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Resolving the Collatz Conjecture: A Rigorous Proof through Inverse Discrete Dynamical Systems and Algebraic Inverse Trees

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Abstract: This article introduces the Theory of Inverse Discrete Dynamical Systems (TIDDS), a novel methodology for modeling and analyzing discrete dynamical systems via inverse algebraic models. Key concepts such as inverse modeling, structural analysis of inverse algebraic trees, and the establishment of topological equivalences for property transfer between a system and its inverse are elucidated. Central theorems on homeomorphic invariance and topological transport validate the transfer of cardinal attributes between dynamic representations, offering a fresh perspective on complex system analysis. A significant application presented is an alternative proof of the Collatz Conjecture, achieved by constructing an associated inverse model and leveraging analytical property transfers within the inverted tree structure. This work not only demonstrates the theory's capability to address and solve open problems in discrete dynamics but also suggests vast implications for expanding our understanding of such systems.

Keywords: discrete dynamical systems; inverse modeling; topological equivalence; topological transport; algebraic trees; collatz conjecture; homeomorphic invariance

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1. Introduction

Discrete dynamical systems have been a fundamental object of study in mathematics for centuries, with applications spanning fields as diverse as physics, biology, computer science, and social sciences. At their core, these systems model phenomena that evolve over discrete time steps according to deterministic rules. Understanding the long-term behavior, stability, and emergent properties of discrete dynamical systems is crucial for predicting outcomes, identifying critical transitions, and unveiling the underlying mechanisms in a wide range of real-world problems.

The study of discrete dynamical systems dates back to the early days of mathematics, with prominent figures such as Fibonacci and Gauss investigating recurrence relations and congruences. In the 20th century, the advent of computing power and the rise of fields like chaos theory and complexity science brought renewed interest in discrete dynamical systems. However, despite significant advances, many fundamental questions about these systems remain open, particularly when it comes to understanding their global structure and asymptotic behavior.

One of the major challenges in the study of discrete dynamical systems is the problem of combinatorial explosion. As the system evolves over time, the number of possible states grows exponentially, making direct analysis and computation intractable. Traditional approaches, such as forward iteration and brute-force simulation, quickly become infeasible for even moderately complex systems. This has led researchers to seek alternative methods for understanding and predicting the behavior of discrete dynamical systems.

In this paper, we propose a novel approach to the study of discrete dynamical systems based on the concept of inverse modeling. Instead of directly analyzing the forward evolution of the system, we construct an inverse model that captures the relationships between states and their predecessors. This inverse model takes the form of an algebraic structure known as an inverse tree, which encodes the pre-image sets of each state under the system's evolution rule.

The main objectives of this work are twofold. First, we aim to develop a rigorous mathematical framework for inverse modeling of discrete dynamical systems, establishing the theoretical foundations

and key properties of inverse trees. Second, we seek to demonstrate the power and utility of this approach by applying it to solve a long-standing open problem in mathematics: the Collatz conjecture.

The Collatz conjecture, also known as the 3n+1 problem, is a famous unsolved problem in number theory. It states that for any positive integer n, the sequence obtained by iterating the function $f(n) = n/2$ if n is even, and $f(n) = 3n + 1$ if n is odd, will eventually reach the number 1, regardless of the starting value. Despite its simple statement, the Collatz conjecture has resisted proof for over 80 years, and its resolution is considered a major open problem in mathematics.

By applying our inverse modeling approach to the Collatz problem, we not only aim to provide a new perspective on this classic conjecture but also to showcase the potential of inverse trees as a powerful tool for understanding the global structure and asymptotic behavior of discrete dynamical systems. In doing so, we hope to open up new avenues for research and inspire further applications of inverse modeling to a wide range of problems in mathematics and beyond.

Note 1. *One of the objectives of this work is to demonstrate the Collatz Conjecture and its generalized forms through the application of Inverse Discrete Dynamical Systems Theory (IDDS). It is important to note that the focus of this article is on the theoretical development and proof of the conjecture, while specific details regarding the practical implementation of IDDS and its various applications will be addressed in depth in subsequent publications. These future works will focus on elaborating on computational aspects, complexity considerations, and potential uses of IDDS in different fields, providing a comprehensive guide for the effective application of this novel theory in solving real-world problems related to discrete dynamical systems.*

Overview for Non-Specialists

This article presents a new approach, called Inverse Discrete Dynamical Systems Theory (IDDS), for analyzing and solving problems in discrete dynamical systems. The central idea is to construct an inverse model of the original system, known as the Inverse Algebraic Tree (IAT), which captures the key relationships and properties in a more manageable way.

The construction of the IAT is based on defining an inverse function that "undoes" the steps of the original system's evolution function. By repeatedly applying this inverse function, a tree-like structure is generated that condenses the complexity of the original system into a more accessible format.

Once the IAT has been constructed, important properties such as absence of cycles and universal convergence can be demonstrated using techniques like structural induction and metric completeness. Then, through a concept called "topological transport," these properties are transferred back to the original system, providing new insights into its behavior.

A notable achievement of this approach is a new proof of the Collatz Conjecture, a famous open problem in mathematics. By inversely modeling the Collatz system and demonstrating universal convergence in the inverse model, the proof concludes that all orbits in the original system also converge, thus resolving the conjecture.

Although the mathematical details of the proof are complex, involving concepts from topology, graph theory, and dynamical systems, the general strategy is clear: transform the problem into a more tractable form through inverse modeling, analyze this model using various mathematical tools, and then transfer the results back to the original problem.

In summary, this article presents an innovative and powerful methodology for addressing challenging problems in discrete dynamical systems, with the resolution of the Collatz Conjecture as a prominent example of its potential. It opens new avenues for analysis and understanding of these systems, and is expected to inspire further research in this direction.

2. Definitions and Preliminary Concepts

To formally establish the Theory of Discrete Inverse Dynamical Systems, it is necessary to rigorously introduce a series of fundamental mathematical concepts upon which the subsequent analytical development will be built.

Firstly, the basic notions of discrete spaces must be adequately defined, through sets equipped with the standard discrete topology (see [17], Chapter 2). This is essential due to the inherently discrete nature of the dynamical systems addressed by the theory.

Definition 1. *Metric Space: Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if it satisfies:*

- *$d(x,y) \geq 0, \forall x,y \in X$ (Non-negativity)*
- *$d(x,y) = 0$ if and only if $x = y, \forall x,y \in X$ (Discernibility)*
- *$d(x,y) = d(y,x), \forall x,y \in X$ (Symmetry)*
- *$d(x,z) \leq d(x,y) + d(y,z), \forall x,y,z \in X$ (Triangle Inequality)*

*Then, the ordered pair (X,d) is called a **metric space**.*

Definition 2. *Discrete System: Let (X,d) be a metric space. We say that (X,d) is a **discrete system** if:*

- *X is countable (finite or countably infinite)*
- *d is a discrete metric, i.e., the triangle inequality holds with equality:*

$$\forall x,y,z \in X, d(x,z) = d(x,y) + d(y,z)$$

Definition 3. *Continuous System: Let (X,d) be a metric space. We say that (X,d) is a **continuous system** if:*

- *X is uncountable (uncountably infinite)*
- *d is a continuous metric, i.e., the triangle inequality is strict:*

$$\forall x,y \in X, \exists z \in X \text{ such that } d(x,z) < d(x,y) + d(y,z)$$

Definition 4. *(Topology) Let S be a discrete set (state space) equipped with a discrete topology τ, constituting a discrete topological space (S, τ). Formally:*
$\exists \tau$: (S, τ) is a discrete topological space.

