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Article

An Investigation on Fractal Characteristics of the Superposition of Fractal Surfaces

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Abstract: In this paper, we make research on fractal characteristics of the superposition of fractal surfaces in the view of fractal dimension. We give the upper bound of the lower and upper Box dimension of the graph of the sum of two bivariate continuous functions and calculate the exact values of them under some particular conditions. Further, it has been proved that the superposition of two continuous surfaces cannot keep the fractal dimensions invariable unless both of them are two-dimensional. A concrete example of numerical experiment has been provided to verify our theoretical results. This study can be applied to the fractal analysis of metal fracture surfaces or computer image surfaces.

Keywords: bivariate continuous functions; fractal dimension; the Box dimension; superposition of fractal surfaces

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1. Introduction

Fractal surface, as a class of fractal sets in the three-dimensional Euclidean space, is an important research object in fractal geometry [1]. At present, fractal surface has been extensively applied in a variety of academic fields such as metal materials [2], geology [3], computer graphics [4] and so on. One of the most concerned problems is to investigate how to measure the geometric complexity of a fractal surface, like the texture roughness of a metal fracture surface or a computer image surface. Fractal dimension [5] is a common measure of the geometric complexity of a surface, which can be used to describe its fractal characteristics well. It is well known that the topological dimension of a surface is two. Nevertheless, its fractal dimension increases with larger amounts of complexity or roughness, which is usually greater than its topological dimension. For instance, the fractal dimension of the relief on the earth has been found to be 2.3 in general [6]. Beyond that, many scholars have used iterative function systems (IFS) to construct fractal surfaces that are attractors of certain IFS in fact. More details about fractal surfaces and relevant studies of their fractal dimensions can be found in [7–10].

In recent years, exploring the fractal dimension of the graph of fractal curves has drawn the attention of numerous researchers. There are some commonly used definitions of the fractal dimension, such as the Box dimension, the Packing dimension, the Hausdorff dimension and the Assouad dimension, which are denoted as $\dim_B$, $\dim_P$, $\dim_H$ and $\dim_A$ throughout this paper, respectively. Of the diverse fractal dimensions, the Box dimension mainly considered in the present paper shows its advantage of relatively easy calculation. Up to now, a lot of meaningful work have been done, including fractal interpolation functions [11–14], α-Hölder continuous functions [15,16], self-similar curves like the Von Koch curve [17,18], and some specific fractal functions like the Weierstrass function [19–23] and the Besicovitch function [24–26]. For more details of latest work, we refer the interested readers to [27–32].

Another essential issue involved recently is to estimate the fractal dimension of the superposition of two fractal curves, namely, the sum of two continuous functions of one variable. This problem can be traced back to the research made first by Falconer [33] who showed that the Box dimension of the sum of two continuous functions equals to the greater of the Box dimensions of them. On this basic, a group of academic workers have pushed this study forward and obtained a series of preliminary

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conclusions, whose related progress can be found in [34–39]. So in this paper, we shall focus on the fractal dimension of the superposition of two fractal surfaces and investigate whether it has the same result with that of fractal curves. Based on three-dimensional Cartesian coordinate system, a fractal surface can be looked upon as a bivariate continuous function, whose fractal dimension and fractional calculus has been established in [40]. This work will contribute to enriching the theory with regards to the fractal dimension of fractal surfaces and can be applied to the research on fractal characteristics analysis of the superposition of two metal fracture surfaces or two computer image surfaces.

The outline of the remainder of this paper is organized as follows: In upcoming Section 2, we cover required notations, concepts and results on fractal dimensions of the graph of bivariate continuous functions for subsequent sections. Then in Section 3, we present our main results obtained in this paper. Firstly, we give the upper bound of the lower and upper Box dimension of the graph of the sum of two bivariate continuous functions. Secondly, we calculate the exact value of the lower and upper Box dimension of the graph of the sum of two bivariate continuous functions under certain particular circumstances. Thirdly, we explore some concrete situations when the two bivariate continuous functions have the Box dimension or not and also consider the case when one of these two functions is Lipschitz. Later in Section 4, we provide a specific example and do numerical experiments to verify the theoretical results in Section 3. Finally in Section 5, we sum up our conclusions and discuss the further research in the future.

2. Preliminaries

In the present paper, all the subjects we discuss are entirely real. Given a non-empty subset $\mathcal{D} \subset \mathbb{R}^2$ and a bivariate function $f : \mathcal{D} \to \mathbb{R}$, the oscillation of f over the rectangular region $\mathcal{R}$ is written as

$$\mathrm{OSC}(f, \mathcal{R}) = \sup_{(x,y),(u,v) \in \mathcal{R} \cap \mathcal{D}} |f(x,y) - f(u,v)| \tag{1}$$

and the graph of $f(x)$ on $\mathcal{D}$ is defined as

$$G_f = \{((x,y), f(x,y)) : (x,y) \in \mathcal{D}\} \subseteq \mathcal{D} \times \mathbb{R}.$$

We denote ϑ as the function which is always equal to 0 on $\mathcal{D}$. Let $\|\cdot\|_2$ be the usual Euclidean norm in $\mathbb{R}^n$. For any $\tau_1, \tau_2, \cdots, \tau_n \in \mathbb{Z}$ and $\delta > 0$, we call $\prod_{i=1}^{n} [\tau_i \delta, (\tau_i + 1)\delta]$ a δ-coordinate cube in $\mathbb{R}^n$.

Below we shall briefly introduce the definition of the Box dimension. For more details about other kinds of fractal dimensions, we consult the interested readers to [1,5,33,37,41], for example.

Definition 1 ([33]). *Let $X \neq \emptyset$ be a bounded subset of $\mathbb{R}^n$ and let $\mathcal{N}_\delta(X)$ be the smallest number of δ-coordinate cubes that intersect X. Then the lower and upper Box dimension of X are defined as, respectively,*

$$\underline{\dim}_B(X) = \underline{\lim}_{\delta \to 0} \frac{\log \mathcal{N}_\delta(X)}{-\log \delta}$$

and

$$\overline{\dim}_B(X) = \overline{\lim}_{\delta \to 0} \frac{\log \mathcal{N}_\delta(X)}{-\log \delta}.$$

If the above two are equal, we define the Box dimension of X as the common value, that is,

$$\dim_B(X) = \lim_{\delta \to 0} \frac{\log \mathcal{N}_\delta(X)}{-\log \delta}.$$

Remark 1. *The notation $\mathcal{N}_\delta(X)$ in Definition 1 can also be replaced by one of the following:*

(1) The smallest number of sets of diameter at most δ that cover X;
(2) The smallest number of cubes of side δ that cover X;

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(3) The largest number of disjoint balls of radius δ with centres in X;
(4) The smallest number of closed balls of radius δ that cover X.

Now we provide some fundamental results, which will be used in subsequent research. The forthcoming two lemmas can be essential approaches to estimating the Box dimension of the graph of a bivariate continuous function.

Lemma 1 ([33]). *Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^n$.*

(1) If f is a Lipschitz map, that is,
$$\|f(x) - f(y)\|_2 \leq C\,\|x - y\|_2$$
for $\forall x, y \in X$ and certain $0 < C < +\infty$. Then
$$\dim\left(f(X)\right) \leq \dim(X).$$

(2) If f is a bi-Lipschitz map, that is,
$$C_1\,\|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq C_2\,\|x - y\|_2$$
for $\forall x, y \in X$ and certain $0 < C_1 \leq C_2 < +\infty$. Then
$$\dim\left(f(X)\right) = \dim(X).$$

Here $\dim$ *denotes any one of* $\dim_B$, $\underline{\dim}_B$ *and* $\overline{\dim}_B$.

Lemma 2 ([33]). *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and $0 < \delta < \min\{b-a, d-c, 1\}$. Suppose that m and n, respectively, are the least integer greater than or equal to $\frac{b-a}{\delta}$ and $\frac{d-c}{\delta}$. Then the range of $\mathcal{N}_\delta(G_f)$ can be estimated as*
$$\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} \max\left\{1, \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1}\right\} \leq \mathcal{N}_\delta(G_f) \leq \sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\left\{2 + \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1}\right\}$$
where $\mathcal{R}_{i,j} = [a + i\delta, a + (i+1)\delta] \times [c + j\delta, c + (j+1)\delta]$.

