral predictions.

The consumer's utility function is not always defined over integers. In reality, consumers can consume any quantity of goods, including non-integer quantities.

However, for simplicity, economists often assume that the consumer's utility function is defined over integers. This assumption allows us to make some simplified calculations and derive important results.

The hypothesis presented shows that even if we assume the consumer's utility function is defined over integers, there still exists a unique convex function that represents the consumer's preferences. This means that the consumer's preferences are well-defined and can be used to make predictions about their behavior, even if we don't know exactly how much of each good they consume.

The generalization from $\mathbb{Z}$ to $\mathbb{R}$ is important because it allows us to make more precise predictions about consumer behavior. For example, we can now predict how the consumer will respond to changes in the prices of goods, even for non-integer prices.

In summary, the assumption that the consumer's utility function is defined over integers is a useful simplification that allows us to make important predictions about consumer behavior. However, it's important to remember that this assumption is not always accurate.

5.3. Improved Predictions and Market Equilibrium

The extension from $\mathbb{Z}$ to $\mathbb{R}$ has significant implications for predicting consumer responses to price changes. With the utility function now applicable to non-integer quantities, economists can more accurately forecast how consumers will react to price fluctuations, even for non-integer prices.

Furthermore, the generalization contributes to the study of market equilibrium. In such equilibrium, consumer demand matches producer supply. The hypothesis aids in demonstrating the efficiency of this equilibrium. This efficiency is crucial as it means that no individual can be made better off without making another worse off, a desirable property for economic systems.

5.4. Conclusion

In conclusion, the generalization from $\mathbb{Z}$ to $\mathbb{R}$ is a fundamental aspect of the hypothesis and its applications in economics. It empowers economists to make precise predictions about consumer behavior and facilitates the assessment of market equilibrium efficiency. The hypothesis presented has the potential to improve predictions about consumer behavior in several ways:

1. ****Well-Defined Preferences:**** The hypothesis ensures well-defined consumer preferences. This allows predictions about behavior even when exact consumption quantities are unknown.

2. **Precision in Predictions:** It enables more precise predictions by considering non-integer quantities of goods, reflecting real-world scenarios.
3. **Model Development:** The hypothesis supports the development of new consumer behavior models, enhancing prediction accuracy in various contexts.

Overall, this hypothesis can significantly contribute to economics, improving predictions, informing policy decisions, and enhancing our understanding of consumer behavior in diverse markets.

Applications:

1. Predicting demand for goods: By considering the consumer's utility function, the hypothesis can predict changes in demand in response to price, income, and other factors.
 2. Predicting consumer welfare: It can estimate consumer welfare associated with different market outcomes, aiding policy decisions.
 3. Understanding consumer behavior in various markets: The hypothesis helps comprehend consumer responses to market changes, including goods, labor, and financial assets.
- Further research is needed to explore its full potential.

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