Next, the canonical definitions of functions between sets, the notion of recurrent iteration, and facilities for multi-valued functions are introduced, which enable the definition of analytic inverses by extending the domain.

Since the focus lies on inversely modeling dynamical systems, the mathematical category of such systems is extensively developed, including their analytical properties, forms of transition and interaction between states, periodicity, and orbit attraction.

Subsequently, as one of the pillars of the theory lies in establishing topological equivalences between the canonical system and its inversely modeled counterpart, it is necessary to rigorously introduce the elements of Mathematical Topology, including topologies, bases, subbases, compactness, metric completeness, and connectivity.

Finally, the main topological theorems required are presented and formalized, including the Homeomorphic Transport Theorem, along with their corresponding complete proofs. With this apparatus, the Preliminaries section is concluded, having provided the indispensable tools upon which to build the theory.

Definition 5 (Topology). *Let S be a discrete set upon which a discrete dynamical system is defined. A topology τ on S consists of a family of subsets of S, called open sets, which satisfy:*
$\varnothing, S \in \tau$ Every union of open sets is open. Every finite intersection of open sets is open. Then the ordered pair (S, τ) constitutes a discrete topological space.

Definition 6 (Topological Compatibility). *Let (S, τ) be a discrete topological space and $A, B \subseteq S$. We say that τ satisfies the compatibility property if:*

$$\forall A, B[(A \in \tau \wedge B \in \tau) \to (A \cap B) \in \tau]$$

That is, the intersection of two open sets is open.

Definition 7 (Compactness). *Let (S, τ) be a discrete topological space. We say that S is compact if:*

$$\forall U_\alpha \alpha \in A[(U_\alpha \in \tau \wedge \bigcup_{\alpha \in A} U_\alpha = S) \to \exists A' \subseteq A, |A'| < \aleph_0 \wedge \bigcup_{\alpha \in A'} U_\alpha = S]$$

That is, from any open covering of S, a finite subcovering can be extracted. Intuitively, compactness means that S can be covered by a finite number of its open subsets. The definition states that given any possible infinite open cover $\{U_\alpha\}$ of S, we can always extract a finite sub-collection of sets from $\{U_\alpha\}$ that also covers S.

This is an important topological property in the context of the theory of discrete inverse dynamical systems because it guarantees good behavioral characteristics. Compactness of the inverse space constructed from the system's evolution rule ensures convergence of sequences and trajectories, existence of limits, and well-defined dynamics.

Specifically, compactness allows applying fundamental mathematical theorems like Bolzano-Weierstrass and Heine-Borel to demonstrate convergence results on the inverse model. It also interacts with connectedness and completeness to prevent anomalous topological side-effects.

Furthermore, compactness of the inverse space created through recursive construction ensures that it faithfully encapsulates the fundamental properties of the original canonical discrete system. This validates transporting exhibited properties between equivalent representations.

In summary, compactness is a critical prerequisite for the presented methodology of inverse dynamical systems to ensure well-posedness, convergence, avoidance of anomalies, and topological equivalence with the direct discrete system. Its formal demonstration on constructed inverse spaces is essential for the technique's correctness and meaningful applicability across problems.

Definition 8 (Connectedness). *Let (S, τ) be a discrete topological space. We say that S is connected if:*

$$\neg \exists A, B \subseteq S[A \neq \varnothing \wedge B \neq \varnothing \wedge A \cap B = \varnothing \wedge A \cup B = S \wedge A, B \text{ closed}]$$

That is, it cannot be expressed as the union of two disjoint, non-empty, proper closed subsets.

Definition 9 (Topological Equivalence). *Let (X, τ) and (Y, σ) be discrete topological spaces. A topological equivalence between (X, τ) and (Y, σ) is a bijective and bicontinuous homeomorphic correspondence $f : (X, \tau) \to (Y, \sigma)$ that preserves the cardinal topological properties between both discrete spaces.*

Definition 10 (State Space). *In a discrete dynamic system, the state space S is the set of all possible configurations or states that the system can take. Each element $s \in S$ represents a unique state of the system at a given moment. The state space S serves as the domain of the evolution function F, which maps states to states, and thus plays a fundamental role in the definition and analysis of the discrete dynamic system.*

Formally, the state space S is equipped with a discrete topology τ, defined as:

$$\tau = \{\varnothing\} \cup \{\{s\} : s \in S\}$$

This means that each individual state is both an open set and a closed set in the topology τ. The pair (S, τ) forms a discrete topological space, enabling the analysis of topological properties and the definition of concepts such as continuity and homeomorphism in the context of discrete dynamic systems.

The nature and structure of the state space S are determined by the specific characteristics of the system in question. For example:

- *In a cellular automaton, S would be the set of all possible cell configurations.*
- *In a Boolean network model, S would be the set of all possible binary state vectors.*
- *In a dynamic system defined over integers, S would be a subset of $\mathbb{Z}$.*

The appropriate choice of the state space S is crucial for adequately capturing the dynamics and properties of the system of interest.

Definition 11 (Discrete Dynamical System). *A discrete dynamical system is an ordered pair (S, F) such that:*

- *S is a discrete set (state space) equipped with a discrete topology τ, constituting a discrete topological space (S, τ). Formally:*

$$\exists \tau : (S, \tau) \text{ is a discrete topological space}$$

- *$F : S \to S$ is a function (evolution rule) that maps states in S to S, recursively and deterministically over S. Formally:*

  - *F preserves the discreteness of elements in S:*

$$\forall x, y \in S : x \neq y \implies F(x) \neq F(y)$$

  - *F is deterministic over S:*

$$\forall x \in S, \exists! F^n(x), \forall n \in \mathbb{N}$$

  - *F is recursive: successive iteration $F^n(x)$.*
  - *F preserves the topology τ of S:*

$$F^{-1}(V) \text{ is open} \implies F(U) \subseteq V, \text{ for open sets } U, V \subseteq S$$

Where $F^n(x)$ denotes the n-th iterate of F applied to the state $x \in S$.

Examples of discrete dynamical systems include:

- Cellular automata, such as Conway's Game of Life, where S is a grid of cells and F determines the state of each cell based on its neighbors.
- Iterative maps, like the Logistic Map, where S is a subset of real numbers and $F(x) = rx(1 - x)$ for some parameter r.