Proof. Since $f(x)$ is continuous on $[a,b] \times [c,d]$, the estimation of $\mathcal{N}_\delta(G_f)$ can be transformed into the oscillation of $f(x)$ on the above subregions. We note that the number of cubes of side length δ in the part above the rectangular region $\mathcal{R}_{i,j}$ that intersect G_f is no less than
$$\max\left\{1, \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1}\right\}$$
and no more than
$$2 + \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1}.$$
Summing over all the subregions just leads to the present lemma. □

The next proposition reveals several basic properties relating to the fractal dimensions of the graph of a bivariate continuous function.

Proposition 1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Given a constant $r \in \mathbb{R}$, the following three statements hold.*

(1) It holds
$$2 \leq \underline{\dim}_B(G_f) \leq \overline{\dim}_B(G_f) \leq 3.$$

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(2) *For a constant bivariate function $f(x,y) \equiv r$ on $[a,b] \times [c,d]$, we have*

$$\underline{\dim}_B(G_f) = \overline{\dim}_B(G_f) = \dim_B(G_f) = 2.$$

(3) *If $r \neq 0$, then*

$$\underline{\dim}_B(G_{r \cdot f}) = \underline{\dim}_B(G_f) \quad \text{and} \quad \overline{\dim}_B(G_{r \cdot f}) = \overline{\dim}_B(G_f).$$

Proof. The following arguments for (1)–(3) are all based on Definition 1, Lemma 1 and 2.

(1) Assume that $\max_{(x,y)\in[a,b]\times[c,d]} |f(x,y)| = \mathcal{M} > 0$. On one hand, it follows from Lemma 2 that

$$\begin{aligned}
\mathcal{N}_\delta(G_f) &\leq \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} \left\{2 + \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1}\right\} \\
&\leq mn\left(2 + 2\mathcal{M}\delta^{-1}\right) \\
&\leq 2\left((b-a)\delta^{-1} + 1\right)\left((d-c)\delta^{-1} + 1\right)\left(1 + \mathcal{M}\delta^{-1}\right) \\
&\leq 2(b-a+1)(d-c+1)(\mathcal{M}+1)\delta^{-3}.
\end{aligned}$$

Thus by Definition 1,

$$\begin{aligned}
\overline{\dim}_B(G_f) &= \overline{\lim_{\delta\to 0}} \frac{\log \mathcal{N}_\delta(G_f)}{-\log\delta} \\
&\leq \overline{\lim_{\delta\to 0}} \frac{\log\left[2(b-a+1)(d-c+1)(\mathcal{M}+1)\delta^{-3}\right]}{-\log\delta} \\
&= \overline{\lim_{\delta\to 0}} \frac{\log\delta^3}{\log\delta} + \overline{\lim_{\delta\to 0}} \frac{\log\left[2(b-a+1)(d-c+1)(\mathcal{M}+1)\right]}{-\log\delta} \\
&= 3.
\end{aligned}$$

On the other hand, it is observed that

$$\begin{aligned}
\mathcal{N}_\delta(G_f) &\geq \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} 1 \\
&= mn \\
&= \left((b-a)\delta^{-1} + 1\right)\left((d-c)\delta^{-1} + 1\right) \\
&\geq (b-a)(d-c)\delta^{-2}.
\end{aligned}$$

So by Definition 1, we can get

$$\begin{aligned}
\underline{\dim}_B(G_f) &= \underline{\lim_{\delta\to 0}} \frac{\log \mathcal{N}_\delta(G_f)}{-\log\delta} \\
&\geq \underline{\lim_{\delta\to 0}} \frac{\log\left[(b-a)(d-c)\delta^{-2}\right]}{-\log\delta} \\
&= \underline{\lim_{\delta\to 0}} \frac{\log\delta^2}{\log\delta} + \underline{\lim_{\delta\to 0}} \frac{\log\left[(b-a)(d-c)\right]}{-\log\delta} \\
&= 2.
\end{aligned}$$

Obviously, we can assert from Definition 1 that $\underline{\dim}_B(G_f) \leq \overline{\dim}_B(G_f)$, which leads to the conclusion of (1).

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(2) Note that $\mathrm{OSC}(f, \mathcal{R}_{i,j}) = 0$ when $f(x,y) \equiv r$ on $[a,b] \times [c,d]$. Consequently,

$$\begin{aligned}\mathcal{N}_\delta(\mathrm{G}_f) &\leq 2mn + \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} \mathrm{OSC}(f, \mathcal{R}_{i,j}) \cdot \delta^{-1} \\ &\leq 2\left((b-a)\delta^{-1}+1\right)\left((d-c)\delta^{-1}+1\right) \\ &\leq 2(b-a+1)(d-c+1)\delta^{-2}.\end{aligned}$$

At this time, we obtain

$$\begin{aligned}\overline{\dim}_B(\mathrm{G}_f) &= \overline{\lim_{\delta\to 0}} \frac{\log \mathcal{N}_\delta(\mathrm{G}_f)}{-\log\delta} \\ &\leq \overline{\lim_{\delta\to 0}} \frac{\log\left[2(b-a+1)(d-c+1)\delta^{-2}\right]}{-\log\delta} \\ &= \overline{\lim_{\delta\to 0}} \frac{\log\delta^2}{\log\delta} + \overline{\lim_{\delta\to 0}} \frac{\log\left[2(b-a+1)(d-c+1)\right]}{-\log\delta} \\ &= 2.\end{aligned}$$

Combining (1) of Proposition 1,

$$2 \leq \underline{\dim}_B(\mathrm{G}_f) \leq \overline{\dim}_B(\mathrm{G}_f) \leq 2.$$

That is,

$$\underline{\dim}_B(\mathrm{G}_f) = \overline{\dim}_B(\mathrm{G}_f) = \dim_B(\mathrm{G}_f) = 2,$$

finishing the proof of (2).

(3) Let us define a mapping $\Gamma : \mathrm{G}_f \to \mathrm{G}_{r\cdot f}$ by

$$\Gamma((x,y), f(x,y)) = ((x,y), (r\cdot f)(x,y)), \quad (x,y) \in [a,b]\times[c,d]$$

for $\forall r \in \mathbb{R}\setminus\{0\}$. By using the simple properties of norm, one can show that

$$\begin{aligned}&\|\Gamma((x,y), f(x,y)) - \Gamma((u,v), f(u,v))\|_2 \\ &\leq \sqrt{1+r^2}\,\|((x,y), f(x,y)) - ((u,v), f(u,v))\|_2\end{aligned}$$

and

$$\begin{aligned}&\|\Gamma((x,y), f(x,y)) - \Gamma((u,v), f(u,v))\|_2 \\ &\geq \frac{|r|}{\sqrt{r^2+1}}\,\|((x,y), f(x,y)) - ((u,v), f(u,v))\|_2\end{aligned}$$

for $\forall (x,y), (u,v) \in [a,b]\times[c,d]$. With Lemma 1, we can claim that Γ is a bi-Lipschitz mapping and then the result of (3) holds.

□

Remark 2. *In Proposition 1, if the Box dimension of* G_f *exists on* $[a,b]\times[c,d]$*, then*

$$2 \leq \dim_B(\mathrm{G}_f) \leq 3$$

and for $\forall r \in \mathbb{R}\setminus\{0\}$,

$$\dim_B(\mathrm{G}_{r\cdot f}) = \dim_B(\mathrm{G}_f).$$

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In particular, if $r = 0$, we have

$$\underline{\dim}_B(G_\vartheta) = \overline{\dim}_B(G_\vartheta) = \dim_B(G_\vartheta) = 2$$

by (2) of Proposition 1. Thus for any continuous function $f : [a, b] \times [c, d] \to \mathbb{R}$, $0 \cdot f$ must be a two-dimensional continuous function on $[a, b] \times [c, d]$.

Up to now, some particular bivariate continuous functions with non-integer fractal dimensions have been constructed. For instance, Yu [42] had given the following facts.

Example 1 ([42])**.** *For $0 < \alpha < 1$ and $\lambda \geq 2$, let*

$$\mathcal{W}(x, y) = \sum_{j=1}^{\infty} \lambda^{-\alpha j} \sin(\lambda^j x), \quad (x, y) \in [a, b] \times [c, d].$$

Then

$$\dim_B(G_{\mathcal{W}}) = 3 - \alpha.$$

Example 2 ([42])**.** *For $1 < s < 2$, let*

$$\mathcal{B}(x, y) = \sum_{j=1}^{\infty} \lambda_j^{s-2} \cos(\lambda_j x), \quad (x, y) \in [a, b] \times [c, d]$$

where $\frac{\lambda_{j+1}}{\lambda_j} \geq \lambda > 1$ for $\forall j \in \mathbb{N}^*$. If $\frac{\lambda_{j+1}}{\lambda_j} \nearrow \infty$, then $\underline{\dim}_B(G_{\mathcal{B}})$ and $\overline{\dim}_B(G_{\mathcal{B}})$ could be any numbers satisfying

$$2 \leq \underline{\dim}_B(G_{\mathcal{B}}) < \overline{\dim}_B(G_{\mathcal{B}}) < 3.$$

3. Main Results

In this section, we present our main results for the fractal dimensions the graph of the sum of two bivariate continuous functions. For two bivariate continuous functions $f, g : [a, b] \times [c, d] \to \mathbb{R}$, our motivation is to explore the values of $\underline{\dim}_B(G_{f+g})$ and $\overline{\dim}_B(G_{f+g})$. According to Definition 1, we can notice that the estimation of $\mathcal{N}_\delta(G_{f+g})$ is key to calculate them. Hence, we begin by investigating how to attain the range of $\mathcal{N}_\delta(G_{f+g})$. The upcoming result about the oscillation is basic.

Theorem 1. *Let $f, g : [a, b] \times [c, d] \to \mathbb{R}$ be continuous. Then the range of $\mathrm{OSC}(f + g, \mathcal{R}_{i,j})$ can be estimated as*

$$\begin{aligned}\left| \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \mathrm{OSC}(f, \mathcal{R}_{i,j}) - \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \mathrm{OSC}(g, \mathcal{R}_{i,j}) \right| &\leq \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \mathrm{OSC}(f + g, \mathcal{R}_{i,j}) \\ &\leq \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \mathrm{OSC}(f, \mathcal{R}_{i,j}) + \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} \mathrm{OSC}(g, \mathcal{R}_{i,j})\end{aligned}$$

where $m, n, \mathcal{R}_{i,j}$ have been defined in Lemma 2.

Proof. From Equation (1), we can obtain

$$\mathrm{OSC}(-f, \mathcal{R}_{i,j}) = \mathrm{OSC}(f, \mathcal{R}_{i,j}) \tag{2}$$

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and

$$\begin{aligned} \mathrm{OSC}(f+g,\mathcal{R}_{i,j}) &= \sup_{(x,y),(u,v)\in\mathcal{R}_{i,j}} |(f+g)(x,y)-(f+g)(u,v)| \\ &\le \sup_{(x,y),(u,v)\in\mathcal{R}_{i,j}} \{|f(x,y)-f(u,v)|+|g(x,y)-g(u,v)|\} \\ &\le \sup_{(x,y),(u,v)\in\mathcal{R}_{i,j}} |f(x,y)-f(u,v)| + \sup_{(x,y),(u,v)\in\mathcal{R}_{i,j}} |g(x,y)-g(u,v)| \\ &\le \mathrm{OSC}(f,\mathcal{R}_{i,j})+\mathrm{OSC}(g,\mathcal{R}_{i,j}). \end{aligned} \tag{3}$$

Summing over all the rectangular regions in Equation (3) just leads to the right end of the required inequality. Then combining Equations (2) and (3), we estimate

$$\mathrm{OSC}(f,\mathcal{R}_{i,j}) = \mathrm{OSC}(f+g-g,\mathcal{R}_{i,j}) \le \mathrm{OSC}(f+g,\mathcal{R}_{i,j})+\mathrm{OSC}(g,\mathcal{R}_{i,j})$$

and

$$\mathrm{OSC}(g,\mathcal{R}_{i,j}) = \mathrm{OSC}(f+g-f,\mathcal{R}_{i,j}) \le \mathrm{OSC}(f+g,\mathcal{R}_{i,j})+\mathrm{OSC}(f,\mathcal{R}_{i,j}).$$

Thus

$$\mathrm{OSC}(f+g,\mathcal{R}_{i,j}) \ge \left|\mathrm{OSC}(f,\mathcal{R}_{i,j})-\mathrm{OSC}(g,\mathcal{R}_{i,j})\right|. \tag{4}$$

Summing over all the rectangular regions in Equation (4) and using absolute value inequality, one can get the left end of our required inequality as well. □

In the light of Theorem 1 and Lemma 2, $\mathcal{N}_\delta(\mathrm{G}_{f+g})$ seems to have a certain relationship with $\mathcal{N}_\delta(\mathrm{G}_f)$ and $\mathcal{N}_\delta(\mathrm{G}_g)$. The next important theorem establishes a connection among the above three.

Theorem 2. *Let $f,g:[a,b]\times[c,d]\to\mathbb{R}$ be continuous. Then the range of $\mathcal{N}_\delta(\mathrm{G}_{f+g})$ can be estimated as*

$$\left|\mathcal{N}_\delta(\mathrm{G}_f)-\mathcal{N}_\delta(\mathrm{G}_g)\right| - \rho\delta^{-2} \le \mathcal{N}_\delta(\mathrm{G}_{f+g}) \le \mathcal{N}_\delta(\mathrm{G}_f)+\mathcal{N}_\delta(\mathrm{G}_g)+\rho\delta^{-2}$$

where $0<\delta<\min\{b-a,d-c,1\}$ and $\rho=2(b-a+1)(d-c+1)\delta^{-2}$.

Proof. It follows from Theorem 1 and Lemma 2 that

$$\begin{aligned} \mathcal{N}_\delta(\mathrm{G}_{f+g}) &\le 2mn+\delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(f+g,\mathcal{R}_{i,j}) \\ &\le 2\left((b-a)\delta^{-1}+1\right)\left((d-c)\delta^{-1}+1\right) \\ &\quad +\delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(f,\mathcal{R}_{i,j})+\delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(g,\mathcal{R}_{i,j}) \\ &\le 2(b-a+1)(d-c+1)\delta^{-2}+\mathcal{N}_\delta(\mathrm{G}_f)+\mathcal{N}_\delta(\mathrm{G}_g) \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_\delta(\mathrm{G}_{f+g}) &\ge \delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(f+g,\mathcal{R}_{i,j}) \\ &\ge 2mn+\left|\delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(f,\mathcal{R}_{i,j})-\delta^{-1}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}\mathrm{OSC}(g,\mathcal{R}_{i,j})\right|-2mn \\ &\ge \left|\mathcal{N}_\delta(\mathrm{G}_f)-\mathcal{N}_\delta(\mathrm{G}_g)\right|-2(b-a+1)(d-c+1)\delta^{-2}. \end{aligned}$$

This concludes the proof of Theorem 2. □

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With the help of Theorem 2, we shall prove the following several conclusions. Theorem 3 and 4 give the upper bound of $\overline{\dim}_B(G_{f+g})$ and $\underline{\dim}_B(G_{f+g})$, respectively.