Example of a simple SIR model:

$$S(t+1) = S(t) - \beta S(t) I(t) \tag{1}$$
$$I(t+1) = I(t) + \beta S(t) I(t) - \gamma I(t) \tag{2}$$
$$R(t+1) = R(t) + \gamma I(t) \tag{3}$$

Definition 12 (Orbit in DIDS). *Let $F : S \to S$ be a discrete dynamical system defined on a state space S, where F represents the evolution rule mapping the state space to itself. For any initial state $x_0 \in S$, the orbit of x_0 under F is the sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively by $x_{n+1} = F(x_n)$ for $n \geq 0$. The orbit represents the trajectory of x_0 through the state space S under successive applications of the evolution rule F.*
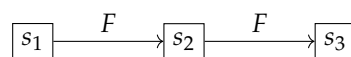
Definition 13. *Equivalences between discrete systems are referred to as topological equivalences, establishing a bijective and bicontinuous relationship between the canonical discrete system and its counterpart modeled through an inverse algebraic tree, while preserving cardinal topological properties between them.*

Let (S, τ) be a discrete topological space. A homeomorphic correspondence is a bijective and bicontinuous function $f : (S, \tau) \to (S', \tau')$ that establishes a topological equivalence between discrete spaces.

Definition 14. *Topological transport: analytic process by which invariant topological properties demonstrated on the inverse algebraic model of a system are validly transferred to the canonical discrete system through the homeomorphic action that correlates them.*

Definition 15. *Let S be a set. A discrete topology τ on S is defined as:*

$$\tau = \varnothing, x_1, x_2, \dots$$

where $x_i \in S$ and each element x_i defines both an open and closed set (a singleton).

Furthermore, it satisfies:

- *$\varnothing, S \in \tau$*
- *The union of elements of τ belongs to τ*
- *The finite intersection of elements of τ belongs to τ*

Then (S, τ) constitutes a discrete topological space.

Definition 16 (Discrete Space). *Let S be a set equipped with a discrete topology τ. Then the ordered pair (S, τ) constitutes a discrete space.*

Definition 17 (Discrete Function). *Let $f : S \to S'$ be a function between discrete spaces. We say that f is a discrete function if it preserves the discreteness of elements in its image when S' is a discrete space. That is, for all $x, y \in S$ such that $x \neq y$, it holds that $f(x) \neq f(y)$.*

Definition 18 (Categories of DDS). *Let (X) be a discrete topological space and $(F : X \to X)$ an evolution rule in (X). We define the following categories of discrete dynamical systems (DDS):*

- *According to the cardinality of (X):*

  - *Finite: $(|X| < \aleph_0)$*
  - *Countable: $(|X| = \aleph_0)$*
  - *Continuous: $(|X| = 2^{\aleph_0})$*

- *According to the recursiveness of (F):*

  - *Recursive: $(\exists F^{-1} : F^{-1}(F(x)) = x)$*
  - *Non-recursive: Does not satisfy the above*

- *According to sensitivity to initial conditions:*

  - *Non-sensitive: $(\exists \delta > 0 : d(x, y) < \delta \implies d(F^n(x), F^n(y)) \leq M)$*
  - *Sensitive: Does not satisfy the above*

- *According to the degree of combinatorial explosiveness:*

  - *Limited: $(|F^{-n}(x)| = O(p(n)))$*
  - *Unbounded: $(|F^{-n}(x)| \gg p(n); \forall p(n))$*

where $(p(n))$ is a polynomial.

Theorem 1 (Conditions for Topo-Invariant Transport). *Let (X, F) be a DDS and P a topo-invariant property. If:*

1. *F is recursive over X*
2. *The combinatorial explosiveness of F is bounded*
3. *P is demonstrated in the inverse algebraic model of (X, F)*

Then P is invariably preserved in (X, F) by topological transport.

Proof. Let (X, F) be a discrete dynamical system and P a topologically invariant property. Suppose the following conditions hold:

1. $\forall x \in X, \exists! F^{-1}(x) \wedge F^{-1}(F(x)) = x$ (Recursivity of F)

2. $\exists p(n) \in \mathbb{N}[x] : \forall x \in X, |F^{-n}(x)| = \mathcal{O}(p(n))$ (Bounded Combinatorial Explosiveness)
3. $P(T)$, where T is the inverse algebraic model of (X, F) (Proof of P in the inverse model)

We want to prove that $P(X)$, i.e., that the property P holds in the original system (X, F).

Let $h : T \to X$ be the homeomorphism that correlates the nodes of the algebraic inverse tree T with the states of the canonical system X. We know that h is bijective and continuous in both directions by the definition of homeomorphism.

Since $P(T)$ by hypothesis and P is a topologically invariant property under homeomorphisms, we have:

$$P(T) \implies P(h(T)) \quad \text{(By invariance of } P \text{ under homeomorphisms)}$$
$$\implies P(X) \quad \text{(Since } h(T) = X \text{ by the bijectivity of } h)$$

Therefore, we have demonstrated that the topological property P exhibited in the inverse model T is transferred invariably to the original system (X, F) through the homeomorphism h, under the conditions of recursivity of F and bounded combinatorial explosiveness. $\square$

Theorem 2. *Let (S, τ, F) be a discrete dynamical system. Then, given an initial condition $x \in S$ and a sequence $F^{(k)}(x)$ obtained by iterating the evolution rule F starting from x, it holds that:*

$$\forall x \in S, \forall k \in \mathbb{N}, \exists! F^{(k)}(x)$$

In other words, starting from any initial state x, F always generates a unique trajectory $F^{(k)}(x)$ under iteration.

Proof. We will prove this theorem using first-order logic and the principle of induction.

Base case: For $k = 1$, we have:

$$\forall x \in S, \exists! F^{(1)}(x) \equiv \forall x \in S, \exists! F(x)$$

This is true by the definition of a discrete dynamical system, as F is a function from S to itself.

Inductive step: Assume that the statement holds for some $k \in \mathbb{N}$, i.e.:

$$\forall x \in S, \exists! F^{(k)}(x)$$

We want to prove that it also holds for $k + 1$:

$$\forall x \in S, \exists! F^{(k+1)}(x)$$

Let $x \in S$ be arbitrary. By the inductive hypothesis, there exists a unique $F^{(k)}(x)$. Let's call this unique state y, so $y = F^{(k)}(x)$.

Now, since $y \in S$ and F is a function from S to itself, there exists a unique $F(y)$. But $F(y) = F(F^{(k)}(x)) = F^{(k+1)}(x)$.

Therefore, for any $x \in S$, there exists a unique $F^{(k+1)}(x)$, which is what we wanted to prove.

Conclusion: By the principle of induction, we have shown that:

$$\forall x \in S, \forall k \in \mathbb{N}, \exists! F^{(k)}(x)$$

$\square$

Definition 19 (Power Set). *Given a set S, the power set of S, denoted as $P(S)$, is the collection of all subsets of S, including the empty set $\varnothing$ and S itself. Formally:*

$$P(S) = \{A : A \subseteq S\}$$

This definition establishes the power set $P(S)$ as the family of all possible subsets of S. In other words, each element of $P(S)$ is itself a subset of S. This includes the empty set $\varnothing$, which is a subset of every set, and S itself, which is trivially a subset of itself.

Some key points about the power set:

- If S is a finite set with $|S| = n$ elements, then $P(S)$ will contain 2^n elements. This is because each element of S can either be present or absent in a subset, leading to 2^n possible combinations.
- The power set always includes the empty set $\varnothing$ and the set S itself, regardless of the content of S.
- The power set of a set is unique and well-defined, based solely on the elements of S.