Theorem 3. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Then*

$$\overline{\dim}_B(G_{f+g}) \leq \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}.$$

Proof. Assume that $\overline{\dim}_B(G_f) = s_1$ and $\overline{\dim}_B(G_g) = s_2$. Given $\forall \varepsilon > 0$, by Definition 1 there must exist a certain number $\delta_0 \in (0, \min\{b-a, d-c, 1\})$ such that

$$\mathcal{N}_\delta(G_f) \leq \delta^{-s_1-\varepsilon},$$

$$\mathcal{N}_\delta(G_g) \leq \delta^{-s_2-\varepsilon}$$

for $\forall \delta \in (0, \delta_0]$. Then by Theorem 2, we get

$$\begin{aligned}
\mathcal{N}_\delta(G_{f+g}) &\leq \mathcal{N}_\delta(G_f) + \mathcal{N}_\delta(G_g) + \rho\delta^{-2} \\
&\leq \delta^{-s_1-\varepsilon} + \delta^{-s_2-\varepsilon} + \rho\delta^{-2} \\
&\leq \left(\delta^{\max\{s_1,s_2\}-s_1} + \delta^{\max\{s_1,s_2\}-s_2} + \rho\delta^{\max\{s_1,s_2\}-2+\varepsilon}\right)\delta^{-\max\{s_1,s_2\}-\varepsilon} \\
&\leq (\rho+2)\delta^{-\max\{s_1,s_2\}-\varepsilon}
\end{aligned}$$

for $\forall \delta \in (0, \delta_0]$. From Definition 1, we can conclude that

$$\begin{aligned}
\overline{\dim}_B(G_{f+g}) &= \overline{\lim_{\delta\to 0}} \frac{\log \mathcal{N}_\delta(G_{f+g})}{-\log\delta} \\
&\leq \overline{\lim_{\delta\to 0}} \frac{\log\left[(\rho+2)\delta^{-\max\{s_1,s_2\}-\varepsilon}\right]}{-\log\delta} \\
&= \overline{\lim_{\delta\to 0}} \frac{\log(\rho+2)}{-\log\delta} + \overline{\lim_{\delta\to 0}} \frac{\log\delta^{\max\{s_1,s_2\}+\varepsilon}}{\log\delta} \\
&= \max\{s_1, s_2\} + \varepsilon.
\end{aligned}$$

Since the above formula is true for $\forall \varepsilon > 0$, we have

$$\overline{\dim}_B(G_{f+g}) \leq \max\{s_1, s_2\} = \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\},$$

which completes the proof of Theorem 3. □

Theorem 4. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Then*

$$\underline{\dim}_B(G_{f+g}) \leq \max\left\{\overline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\}.$$

Proof. Assume that

$$\overline{\dim}_B(G_f) = \alpha_1 \quad \text{and} \quad \underline{\dim}_B(G_g) = \alpha_2.$$

From the definition of $\underline{\dim}_B(G_g)$, there exists a positive subsequence $\{\delta_{\lambda_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \delta_{\lambda_k} = 0$ and meanwhile

$$\lim_{k\to\infty} \frac{\log \mathcal{N}_{\delta_{\lambda_k}}(G_g)}{-\log\delta_{\lambda_k}} = \alpha_2.$$

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So given $\forall \varepsilon > 0$, there exists a $\kappa_1 \in \mathbb{N}^*$ such that

$$\mathcal{N}_{\delta_{\lambda_k}}(G_g) \leq \delta_{\lambda_k}^{-\alpha_2-\varepsilon} \tag{5}$$

when $k \geq \kappa_1$. Then by the definition of $\overline{\dim}_B(G_f)$, there exists a $\kappa_2 \in \mathbb{N}^*$ such that

$$\mathcal{N}_{\delta_{\lambda_k}}(G_f) \leq \delta_{\lambda_k}^{-\alpha_1-\varepsilon} \tag{6}$$

when $k \geq \kappa_2$. Combining Theorem 2, Equations (5) and (6), we can obtain

$$\begin{aligned}
\mathcal{N}_{\delta_{\lambda_k}}(G_{f+g}) &\leq \mathcal{N}_{\delta_{\lambda_k}}(G_f) + \mathcal{N}_{\delta_{\lambda_k}}(G_g) + \rho\delta_{\lambda_k}^{-2} \\
&\leq \delta_{\lambda_k}^{-\alpha_1-\varepsilon} + \delta_{\lambda_k}^{-\alpha_2-\varepsilon} + \rho\delta_{\lambda_k}^{-2} \\
&\leq \left(\delta_{\lambda_k}^{\max\{\alpha_1,\alpha_2\}-\alpha_1} + \delta_{\lambda_k}^{\max\{\alpha_1,\alpha_2\}-\alpha_2} + \rho\delta_{\lambda_k}^{\max\{\alpha_1,\alpha_2\}-2+\varepsilon}\right)\delta_{\lambda_k}^{-\max\{\alpha_1,\alpha_2\}-\varepsilon} \\
&\leq (\rho+2)\delta_{\lambda_k}^{-\max\{\alpha_1,\alpha_2\}-\varepsilon}
\end{aligned}$$

when $k \geq \max\{\kappa_1, \kappa_2\}$. Thus by Definition 1, we have

$$\begin{aligned}
\underline{\dim}_B(G_{f+g}) &= \underline{\lim}_{\delta\to 0} \frac{\log \mathcal{N}_\delta(G_{f+g})}{-\log\delta} \\
&\leq \underline{\lim}_{k\to\infty} \frac{\log\left[(\rho+2)\delta_{\lambda_k}^{-\max\{\alpha_1,\alpha_2\}-\varepsilon}\right]}{-\log\delta_{\lambda_k}} \\
&= \lim_{k\to\infty} \frac{\log(\rho+2)}{-\log\delta_{\lambda_k}} + \lim_{k\to\infty} \frac{\log\delta_{\lambda_k}^{\max\{\alpha_1,\alpha_2\}+\varepsilon}}{\log\delta_{\lambda_k}} \\
&= \max\{\alpha_1,\alpha_2\} + \varepsilon.
\end{aligned}$$

In the light of the arbitrariness of ε, we immediately get our required result. □

Under certain particular circumstances, the previous two formulae could take an equal sign, shown in the undermentioned two theorems.

Theorem 5. *Let $f, g : [a,b]\times[c,d] \to \mathbb{R}$ be continuous. If*

$$\overline{\dim}_B(G_f) \neq \overline{\dim}_B(G_g),$$

then

$$\overline{\dim}_B(G_{f+g}) = \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}.$$

Proof. Let $H = f + g$. Without loss of generality, we can assume that

$$\overline{\dim}_B(G_f) > \overline{\dim}_B(G_g). \tag{7}$$

Suppose that

$$\overline{\dim}_B(G_H) \neq \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\} = \overline{\dim}_B(G_f).$$

From Theorem 3, it follows that

$$\overline{\dim}_B(G_H) < \overline{\dim}_B(G_f). \tag{8}$$

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Then combining Equations (7) and (8), we have

$$\begin{aligned}\overline{\dim}_B(G_{H-g}) &= \overline{\dim}_B(G_f)\\ &> \max\left\{\overline{\dim}_B(G_H), \overline{\dim}_B(G_g)\right\}\\ &= \max\left\{\overline{\dim}_B(G_H), \overline{\dim}_B(G_{-g})\right\},\end{aligned}$$

which is a contradiction to Theorem 3. Therefore, we can conclude that

$$\overline{\dim}_B(G_{f+g}) = \overline{\dim}_B(G_H) = \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}.$$

This means the conclusion of Theorem 5 holds. □

Theorem 6. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. If*

$$\max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\} > \min\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\},$$

then

$$\underline{\dim}_B(G_{f+g}) = \max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\}.$$

Proof. Without loss of generality, we suppose that

$$\eta_1 = \underline{\dim}_B(G_g) > \overline{\dim}_B(G_f) = \eta_2.$$

At this time, we know that

$$\max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\} = \eta_1 > \eta_2 = \min\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}.$$

From Theorem 4, it follows that

$$\underline{\dim}_B(G_{f+g}) \le \max\left\{\overline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\} = \eta_1. \tag{9}$$

Next, we prove that $\underline{\dim}_B(G_{f+g}) \ge \eta_1$ as below. By the definition of $\underline{\dim}_B(G_g)$ and $\overline{\dim}_B(G_f)$, given $\forall \varepsilon \in \left(0, \frac{\eta_1-\eta_2}{2}\right)$, there exists a $\delta_1 \in (0, \min\{b-a, d-c, 1\})$ such that

$$\mathcal{N}_\delta(G_f) \le \delta^{-\eta_2-\varepsilon} < \delta^{-\eta_1+\varepsilon} \le \mathcal{N}_\delta(G_g)$$

for $\forall \delta \in (0, \delta_1]$. Note that $\eta_1 - \eta_2 - 2\varepsilon > 0$ and $\eta_1 - 2 - \varepsilon > 0$, thus there exists a $\delta_2 \in (0, \min\{b-a, d-c, 1\})$ such that

$$\delta^{\eta_1-\eta_2-2\varepsilon} \le \frac{1}{3} \quad \text{and} \quad \delta^{\eta_1-2-\varepsilon} \le \frac{1}{3\rho}$$

for $\forall \delta \in (0, \delta_2]$. Then by Theorem 2, we estimate

$$\begin{aligned}\mathcal{N}_\delta(G_{f+g}) &\ge \left|\mathcal{N}_\delta(G_f) - \mathcal{N}_\delta(G_g)\right| - \rho\delta^{-2}\\ &\ge \delta^{-\eta_1+\varepsilon} - \delta^{-\eta_2-\varepsilon} - \rho\delta^{-2}\\ &\ge \left(1 - \delta^{\eta_1-\eta_2-2\varepsilon} - \rho\delta^{\eta_1-2-\varepsilon}\right)\delta^{-\eta_1+\varepsilon}\\ &\ge \frac{1}{3}\delta^{-\eta_1+\varepsilon}\end{aligned}$$

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for $\forall\delta \in (0, \min\{\delta_1, \delta_2\}]$. Consequently, one can get

$$\underline{\dim}_B(G_{f+g}) = \underline{\lim}_{\delta\to 0} \frac{\log \mathcal{N}_\delta(G_{f+g})}{-\log\delta} \geq \underline{\lim}_{\delta\to 0} \frac{\log \frac{1}{3}\delta^{-\eta_1+\varepsilon}}{-\log\delta} = \eta_1 - \varepsilon$$

by Definition 1. Since ε in the above formula is arbitrary, we have $\underline{\dim}_B(G_{f+g}) \geq \eta_1$. Combining Equation (9), we just obtain the required result. □

Now we shall deal with some concrete examples on the fractal dimensions of the graph of the sum of two bivariate continuous functions. To this end, we first need to state the definition of function spaces as follows.

Definition 2. *Spaces of bivariate continuous functions.*

(1) *Let $\mathcal{S}^d$ be the space of all bivariate continuous functions whose Box dimension exists and is equal to d on $[a,b]\times[c,d]$ as $2 \leq d \leq 3$. Namely, $\mathcal{S}^d$ is the space of d-dimensional bivariate continuous functions on $[a,b]\times[c,d]$.*
(2) *Let $\mathcal{S}_{d_1}^{d_2}$ as the space of all bivariate continuous functions whose Box dimension does not exist on $[a,b]\times[c,d]$. Here d_1, d_2 are the lower and upper Box dimension of the function on $[a,b]\times[c,d]$ as $2 \leq d_1 < d_2 \leq 3$, respectively.*

Below we start by the case when the two bivariate continuous functions have the different Box dimension.

Proposition 2. *Let $f(x,y) \in \mathcal{S}^{d_1}$ and $g(x,y) \in \mathcal{S}^{d_2}$. If $d_1 \neq d_2$, then*

$$f(x,y) + g(x,y) \in \mathcal{S}^{\max\{d_1,d_2\}}.$$

Proof. Without loss of generality, suppose that $d_1 > d_2$. At this time, we observe that

$$\underline{\dim}_B(G_g) = \overline{\dim}_B(G_g) < \underline{\dim}_B(G_f) = \overline{\dim}_B(G_f).$$

Then it follows from Theorem 5 and 6 that

$$\overline{\dim}_B(G_{f+g}) = \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\} = \max\{d_1, d_2\}$$

and

$$\underline{\dim}_B(G_{f+g}) = \max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\} = \max\{d_1, d_2\}.$$

That is,

$$\dim_B(G_{f+g}) = \max\{d_1, d_2\},$$

completing the proof of Proposition 2. □

The upcoming two corollaries discuss a few situations when at least one of two bivariate continuous functions does not have the Box dimension on $[a,b]\times[c,d]$. These results can easily be obtained from Theorem 5 and 6 with their proofs omitted.

Corollary 1. *Let $f(x,y) \in \mathcal{S}_{d_1}^{d_2}$ and $g(x,y) \in \mathcal{S}^d$.*

(1) *If $d_1 < d_2 < d$,*

$$f(x,y) + g(x,y) \in \mathcal{S}^d.$$

(2) *If $d < d_1 < d_2$,*

$$f(x,y) + g(x,y) \in \mathcal{S}_{d_1}^{d_2}.$$

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Corollary 2. *Let* $f(x,y) \in \mathcal{S}_{d_1}^{d_2}$, $g(x,y) \in \mathcal{S}_{d_3}^{d_4}$.

(1) If $d_1 < d_2 < d_3 < d_4$,
$$f(x,y) + g(x,y) \in \mathcal{S}_{d_3}^{d_4}.$$

(2) If $d_3 < d_4 < d_1 < d_2$,
$$f(x,y) + g(x,y) \in \mathcal{S}_{d_1}^{d_2}.$$

If the two bivariate continuous functions have the same Box dimension d, the result will become more complicate. Here we discuss two situations according to whether d equals to two or not. If $d \neq 2$, we can arrive at the following two conclusions.

Proposition 3. *Let* $f(x,y), g(x,y) \in \mathcal{S}^d$ *for* $2 < d \leq 3$. *If the Box dimension of* G_{f+g} *exists, then*

$$f(x,y) + g(x,y) \in \bigoplus_{t \in [2,d]} \mathcal{S}^t.$$

Proof. Firstly, let
$$f(x,y) = -g(x,y) + \mathcal{W}(x,y)$$
where $\mathcal{W}(x,y)$ is the function given in Example 1 and $\dim_B(G_{\mathcal{W}}) = 3 - \alpha$ could be any number belonging to $(2,d)$ by choosing suitable α. Then from Proposition 1 and 2, it follows that

$$\begin{aligned}
\dim_B(G_f) &= \dim_B(G_{-g+\mathcal{W}}) \\
&= \max\{\dim_B(G_g), \dim_B(G_{\mathcal{W}})\} \\
&= \max\{d, 3-\alpha\} \\
&= d.
\end{aligned}$$

Secondly, let
$$f(x,y) = -g(x,y) + H(x,y)$$
where $H(x) \in \mathcal{S}^2$. At this time,

$$\dim_B(G_f) = \max\{\dim_B(G_g), \dim_B(G_H)\} = \max\{d, 2\} = d.$$

Thirdly, let $f(x,y) = g(x,y)$. Then we know from Proposition 1 that

$$\dim_B(G_{f+g}) = \dim_B(G_{2f}) = \dim_B(G_f) = d.$$

According to the above discussion, we just finish the proof of the present proposition. □

Proposition 4. *Let* $f(x,y), g(x,y) \in \mathcal{S}^d$ *for* $2 < d \leq 3$. *If the Box dimension of* G_{f+g} *does not exist, then*

$$f(x,y) + g(x,y) \in \bigoplus_{\substack{t_1, t_2 \in [2,d) \\ t_1 < t_2}} \mathcal{S}_{t_1}^{t_2}.$$

Proof. Let
$$f(x,y) = -g(x,y) + \mathcal{B}(x,y)$$
where $\mathcal{B}(x,y)$ is the function given in Example 2 and $\underline{\dim}_B(G_{\mathcal{B}}), \overline{\dim}_B(G_{\mathcal{B}})$ could be any numbers satisfying
$$2 \leq \underline{\dim}_B(G_{\mathcal{B}}) < \overline{\dim}_B(G_{\mathcal{B}}) < d \leq 3. \tag{10}$$

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From Theorem 5 and 6, we can get

$$\overline{\dim}_B(G_f) = \overline{\dim}_B(G_{-g+\mathcal{B}}) = \max\left\{\overline{\dim}_B(G_g), \overline{\dim}_B(G_{\mathcal{B}})\right\} = d$$

and

$$\underline{\dim}_B(G_f) = \underline{\dim}_B(G_{-g+\mathcal{B}}) = \underline{\dim}_B(G_g) = d,$$

which implies that

$$\dim_B(G_f) = d.$$

Then by Equation (10), we just obtain our required result. □

If $d = 2$, the next result manifests that the sum of two two-dimensional bivariate continuous functions can keep the fractal dimension closed.