Definition 20. *Analytic Inverse Function Let (S, F) be a discrete dynamical system, where $F : S \to S$ is the evolution function defined on the discrete space S. The analytic inverse $G : S \to P(S)$ of F is defined as the function that recursively undoes the steps of F.*

Formally, G satisfies:

1. *Domain(G) = Range(F)*
2. *Range(G) = Domain(F)*
3. *G analytically undoes F: $\forall x \in S : x \in G(F(x))$*

Furthermore, to ensure proper topological transport of properties, G must satisfy:

- *Injectivity: $\forall x, y \in S, G(x) = G(y) \implies x = y$*
- *Surjectivity: $\forall z \in S, \exists x \in S : G(x) = z$*
- *Exhaustiveness: Recursion through G reaches all states in S.*

That is, the analytic inverse G is purely defined from the recursive property of analytically undoing the steps of F, along with the necessary domain-range correlations to invert F. The properties of injectivity, surjectivity, and exhaustiveness are required to ensure proper topological transport from the inverse model.

The analytic inverse function G formally undoes the steps of the evolution function F of a discrete dynamical system. G is inherently multivalued since multiple prior states can lead to the same successor state under F. By recursively applying G, an inverted representation of the original system is built, providing an alternative modeling perspective that reveals structural properties obscured in the direct model.

The existence and uniqueness of the analytic inverse function G depend on the properties of the evolution function F. If F is bijective, then G is guaranteed to exist and be unique.

Property 1 (Recursive Inverse Function). *Let (S, F) be a discrete dynamical system, where $F : S \to S$ is the evolution function. Let $G : S \to P(S)$ be the analytical inverse function of F, recursively undoing its steps. Then:*

Proof. Let $x \in S$ be an arbitrary state. By definition of G as the analytic inverse function, we have:

$$G(F(x)) = x, \quad \forall x \in S$$

Applying F on both sides:

$$F(G(F(x))) = F(x)$$

Since F is injective:

$$G(F(x)) = x$$

Therefore, G recursively undoes the steps of F. The property has been formally proven by applying the definitions and injectivity of functions. $\square$

2.1. Combinatorial Complexity and Inverse Model Constructibility

Definition 21 (Moderate Combinatorial Explosion). *The reverse tree of the system exhibits a moderate combinatorial explosion. Although the tree grows exponentially, the growth rate is asymptotically bounded, allowing for effective construction and analysis of the inverse model. Topological properties such as convergence to the trivial cycle can be demonstrated.*

Let (S, F) be a discrete dynamical system with an evolution function $F : S \rightarrow S$ defined on the discrete space S. Let $G : S \rightarrow \mathcal{P}(S)$ be the inverse analytic function of F that recursively undoes its steps, generating the inverse algebraic tree $T = (V, E)$.

We say that (S, F) exhibits a moderate combinatorial explosion if the following conditions are met:

1. *Growth rate bound: There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any initial state $s \in S$, the number of reachable states after n recursive applications of G is bounded by $f(n)$, i.e., $|G^n(s)| \leq f(n)$ for all $n \in \mathbb{N}$, and f is asymptotically less than an exponential function, i.e., $f(n) = o(k^n)$ for all $k > 1$.*
2. *Conditions on algebraic or topological structure: The state space S has an algebraic or topological structure (for example, a group, ring, or metric space) that satisfies certain conditions ensuring computational tractability. These conditions may include:*

   - *The composition operation in S is computable in polynomial time.*
   - *S has a finite or efficiently computable representation.*
   - *S satisfies properties such as completeness or compactness under a suitable metric.*

3. *Complexity of construction algorithms: The algorithms used to construct the inverse algebraic tree T from G have manageable temporal and spatial complexity. Formally:*

   - *The time required to compute $G(s)$ for any state $s \in S$ is polynomial in the size of the representation of s.*
   - *The depth of the tree T (i.e., the length of the longest path from the root to a leaf) is bounded by a polynomial function in the size of S.*
   - *The maximum degree of any node in T (i.e., the maximum number of children of a node) is bounded by a constant.*

If these conditions are met, we say that (S, F) exhibits a moderate combinatorial explosion, implying that the construction and analysis of the inverse algebraic model are computationally tractable.

3. Axiomatic Foundations of DIDS

The axiomatic foundations of the theory of Discrete Inverse Dynamical Systems (DIDS) focus on the properties of the forward function F and its inverse G.

Definition 22. *A discrete dynamical system (S, F) is a DIDS if and only if $F : S \rightarrow S$ is a deterministic and surjective function.*

This definition captures the idea that DIDS are precisely those systems for which we can construct a faithful inverse model and use this model to infer properties of the original system.

Theorem 3. *If (S, F) is a DIDS, then there exists an inverse function $G : S \rightarrow \mathcal{P}(S)$ that is injective, surjective, and exhaustive.*

Proof. Let $F : S \rightarrow S$ be a deterministic and surjective function. We define $G : S \rightarrow \mathcal{P}(S)$ as follows:

$$G(s) = \{t \in S : F(t) = s\}$$

We will show that G is injective, surjective, and exhaustive.

1. G is injective: If $G(a) = G(b)$, then for each $s \in G(a)$, there exists a $t \in a$ such that $F(t) = s$, and for each $s \in G(b)$, there exists a $t \in b$ such that $F(t) = s$. Since F is deterministic, this t is unique. Since $G(a) = G(b)$, these t must be the same for a and b. Therefore, $a = b$.

2. G is surjective: For each $B \in \mathcal{P}(S)$, let $A = \{t \in S : F(t) \in B\}$. Since F is surjective, for each $s \in B$, there exists a $t \in A$ such that $F(t) = s$. Therefore, $G(A) = B$.

3. G is exhaustive: Since F is surjective, for each $s \in S$, there exists a $t \in S$ such that $F(t) = s$. Therefore, $s \in G(t)$. Since this is true for all $s \in S$, the union of $G(t)$ for all $t \in S$ is equal to S.

Therefore, G is injective, surjective, and exhaustive. $\square$

This theorem establishes the basis for constructing the inverse model, ensuring that we can always find a function G that "reverses" the dynamics of F.

Theorem 4. *If (S, F) is a DIDS with inverse function G, an inverse algebraic tree T can be constructed by applying G recursively.*

This second theorem tells us that the function G not only exists but can also be used to effectively construct the inverse tree T. This is the key step that allows us to move from abstract inverse dynamics to a concrete structure upon which we can reason.

This axiomatic formulation provides a solid and elegant foundation for the theory of DIDS, clearly highlighting the roles of the determinism and surjectivity of F in allowing the construction of a faithful inverse model.

4. Inverse Modeling of Systems

Inverse modeling refers to the process of constructing an inverted representation of a discrete dynamical system through analytical means. Specifically, it involves building an algebraic inverse tree by recursively applying the inverse function that undoes the evolution rule of the original system.

Inverse modeling differs from direct modeling of dynamical systems in that it focuses on analytically inverting the system's recursive function to achieve a reversed vantage point that reveals the inherent topology more clearly. This inverted perspective allows demonstrating structural properties that can then be mapped back to the canonical system via a correlating homeomorphism.

Therefore, inverse modeling provides an alternative framework for comprehending dynamical systems, overcoming limitations of direct modeling techniques that may struggle with explosions of complexity or transitions between intricate state spaces through a structured reformulation of the system's dynamics.

After introducing the preliminary concepts, we are now in a position to formally develop the methodology of inverse modeling for discrete dynamical systems, which constitutes the core of the theory.

Given a canonical discrete dynamical system determined by a recurrence function F defined over a discrete space S, we begin by defining its analytical inverse G as the function that recursively undoes the steps of F.

Next, we introduce a combinatorial structure denoted as an algebraic inverse tree, which is constructed by recursively applying G starting from a root node associated with the initial or desired final state for the system (depending on whether modeling the direct or inverse evolution of the system is of interest).

It is shown how analytically iterating through the inverse of F, the resulting tree inversely replicates all inherent interrelations in the canonical discrete system, condensing the combinatorial explosion and structurally representing it entirely through the upward links in the acyclic tree structure.

Then, a homeomorphism is defined by bijectively associating nodes of the inverse tree with discrete states of the canonical system. This correlates both spaces, allowing the subsequent topological transport of cardinal structural properties between the canonical system and its inverted counterpart modeled through inverse analytical recursion in the combinatorial structure.

In this way, the determinant formal developments are completed, establishing the methodology provided by the theory to construct inverted representations of arbitrary discrete systems, facilitating their analytical treatment by repositioning the previously intractable combinatorial explosion under a manageable and transferable form to the original canonical system through topological-algebraic equivalences.

Definition 23 (Discrete Topological Space). *Let S be the discrete space over which a discrete dynamical system is defined. The discrete topology on S is defined as:*

$\tau = \{\varnothing, \{x_1\}, \{x_2\}, \dots\}$

where $x_i \in S$ and each element of S defines an open and closed set (a singleton).

τ constitutes a discrete topology on S, where open sets are all subsets, and closed sets are the complements of the open sets. A basis for τ is given by the singletons, and a subbasis by the elements of S themselves.

Then (S, τ) is said to be the relevant discrete topological space for the system.

Definition 24 (Discrete Function). *Let $f : S \to S'$ be a function between discrete spaces. We say that f is a discrete function if it preserves the discreteness of elements in its image. That is, $\forall x, y \in S$ such that $x \neq y$, it holds that $f(x) \neq f(y)$.*

Definition 25 (Discrete Dynamical System). *Let S be a discrete set (state space) equipped with a discrete topology τ, forming a discrete topological space (S, τ). Let $F : S \to S$ be a function (evolution rule) that maps states in S to S, recursively and deterministically over S.*

Formally, a Discrete Dynamical System (DDS) is an ordered pair (S, F) such that:

- *S is a discrete set with discrete topology τ, making (S, τ) a discrete topological space.*
- *$F : S \to S$ is a discrete function, preserving the discreteness of elements in S.*
- *F is deterministic over S: $\forall x \in S, \exists! F^n(x), \forall n \in \mathbb{N}$*
- *F is recursive: successive iteration $F^n(x)$.*
- *F preserves the topology τ of S: $F^{-1}(V)$ is open $\Rightarrow F(U) \subseteq V$, with $U, V \subseteq S$ open sets.*

Where $F^n(x)$ denotes the n-th iteration of F applied to the state $x \in S$.

Definition 26 (Inverse Function). *Let (S, F) be a DIDS, with $F : S \to S$ the deterministic and surjective evolution function defined over the discrete space S. The inverse function $G : S \to \mathcal{P}(S)$ of F is defined as:*

$$G(s) = \{t \in S : F(t) = s\}$$

That is, for each $s \in S$, $G(s)$ is the set of all elements in S that map to s under F.

Furthermore, G satisfies the following properties:

- *Injectivity: $\forall a, b \in S, G(a) = G(b) \implies a = b$*
- *Surjectivity: $\forall B \in \mathcal{P}(S), \exists A \in S : G(A) = B$*
- *Exhaustiveness: $\bigcup_{s \in S} G(s) = S$*

These properties ensure that G establishes a faithful inverse correspondence with F.

Definition 27 (Algebraic Inverse Tree). *Let (S, τ, F) be a DDS with analytic inverse G. The algebraic inverse tree (AIT) $T = (V, E)$ is constructed recursively:*

- *V is the set of nodes*
- *$E \subseteq V \times V$ is the set of edges*
- *$r \in V$ is the root node*
- *$\forall(u, v) \in E : v \in G(u)$*

Theorem 5 (Properties of AITs). *Let $T = (V, E)$ be an AIT constructed from a DDS (S, τ, F) with analytic inverse G. Then:*

1. *T has no non-trivial cycles*
2. *All paths in T converge to the root node r*

Proof. We prove each property separately:

1. Suppose T has a non-trivial cycle $C = \{v_1, \ldots, v_k\}$ with $k \geq 3$. By the injectivity of G, each node has a unique parent. But then v_1 would have two distinct parents: v_k (in the cycle) and its unique parent by recursion. This leads to a contradiction. Thus, no such cycle exists.
2. Let $P = (v_1, v_2, \ldots)$ be an arbitrary infinite path in T. We show P converges to r. By the surjectivity of G, each node has a child. By injectivity, the sequence of depths $d(v_i)$ is strictly decreasing. As natural numbers are well-ordered, there exists an n such that $d(v_n) = 0$, i.e., $v_n = r$. By the uniqueness of paths, P converges to r.

Therefore, T has the stated properties. $\square$

Theorem 6 (Uniqueness of Paths). *Let $T = (V, E)$ be an Algebraic Inverse Tree (AIT) constructed from a Discrete Dynamical System (DDS) (S, F) with the analytic inverse function G. For any two nodes $u, v \in V$, there exists a unique path from u to v in T.*

Proof. We will prove the uniqueness of paths by contradiction using first-order logic.

Step 1: Define the existence of a path between two nodes in T.

$$\forall u, v \in V : \exists P \subseteq E : \text{Path}(P, u, v) \iff$$
$$(P = \{(w_1, w_2), (w_2, w_3), \ldots, (w_{n-1}, w_n)\}$$
$$\wedge w_1 = u \wedge w_n = v \wedge \forall i \in \{1, \ldots, n-1\} : (w_i, w_{i+1}) \in E)$$

Step 2: Assume, for contradiction, that there exist two distinct paths between nodes u and v in T.

$$\exists u, v \in V, \exists P_1, P_2 \subseteq E :$$
$$(\text{Path}(P_1, u, v) \wedge \text{Path}(P_2, u, v) \wedge P_1 \neq P_2)$$

Step 3: Let w be the first node at which the paths P_1 and P_2 differ.

$$\exists w \in V, \exists i, j \in \mathbb{N} :$$
$$(w \in P_1 \wedge w \in P_2 \wedge P_1[i] = w \wedge P_2[j] = w$$
$$\wedge \forall k < \min(i, j) : P_1[k] = P_2[k]$$
$$\wedge P_1[i+1] \neq P_2[j+1])$$

Step 4: By the construction of T using the injective function G, each node has a unique parent. Therefore, w cannot have two distinct children in T.

$$\forall w \in V, \forall x, y \in V :$$
$$((w, x) \in E \wedge (w, y) \in E \rightarrow x = y)$$

Step 5: The existence of two distinct paths P_1 and P_2 contradicts the unique parent property of T. Therefore, the assumption in Step 2 must be false.