Theorem 7. *Let $f(x,y), g(x,y) \in \mathcal{S}^2$. Then $f(x,y) + g(x,y) \in \mathcal{S}^2$.*

Proof. From Theorem 3, it follows that

$$\overline{\dim}_B(G_{f+g}) \leq \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\} = 2.$$

Combining (1) of Proposition 1, we obtain

$$2 \leq \underline{\dim}_B(G_{f+g}) \leq \overline{\dim}_B(G_{f+g}) \leq 2.$$

Thus

$$\dim_B(G_{f+g}) = 2,$$

namely, $f(x,y) + g(x,y) \in \mathcal{S}^2$. □

In particular, if one of the two bivariate continuous functions is Lipschitz, we have the following assertion.

Theorem 8. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. If g is Lipschitz on $[a,b] \times [c,d]$, then*

$$\dim(G_{f+g}) = \dim(G_f)$$

where dim *denotes any one of* $\dim_H$, $\dim_P$, $\dim_A$, $\dim_B$, $\underline{\dim}_B$ *and* $\overline{\dim}_B$.

Proof. Let us define a map $\Upsilon : G_f \to G_{f+g}$ by

$$\Upsilon((x,y), f(x,y)) = ((x,y), f(x,y) + g(x,y)), \quad (x,y) \in [a,b] \times [c,d].$$

Since g is Lipschitz on $[a,b] \times [c,d]$, let

$$L = \sup_{(x,y),(u,t) \in [a,b]\times[c,d]} \frac{|g(x,y) - g(u,t)|}{\|(x,y) - (u,t)\|_2} < +\infty.$$

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For $\forall(x,y),(u,t) \in [a,b] \times [c,d]$, on one hand,

$$\begin{aligned}
&\|\Upsilon((x,y),f(x,y)) - \Upsilon((u,t),f(u,t))\|_2^2 \\
&= \|((x,y),f(x,y)+g(x,y)) - ((u,t),f(u,t)+g(u,t))\|_2^2 \\
&= \|(x,y)-(u,t)\|_2^2 + |(f(x,y)-f(u,t)) + (g(x,y)-g(u,t))|^2 \\
&\leq \|(x,y)-(u,t)\|_2^2 + 2|f(x,y)-f(u,t)|^2 + 2|g(x,y)-g(u,t)|^2 \\
&\leq \|(x,y)-(u,t)\|_2^2 + 2|f(x,y)-f(u,t)|^2 + 2L^2\,\|(x,y)-(u,t)\|_2^2 \\
&= \left(1+2L^2\right)\|(x,y)-(u,t)\|_2^2 + 2|f(x,y)-f(u,t)|^2 \\
&\leq \left(3+2L^2\right)\|((x,y),f(x,y)) - ((u,t),f(u,t))\|_2^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|((x,y),f(x,y)) - ((u,t),f(u,t))\|_2^2 \\
&= \|(x,y)-(u,t)\|_2^2 + |f(x,y)-f(u,t)|^2 \\
&= \|(x,y)-(u,t)\|_2^2 + |(f(x,y)-f(u,t)) + (g(x,y)-g(u,t)) - (g(x,y)-g(u,t))|^2 \\
&\leq \|(x,y)-(u,t)\|_2^2 + 2\,|(f(x,y)-f(u,t)) + (g(x,y)-g(u,t))|^2 + 2|g(x,y)-g(u,t)|^2 \\
&\leq \|(x,y)-(u,t)\|_2^2 + 2\,|(f(x,y)-f(u,t)) + (g(x,y)-g(u,t))|^2 + 2L^2\,\|(x,y)-(u,t)\|_2^2 \\
&= \left(1+2L^2\right)\|(x,y)-(u,t)\|_2^2 + 2\,|(f(x,y)-f(u,t)) + (g(x,y)-g(u,t))|^2 \\
&\leq \left(3+2L^2\right)\|\Upsilon((x,y),f(x,y)) - \Upsilon((u,t),f(u,t))\|_2^2.
\end{aligned}$$

Then by the above two inequalities, we can obtain

$$\begin{aligned}
C_1\,\|((x,y),f(x,y)) - ((u,t),f(u,t))\|_2 &\leq \|\Upsilon((x,y),f(x,y)) - \Upsilon((u,t),f(u,t))\|_2 \\
&\leq C_2\,\|((x,y),f(x,y)) - ((u,t),f(u,t))\|_2
\end{aligned}$$

where $C_1 = \dfrac{1}{\sqrt{3+2L^2}}$ and $C_2 = \sqrt{3+2L^2}$ satisfying $0 < C_1 < C_2 < +\infty$. This means that Υ is a bi-Lipschitz map. With Lemma 1, we just get our required result. □

4. Examples

In this section, we give a concrete example to verify the result acquired in Section 3.

Example 3. *For $0 < \alpha < 1$, let*

$$\mathcal{W}^*(x,y) = \sum_{j=1}^{\infty} 2^{-\alpha j}\sin(2^j x), \quad (x,y) \in [0,1]\times[0,1]$$

and

$$\mathcal{B}^*(x,y) = \sum_{j=1}^{\infty} (2j)^{-\frac{9}{10}\times 2^j}\cos\left((2j)^{2^j}x\right), \quad (x,y) \in [0,1]\times[0,1].$$

By [42], we have

$$\dim_B(G_{\mathcal{W}^*}) = 3-\alpha, \quad \underline{\dim}_B(G_{\mathcal{B}^*}) = \frac{39}{19} \quad \text{and} \quad \overline{\dim}_B(G_{\mathcal{B}^*}) = \frac{21}{10}.$$

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If $0 < \alpha < \frac{9}{10}$, *it follows from Corollary 1 that*

$$\dim_B(G_{\mathcal{W}^*+\mathcal{B}^*}) = \dim_B(G_{\mathcal{W}^*}) = 3 - \alpha.$$

Now we show several graphs and numerical results for Example 3. Figure 1 indicates the graph of $\mathcal{W}^*$ when $\alpha = 0.5$. Figure 2 denotes the graph of $\mathcal{B}^*$. Figure 3 represents the graph of $\mathcal{W}^* + \mathcal{B}^*$. Let α be 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, respectively. Table 1 presents the corresponding numerical results for the Box dimension of the graph of by using computing methods stated in [43]. In addition, Figure 4 supports our theoretical results gained in Section 3 where the minor deviation may be rendered by the approximation of the computer process.

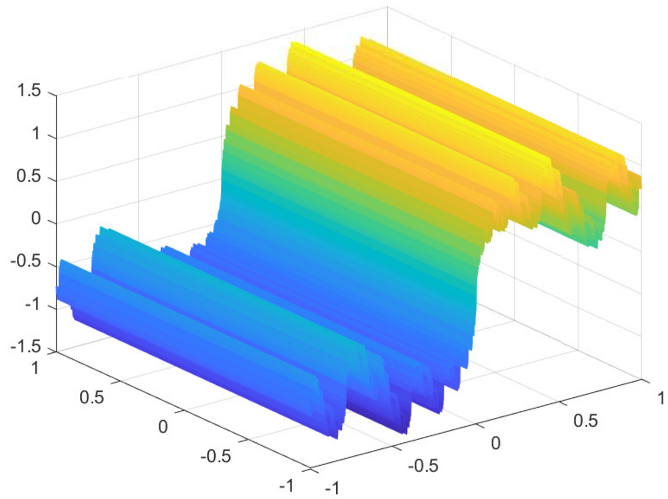

Figure 1. The graph of $\mathcal{W}^*$.

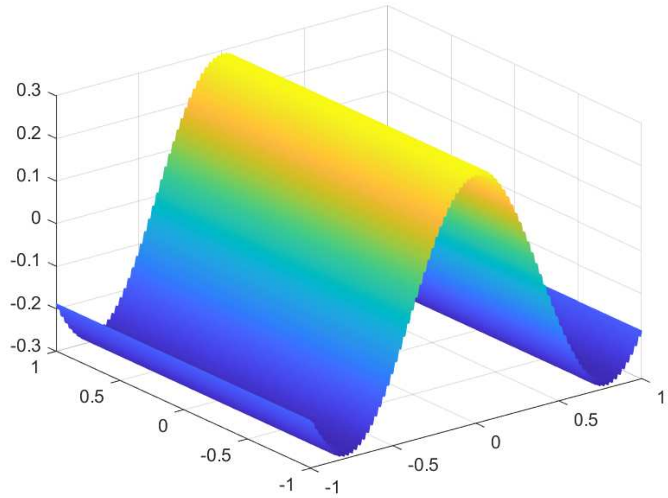

Figure 2. The graph of $\mathcal{B}^*$.