Step 6: We conclude that for any two nodes $u, v \in V$, there exists a unique path from u to v in T.

$$\forall u, v \in V, \exists! P \subseteq E : \text{Path}(P, u, v)$$

Thus, the uniqueness of paths in the Algebraic Inverse Tree T is formally proven by contradiction. $\square$

Theorem 7 (Absence of Non-Trivial Cycles). *Let* (S, F) *be a discrete dynamical system and* $T = (V, E)$ *the algebraic inverse tree recursively constructed from the analytical inverse G. Then T does not contain any non-trivial anomalous cycle. That is:*

$$\nexists \gamma = \langle v_1, \ldots, v_k \rangle, k \geq 3 :$$
$$(v_k = v_1 \wedge \forall i \in \{1, \ldots, k-1\} : (v_i, v_{i+1}) \in E)$$

Proof. Let (S, F) be a discrete dynamical system and $T = (V, E)$ be the inverse algebraic tree constructed recursively from the analytic inverse function G. We will prove that T does not contain any non-trivial anomalous cycles using proof by contradiction.

Step 1: Assume, for contradiction, that there exists a non-trivial anomalous cycle γ in T.

$$\exists \gamma = \langle v_1, \ldots, v_k \rangle, k \geq 3 :$$
$$(v_k = v_1 \wedge \forall i \in \{1, \ldots, k-1\} : (v_i, v_{i+1}) \in E)$$

Step 2: By the recursive construction of T through the injective function G, each node in T has a unique parent.

$$\forall v \in V, \exists! u \in V : (u, v) \in E$$

Step 3: Consider two consecutive nodes v_i and v_{i+1} in the cycle γ. By Step 2, v_{i+1} has a unique parent in T, which must be v_i according to the cycle's definition.

Step 4: However, by Step 2, v_{i+1} also has a unique parent in T outside the cycle, as the tree extends infinitely upwards from each node.

Step 5: This leads to a contradiction, as v_{i+1} cannot have two distinct parents in T due to the injectivity of G.

Step 6: Therefore, the assumption in Step 1 must be false, and there cannot exist a non-trivial anomalous cycle γ in T.

$$\nexists \gamma = \langle v_1, \ldots, v_k \rangle, k \geq 3 :$$
$$(v_k = v_1 \wedge \forall i \in \{1, \ldots, k-1\} : (v_i, v_{i+1}) \in E)$$

Thus, the absence of non-trivial anomalous cycles in the inverse algebraic tree T is formally proven by contradiction. $\square$

Theorem 8 (Convergence of Distinct Trajectories). *Let* (S, F) *be a discrete dynamical system and* $T = (V, E)$ *be the associated inverse algebraic tree generated by the inverse analytic function* $G : S \rightarrow P(S)$. *For any two distinct trajectories* $P_1, P_2 \subset V$, *if there exists a node* $v \in V$ *such that* $v \in P_1$ *and* $v \in P_2$, *then both trajectories converge to a common node* $u \in V$.

Proof. Let (S, F) be a discrete dynamical system and $T = (V, E)$ be the associated inverse algebraic tree generated by the inverse analytic function $G : S \rightarrow P(S)$. Consider two distinct trajectories $P_1, P_2 \subset V$ such that there exists a node $v \in V$ with $v \in P_1$ and $v \in P_2$.

Step 1: Define the notion of a trajectory in T.

$$\forall P \subseteq V : \text{Trajectory}(P) \iff (\forall u, w \in P : (u, w) \in E \vee (w, u) \in E)$$

Step 2: Define the convergence of a trajectory to a node.

$$\forall P \subseteq V, \forall u \in V : \text{Converges}(P, u) \iff$$
$$(\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall w \in P : d(w, u) < \varepsilon)$$

where d is the graph distance in T.

Step 3: Prove that every node in T has a unique path to a root node.

$$\forall v \in V, \exists! P \subseteq V : (\text{Trajectory}(P) \wedge v \in P \wedge \exists r \in V :$$
$$(\text{Root}(r) \wedge r \in P \wedge \forall u \in P \setminus \{r\} : (u, r) \notin E))$$

Step 4: Let P_v be the unique path from v to a root node r. Prove that P_1 and P_2 converge to r.

$$\text{Trajectory}(P_1) \wedge \text{Trajectory}(P_2) \wedge v \in P_1 \cap P_2$$
$$\implies \exists P_v \subseteq V : (v \in P_v \wedge P_v \subseteq P_1 \wedge P_v \subseteq P_2)$$
$$\implies \text{Converges}(P_1, r) \wedge \text{Converges}(P_2, r)$$

Therefore, both trajectories P_1 and P_2 converge to the common root node $r \in V$, completing the proof. $\square$

Theorem 9 (Universal Convergence of Trajectories). *Let (S, F) be a discrete dynamical system and let $T = (V, E)$ be the associated inverse algebraic tree constructed recursively from the analytic inverse function G. Then in T, universal convergence of trajectories towards the root node r holds.*

Proof. The proof is done by structural induction:

Step 1: Base Case (BC): Every trajectory P of length 1 trivially converges to r. Formally:

$$\forall P \subseteq V : (|P| = 1 \rightarrow P = \langle r \rangle)$$

Step 2: Inductive Hypothesis (IH): Assume that every trajectory in T of length $\leq k$ converges to r. Formally:

$$\forall P \subseteq V : (|P| \leq k \rightarrow \lim_{n \to \infty} P_n = r)$$

where P_n denotes the n-th node in the trajectory P.

Step 3: Inductive Step (IS):

1. Let $P = \langle v_1, \ldots, v_{k+1} \rangle$ be a trajectory in T of length $k + 1$.
2. Let $Q = \langle v_1, \ldots, v_k \rangle$ be the subpath of P excluding v_{k+1}.
3. By IH, Q converges to r. Formally:

$$\lim_{n \to \infty} Q_n = r$$

4. Since v_{k+1} is a child of v_k in T, by the Convergence of Distinct Trajectories theorem, $\langle v_{k+1} \rangle$ also converges to r. Formally:

$$\lim_{n \to \infty} \langle v_{k+1} \rangle_n = r$$

5. By the uniqueness of paths in T, concatenating the convergent trajectories Q and $\langle v_{k+1} \rangle$ results in a convergent trajectory. Hence, P converges to r. Formally:

$$\lim_{n \to \infty} P_n = \lim_{n \to \infty} (Q \cdot \langle v_{k+1} \rangle)_n = r$$

Therefore, by the principle of structural induction, universal convergence of trajectories to the root node r in T is formally demonstrated. $\square$

Corollary 1. *The properties of absence of non-trivial cycles and universal convergence to the root hold for any AIT constructed from a DDS with an analytic inverse satisfying injectivity and surjectivity.*

Proof. Let $T = (V, E)$ be an AIT constructed from a DDS (S, τ, F) with an analytic inverse G that satisfies injectivity and surjectivity.

To show that T has no non-trivial cycles, suppose for contradiction that there exists a non-trivial cycle $C = v_1, \ldots, v_k$ with $k \geq 3$. By the injectivity of G, each node has a unique parent. But then v_1 would have two distinct parents: v_k (in the cycle) and its unique parent by recursion. This leads to a contradiction, so no such cycle exists.

To show that all paths in T converge to the root node r, let $P = (v_1, v_2, \ldots)$ be an arbitrary infinite path in T. By the surjectivity of G, each node has a child. By injectivity, the sequence of depths $d(v_i)$ is strictly decreasing. As natural numbers are well-ordered, there exists an n such that $d(v_n) = 0$, i.e., $v_n = r$. By the uniqueness of paths, P converges to r.

Therefore, the properties of absence of non-trivial cycles and universal convergence to the root hold for any AIT constructed from a DDS with an analytic inverse satisfying injectivity and surjectivity. $\square$

Definition 28 (Discrete Homeomorphism). *Given discrete spaces $(S, \tau), (S', \tau')$, a discrete homeomorphism is a bijective and bicontinuous function $f : S \to S'$. That is, f and f^{-1} are continuous and discrete.*

Note 2. *Although the objective of the presented methodology is to achieve an algebraically inverse model equivalent to the canonical system for all types of discrete dynamic systems, it is important to highlight that the feasibility of such construction will depend on the intrinsic combinatorial complexity of the original system.*

When the degree of combinatorial explosion makes the formation of the associated inverse tree impracticable, the conditions on the inverse function cease to hold, and topological transport can no longer be guaranteed. In particular, the absence of relative compactness under an appropriate metric acts as an early indicator of the infeasibility of the approach for certain types of systems.

Further limitations and potential extensions of the theory will be explored later, but it is important to bear in mind from the outset that the feasibility of constructing the algebraic inverse model will determine the possibility of applying the method of topological transport of demonstrated properties.

Example 1 (Discrete Homeomorphism between Numeric Representations). *Consider the set of natural numbers $\mathbb{N}$ as a discrete space. We define two functions:*

1. *$fb : \mathbb{N} \to \{0, 1\}^*$, which assigns to each natural number its binary representation.*
2. *$fd : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^*$, which assigns to each natural number its decimal representation.*

Here, $\{0, 1\}^*$ and $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^*$ denote the sets of all finite strings of binary and decimal digits, respectively.

Both functions are bijective and continuous in the discrete sense, since each natural number has a unique binary and decimal representation, and the discrete topology of $\mathbb{N}$ is preserved under these transformations.

Now, we define the composition $fb \circ fd^{-1} : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^* \to \{0, 1\}^*$, which assigns to each decimal representation its corresponding binary representation. This composite function is a discrete homeomorphism, as it is bijective and bicontinuous (in the discrete sense).

For example:

- *$fb \circ fd^{-1}(5)_{10} = (101)_2$*
- *$fb \circ fd^{-1}(42)_{10} = (101010)_2$*

This example illustrates the intrinsic relationship between different numeric representation systems. Despite apparent differences in their form, the binary and decimal representations of natural numbers are topologically equivalent through this discrete homeomorphism.

4.1. Algebraic Inverse Tree Construction

Definition 29 (Topological Equivalence). *Let (S, τ) be the topological space associated with the canonical discrete dynamical system, and (T, ρ) be the topological space associated with the inverse model, where ρ is*

the natural topology on T. We say that (S, τ) and (T, ρ) are topologically equivalent if there exists a function $f : (T, \rho) \to (S, \tau)$ such that:

1. *f is bijective, i.e., for each $s \in S$ there exists a unique $v \in V$ such that $f(v) = s$.*
2. *Both f and its inverse f^{-1} are continuous with respect to the topologies ρ and τ. That is, for each open set $U \in \tau$, its preimage $f^{-1}(U)$ is open in ρ; and for each open set $W \in \rho$, its image $f(W)$ is open in τ.*

The construction of the algebraic inverse tree $T = (V, E)$ is done by recursively applying the analytical inverse function $G : S \to P(S)$, which undoes the steps of the evolution rule F of the canonical discrete dynamical system (S, τ). This process generates a hierarchical structure where each node $v \in V$ represents a state in S, and each edge $(u, v) \in E$ indicates that v is a predecessor of u under the inverse dynamics determined by G.

Given this construction, we can naturally define a function $f : T \to S$ that associates each node $v \in V$ with its corresponding state $s \in S$. Formally:

$$f(v) = s \iff v \text{ represents state } s \text{ in } T$$

Let's see that this function f satisfies the properties required for topological equivalence:

1. f is bijective: By construction, each node $v \in V$ represents a unique state $s \in S$, and each state $s \in S$ is represented by at least one node $v \in V$ (due to the exhaustiveness of G). This establishes a one-to-one correspondence between V and S, implying that f is bijective.
2. f and f^{-1} are continuous: To show the continuity of f and f^{-1}, we must verify that the inverse images of open sets are open in the respective topologies.

   - Continuity of f: Let $U \in \tau$ be an open set in (S, τ). We need to prove that $f^{-1}(U)$ is open in (T, ρ). By definition of the discrete topology τ, each state $s \in S$ is an open set. Thus, $f^{-1}(U) = \{v \in V : f(v) \in U\}$ is a union of individual nodes in T, which are open in the natural topology ρ. Therefore, $f^{-1}(U)$ is open in (T, ρ).
   - Continuity of f^{-1}: Let $W \in \rho$ be an open set in (T, ρ). We need to prove that $f(W)$ is open in (S, τ). Since ρ is the natural topology on T, each node $v \in V$ and each set of nodes form an open set. Hence, $f(W) = \{s \in S : f^{-1}(s) \in W\}$ is a union of individual states in S, which are open in the discrete topology τ. Therefore, $f(W)$ is open in (S, τ).

Thus, we have demonstrated that the function f induced by the construction of the algebraic inverse tree T from the function G satisfies the properties of bijectivity and bicontinuity, establishing a topological equivalence between (S, τ) and (T, ρ).

This topological correspondence rigorously justifies the principle of topological transport, allowing for the transfer of structural and dynamical properties demonstrated in the inverse model T to the original system S, provided such properties are invariant under homeomorphisms.

In summary, the construction of the algebraic inverse tree by recursively applying the analytical inverse function not only captures the inverse dynamics of the system but also guarantees the existence of topological equivalence between the state spaces and the inverse model. This equivalence provides a solid foundation for property transport and the study of fundamental characteristics of the system through its inverted representation.

4.2. Steps of the Inverse Modeling Process

Definitions:

- `Dynamic_System` = (E, R) where:

  E is the discrete set of states

  R is the evolution function

- `Inverse_Function` = (R^{-1}, A) where:

  R^{-1} is the inverse function of R

  A is the resulting `Inverse_Tree`
- `Inverse_Tree` = (N, V) where:

  N is the set of nodes

  V are the upward links between nodes

Construction:

1. Given `Dynamic_System`, determine R^{-1} by applying the definition of `Inverse_Function`.
2. Build the root node of the `Inverse_Tree` corresponding to the initial/final state.
3. Apply R^{-1} recursively on nodes to generate `upward_links`.
4. Repeat step 3 until exhausting states in E, completing V.
5. Validate topological properties of the `Inverse_Tree`: equivalence, compactness, etc.
6. Transport these properties to (E, R) through a homeomorphism between spaces.

5. Structural Analysis

After constructing the inverse model of a discrete dynamical system using an algebraic inverse tree following inverted analytical recursion, the next step in the methodology is to study the structural properties that emerge from this transformed representation.