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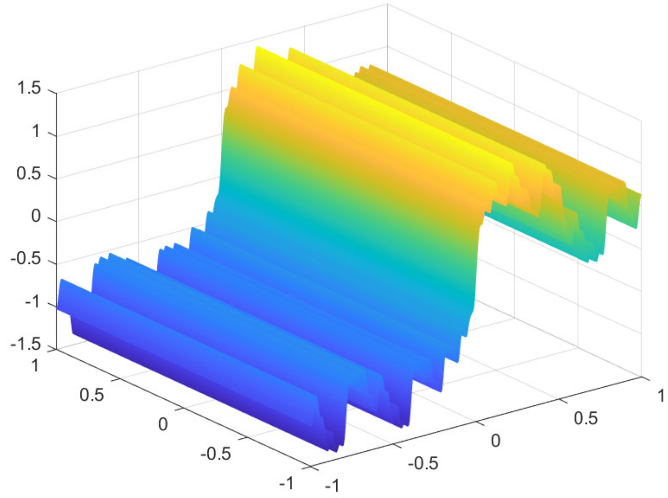

Figure 3. The graph of $\mathcal{B}^* + \mathcal{W}^*$.

Table 1. Connection between α and $\dim_B(G_{\mathcal{W}^*+\mathcal{B}^*})$.

α	$\dim_B(G_{\mathcal{W}^*+\mathcal{B}^*})$
0.1	2.8736
0.2	2.7801
0.3	2.6792
0.4	2.5853
0.5	2.4779
0.6	2.3825
0.7	2.2814
0.8	2.1840

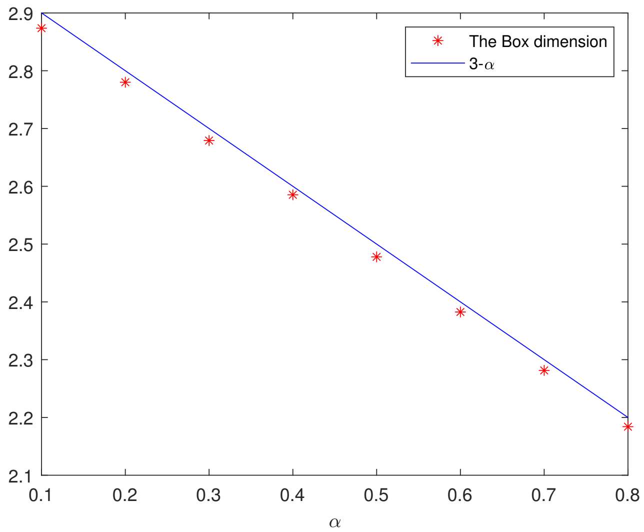

Figure 4. Comparison between numerical results and theoretical results.

5. Conclusions

In this last section, we sum up conclusions obtained in this paper.

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5.1. Conclusions and Remarks

Throughout the present paper, we mainly make research on the fractal dimensions of the graph of the superposition of two continuous surfaces f and g on $[a,b] \times [c,d]$ with certain lower and upper Box dimensions. Our main conclusions can be summarized as the following several aspects:

(1) $\overline{\dim}_B(G_{f+g}) \leq \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}$.
(2) $\underline{\dim}_B(G_{f+g}) \leq \max\left\{\overline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\}$.
(3) When

$$\overline{\dim}_B(G_f) \neq \overline{\dim}_B(G_g),$$

we prove that

$$\overline{\dim}_B(G_{f+g}) = \max\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\}.$$

(4) When

$$\max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\} > \min\left\{\overline{\dim}_B(G_f), \overline{\dim}_B(G_g)\right\},$$

we prove that

$$\underline{\dim}_B(G_{f+g}) = \max\left\{\underline{\dim}_B(G_f), \underline{\dim}_B(G_g)\right\}.$$

(5) It has been proved that the superposition of two continuous surfaces cannot keep the fractal dimensions invariable unless both of them are two-dimensional.
(6) It has been proved that the fractal dimensions of the graph of the sum of a bivariate continuous function and a bivariate Lipschitz function equals to the fractal dimensions of the graph of the former. That is, a bivariate Lipschitz function can be absorbed by any other bivariate continuous function in the sense of fractal dimensions.

Moreover, it is worth mentioning that the previous results can be extended to any closed regain $\mathcal{D} \subset \mathbb{R}^2$. In other words, all the results attained in the present paper still hold for two continuous surfaces f and g defined on $\mathcal{D}$.

5.2. Applications in other fields

In recent years, estimation of the fractal dimensions of the superposition of continuous surfaces has been widely applied in various fields such as metal materials and computer graphics.

In metal materials, the fracture surface topography with regards to the fatigue of metals can be studied by fractal characteristics, which can been found in [44,45]. Furthermore, fractal dimension is closely relevant to the parameters of areal surface of metals, which has been shown in [2]. As is known to all, there exist a good deal of approaches to calculating fractal dimensions and the results under different resolutions and methods will be slightly distinguishing. This work principally investigates how to calculate fractal dimensions by counting boxes and how to estimate fractal dimensions of the superposition of two fractal surfaces, which can just be applied into the research on fracture surface topography regarding to the fatigue of metals.

Besides, in computer graphics, texture roughness is an important visual feature of computer images, which is of great significance to image analysis, recognition and interpretation. A lot of research work has been done on texture analysis and many methods for measuring and describing texture roughness have been proposed (see [46–49], for example). Fractal dimension is one of the mostly used tools to describe the texture roughness of image surfaces, namely, the complexity of image gray surfaces, which can be a representation of image stability. The higher the fractal dimension, the more complex the surface, and then the coarser the image. The results in this paper can also contribute to calculating the fractal dimensions of the surface of the superposition of two computer images.

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5.3. Further Research

In this paper, there still exist some points worthy of improvement and further consideration in the future. Here we present them and put forward several open questions in the following:

(1) This work only deals with the cases when the two bivariate continuous functions have the different upper Box dimension and the lower Box dimension of one function is larger than the upper Box dimension of the other one. People could further explore the other situations later.

Question 1. *Suppose that* $f(x,y) \in \mathcal{S}_{d_1}^{d_2}$, $g(x,y) \in \mathcal{S}_{d_3}^{d_4}$. *What is* $\overline{\dim}_B(G_{f+g})$ *when* $d_2 = d_4$ *and what is* $\underline{\dim}_B(G_{f+g})$ *when* $d_2 \geq d_3$?

(2) In the present paper, we only focus on the Box dimension of the graph of the sum of two bivariate continuous functions. So other kinds of fractal dimensions, such as the Packing dimension, the Hausdorff dimension and the Assouad dimension could be further considered for this problem.

Question 2. *Let* $f(x,y), g(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ *be continuous. What can* $\dim_P(G_{f+g})$, $\dim_H(G_{f+g})$ *and* $\dim_A(G_{f+g})$ *be, respectively?*

(3) This study is only about bivariate continuous functions, which could be generalized to continuous functions of n variables in the future.

Question 3. *Let* $f(\boldsymbol{x}), g(\boldsymbol{x}) : \prod_{i=1}^{n}[a_i, b_i] \to \mathbb{R}$ *be continuous. What can the fractal dimensions of* G_{f+g} *be?*

(4) Apart from addition, people could further investigate the fractal dimensions of the graph of bivariate continuous functions under other operations.

Question 4. *Let* $f(x,y), g(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ *be continuous. What can the fractal dimensions of* $G_{f \cdot g}$ *be?*

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References

1. Mandelbrot, B.B. *The fractal geometry of nature*; Freeman: Sanfrancisco, USA, 1982.
2. Mandelbrot, B.B.; Passoja, D.E.; Paullay, A.J. Fractal character of fracture surfaces of metals. *Nature* **1984**, *308*, 721–722.
3. Turcotte, D.L. Fractals in geology and geophysics, *Pure and Applied Geophysics* **1989**, *131*, 171–196.
4. Kube, P.; Pentland, A. On the imaging of fractal surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **1988**, *10*, 704–707.
5. Massopust, P.R. *Fractal functions, fractal surfaces, and wavelets, 2nd edn*; Academic Press: San Diego, USA, 2016.
6. Pardo-Igúzquiza, E.; Dowd, P.A. Fractal analysis of karst landscapes. *Mathematical Geosciences* **2020**, *52*, 543–563.
7. Massopust, P.R. Fractal surfaces. *Journal of Mathematical Analysis and Applications* **1990**, *151*, 275–290.

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8. Malysz, R. The Minkowski dimension of the bivariate fractal interpolation surfaces. *Chaos, Solitons and Fractals* **2006**, *27*, 1147–1156.
9. Ruan, H.J.; Xu, Q. Fractal interpolation surfaces on rectangular grids. *Bulletin of the Australian Mathematical Society* **2015**, *91*, 435–446.
10. Feng, Z.; Feng, Y.; Yuan, Z. Fractal interpolation surfaces with function vertical scaling factors. *Applied Mathematics Letters* **2012**, *25*, 1896–1900.
11. Barnsley, M.F. Fractal functions and interpolation. *Constructive Approximation* **1986**, *2*, 303–329.
12. Ruan, H.J.; Su, W.Y.; Yao, K. Box dimension and fractional integral of linear fractal interpolation functions. *Journal of Approximation Theory* **2009**, *161*, 187–197.
13. Verma, M.; Priyadarshi, A.; Verma, S. Analytical and dimensional properties of fractal interpolation functions on the Sierpiński gasket. *Fractional Calculus and Applied Analysis* **2023**, *26*, 1294–1325.
14. Yu, B.Y.; Liang, Y.S. Construction of monotonous approximation by fractal interpolation functions and fractal dimensions. *Fractals* **2023**, *31*, accepted.
15. Cui, X.X.; Xiao, W. What is the effect of the Weyl fractional integral on the Hölder continuous functions? *Fractals* **2021**, *29*, 2150026.
16. Wu, J.R. The effects of the Riemann-Liouville fractional integral on the Box dimension of fractal graphs of Hölder continuous functions. *Fractals* **2020**, *28*, 2050052.
17. Bedford, T.J. The box dimension of self-affine graphs and repellers. *Nonlinearity* **1989**, *2*, 53–71.
18. Liang, Y.S.; Su, W.Y. Von Koch curve and its fractional calculus. *Acta Mathematica Sinica, Chinese Series* **2011**, *54*, 227–240.
19. Berry, M.V.; Lewis, Z.V. On the Weierstrass-Mandelbrot fractal function. *Proceedings of the Royal Society of London A* **1980**, *370*, 459–484.
20. Hunt, B.R. The Hausdorff dimension of graphs of Weierstrass functions. *Proceedings of the American Mathematical Society* **1998**, *126*, 791–800.
21. Sun, D.C.; Wen, Z.Y. The Hausdorff dimension of graphs of a class of Weierstrass functions. *Progress in Nature Science* **1996**, *6*, 547–553.
22. Barański, K. On the dimension of graphs of Weierstrass-type functions with rapidly growing frequencies. *Nonlinearity* **2012**, *25*, 193–209.
23. Shen, W.X. Hausdorff dimension of the graphs of the classical Weierstrass functions. *Mathematische Zeitschrift* **2018**, *289*, 223–266.
24. He, G.L.; Zhou, S.P. What is the exact condition for fractional integrals and derivatives of Besicovitch functions to have exact box dimension? *Chaos, Solitons and Fractals* **2005**, *26*, 867–879.
25. Wang, B.; Ji, W.L.; Zhang, L.G.; Li, X. The relationship between fractal dimensions of Besicovitch function and the order of Hadamard fractional integral. *Fractals* **2020**, *28*, 2050128.
26. Liang, Y.S.; Su, W.Y. The relationship between the Box dimension of the Besicovitch functions and the orders of their fractional calculus. *Applied Mathematics and Computation* **2008**, *200*, 297–307.
27. Wang, X.F.; Zhao, C.X.; Yuan, X. A review of fractal functions and applications. *Fractals* **2022**, *30*, 2250113.
28. Chandra, S.; Abbas, S. Box dimension of mixed Katugampola fractional integral of two-dimensional continuous functions. *Fractional Calculus and Applied Analysis* **2022**, *25*, 1022–1036.
29. Verma, M.; Priyadarshi, A. Dimensions of new fractal functions and associated measures. *Numerical Algorithms* **2023**, 2301521.
30. Liang, Y.S. Progress on estimation of fractal dimensions of fractional calculus of continuous functions. *Fractals* **2019**, *27*, 1950084.
31. Verma, M.; Priyadarshi, A.; Verma, S. Vector-valued fractal functions: Fractal dimension and fractional calculus. *Indagationes Mathematicae* **2023**, 2303005.
32. Verma, S.; Massopust, P.R. Dimension preserving approximation. *Aequationes mathematicae* **2022**, *96*, 1233–1247.
33. Falconer, K.J. *Fractal geometry: Mathematical foundations and applications*; John Wiley Sons Inc.: New York, USA, 1990.
34. Yu, B.Y.; Liang, Y.S. On the lower and upper Box dimensions of the sum of two fractal functions. *Fractal and Fractional* **2022**, *6*, 398.
35. Verma, M.; Priyadarshi, A. Graphs of continuous functions and fractal dimensions. *Chaos, Solitons and Fractals* **2023**, *173*, 2311351.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 October 2023 doi:10.20944/preprints202310.0802.v1

36. Yu, B.Y.; Liang, Y.S. Estimation of the fractal dimensions of the linear combination of continuous functions. *Mathematics* **2022**, *10*, 2154.
37. Wen, Z.Y. *Mathematical foundations of fractal geometry*; Science Technology Education Publication House: Shanghai, China, 2000.
38. Wang, X.F.; Zhao, C.X. Fractal dimensions of linear combination of continuous functions with the same Box dimension. *Fractals* **2020**, *28*, 2050139.
39. Yu, B.Y.; Liang, Y.S. Fractal dimension variation of continuous functions under certain operations. *Fractals* **2023**, *31*, 2350044.
40. Verma, S.; Viswanathan, P. Bivariate functions of bounded variation: Fractal dimension and fractional integral. *Indagationes Mathematicae* **2020**, *31*, 294–309.
41. Falconer, K.J. *Techniques in fractal geometry*; John Wiley Sons Inc.: New York, USA, 1997.
42. Yu, B.Y.; Liang, Y.S. On two special classes of fractal surfaces with certain Hausdorff and Box dimensions. *Applied Mathematics and Computation* **2023**, submitted.
43. Wu, J.; Jin, X.; Mi, S.; Tang, J. An effective method to compute the box-counting dimension based on the mathematical definition and intervals. *Results in Engineering* **2020**, *6*, 100106.
44. Hussain Hashmi, M.; Saeid Rahimian Koloor, S.; Foad Abdul-Hamid, M.; Nasir Tamin, M. Exploiting fractal features to determine fatigue crack growth rates of metallic materials. *Engineering Fracture Mechanics* **2022**, *308*, 108589.
45. Macek, W. Correlation between fractal dimension and areal surface parameters for fracture analysis after bending-torsion fatigue. *Metals* **2021**, *11*, 1790.
46. Chen, S.S.; Keller, J.M.; Crownover, R.M. On the calculation of fractal features from images. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **1993**, *15*, 1087–1090.
47. Chan, K.L. Quantitative characterization of electron micrograph image using fractal feature. *IEEE Transactions on Biomedical Engineering* **1995**, *42*, 1033–1037.
48. Martino, G.D.; Riccio, D.; Zinno, I. SAR imaging of fractal surfaces. *IEEE Transactions on Geoscience and Remote Sensing* **2012**, *50*, 630–644.
49. Riccio, D.; Ruello, G. Synthesis of fractal surfaces for remote-sensing applications. *IEEE Transactions on Geoscience and Remote Sensing* **2015**, *53*, 3803–3814.

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