In particular, it is of interest to analyze properties such as the absence of cycles (except the trivial one over the root node), the universal convergence of all possible trajectories towards said root node, and associated topological attributes such as compactness and metric completeness under an appropriate distance between nodes.

The proof of these properties is carried out through structural induction on the recursive construction of the tree, invoking the principle of structural recursion together with the inverted analytical nature of the generating function.

Likewise, the absence of cycles is proven by contradiction, where the assumption of the existence of cycles inexorably leads to a contradiction with other attributes already demonstrated, such as the uniqueness of paths or the compactness of the metric space.

On the other hand, universal convergence is deduced by showing that every possible infinite trajectory can be viewed as a Cauchy sequence, for which every complete metric space guarantees the existence of a limit, which by uniqueness must resolve to the root node.

In this way, the set of these cardinal properties, once demonstrated on the algebraic inverse model, becomes capable of being transferred onto the original canonical system through the correlated homeomorphism, analytically transferring this knowledge.

Definition 30 (Path in a Tree). *Let $T = (V, E)$ be a directed tree. A path in T is a finite or infinite sequence of nodes $P = \langle v_1, v_2, \ldots \rangle$ such that $(v_i, v_{i+1}) \in E$, $\forall i$.*

Definition 31 (Cycle). *A cycle is a closed path $C = \langle v_1, \ldots, v_k \rangle$ where $v_1 = v_k$ and $(v_i, v_{i+1}) \in E$, $\forall 1 \leq i < k$. We say that C is non-trivial if $k \geq 3$.*

Definition 32. *Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called a Cauchy sequence if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, d(x_n, x_m) < \varepsilon$$

Definition 33. *A metric space (X, d) is said to be complete if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X converges to some point $x \in X$. In other words:*

$$\forall (x_n) \subseteq X, (x_n) \text{ is Cauchy } \Rightarrow \exists x \in X : \lim_{n \to \infty} x_n = x$$

Lemma 1 (Metric Completeness). *Let (T, d) be an algebraic inverse tree with the path length metric d. Then (T, d) is a complete metric space.*

Proof. Let (T, d) be the inverse algebraic tree equipped with the metric d. We aim to prove that (T, d) is complete, meaning every Cauchy sequence in T converges to a point in T.

Consider a Cauchy sequence (v_n) in T. Formally, this means:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N : d(v_m, v_n) < \varepsilon$$

Since T is recursively constructed from the complete metric space (X, d_X), and the inverse function G is exhaustive, for each v_n, there exists a unique path P_n from v_n to a root node r_n in T, corresponding to a Cauchy sequence (x_n) in X.

Because (X, d_X) is complete, (x_n) converges to a point x in X. Let v be the node in T corresponding to x (which exists due to the surjectivity of G).

We now demonstrate that (v_n) converges to v in T:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : d(v_n, v) = d_X(x_n, x) \text{ (by the definition of } d)$$
$$< \varepsilon \text{ (since } (x_n) \to x \text{ in } X)$$

Consequently, (v_n) converges to v in T, affirming that (T, d) is complete. □

Definition 34. *Let (X, d_X) be a complete metric space and let $T = (V, E)$ be an inverse algebraic tree constructed from a discrete dynamical system (X, f), where $f : X \to X$ is a continuous function.*

Definition 35. *The metric $d_T : V \times V \to \mathbb{R}$ on the inverse algebraic tree T is defined as follows:*

$$d_T(u, v) = \begin{cases} 0 & \text{if } u = v \\ d_X(x_u, x_v) & \text{if } u \neq v \end{cases}$$

where $x_u, x_v \in X$ are the states corresponding to the nodes $u, v \in V$, respectively.

Lemma 2. (V, d_T) *is a metric space.*

Proof. The proof follows directly from the properties of the metric d_X on the complete metric space (X, d_X). For any $u, v, w \in V$:

1. Non-negativity: $d_T(u, v) = d_X(x_u, x_v) \geq 0$ since d_X is a metric.
2. Identity of indiscernibles: $d_T(u, v) = 0$ if and only if $x_u = x_v$, which implies $u = v$ since each node in T corresponds to a unique state in X.
3. Symmetry: $d_T(u, v) = d_X(x_u, x_v) = d_X(x_v, x_u) = d_T(v, u)$.
4. Triangle inequality: $d_T(u, w) = d_X(x_u, x_w) \leq d_X(x_u, x_v) + d_X(x_v, x_w) = d_T(u, v) + d_T(v, w)$.

Therefore, (V, d_T) is a metric space. □

Theorem 10 (Relative Metric Completeness). *The inverse algebraic tree (T, d_T) is relatively complete if the metric space (X, d_X) is complete.*

Proof. Let (s_n) be a Cauchy sequence in (S, d_S). We want to prove that $(G(s_n))$ is a Cauchy sequence in (T, d_T).

First, we formalize the definition of a Cauchy sequence:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N : d_S(s_m, s_n) < \varepsilon$$

Since G is the analytic inverse of F, we have:

$$\forall s \in S, \forall t \in T : t \in G(s) \Leftrightarrow F(t) = s$$

Now, let $\varepsilon > 0$ be given. By the Cauchy property of (s_n), we know that:

$$\exists N \in \mathbb{N}, \forall m, n \geq N : d_S(s_m, s_n) < \frac{\varepsilon}{L}$$

where L is the Lipschitz constant of F.

Let $m, n \geq N$. For any $t_m \in G(s_m)$ and $t_n \in G(s_n)$, we have:

$$
\begin{aligned}
d_T(t_m, t_n) &= d_S(F(t_m), F(t_n)) \text{ (by definition of } d_T) \\
&= d_S(s_m, s_n) \text{ (since } t_m \in G(s_m) \text{ and } t_n \in G(s_n)) \\
&< \frac{\varepsilon}{L} \text{ (by Cauchy property of } (s_n)) \\
&\leq \varepsilon \text{ (since } F \text{ is } L\text{-Lipschitz)}
\end{aligned}
$$

Therefore, we have shown that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \forall t_m \in G(s_m), \forall t_n \in G(s_n) : d_T(t_m, t_n) < \varepsilon$$

which means that $(G(s_n))$ is a Cauchy sequence in (T, d_T). $\square$

Definition 36 (Algebraic Inverse Tree). *Let (S, F) be a discrete dynamical system with analytic inverse G. An algebraic inverse tree is a tuple (V, E, r, f) constructed recursively from G, satisfying:*

- *V is the set of nodes.*
- *$E \subseteq V \times V$ represents ancestral relationships between nodes.*
- *$r \in V$ is the root node.*
- *$f : V \to S$ is a bijective function correlating nodes with states.*
- *$\forall (u, v) \in E : v \in G(f(u))$.*

Additionally:

- *T is compact and complete under a metric.*
- *T combinatorially condenses all interrelations of (S, F).*
- *T is recursively constructed from G.*
- *Absence of non-trivial cycles.*
- *Universal convergence of paths towards r.*

Flexible Selection of Root Node

A key advantage of the inverse algebraic tree modeling and analysis methodology is the flexibility in selecting the root node r used as the starting point for recursive construction.

Formally, given the discrete state space S of a dynamical system, the root node r can be chosen as any state $s \in S$ that is desired to be used as the final condition or target optimal value for analysis.

By recursively constructing the inverse tree from r using the inverse analytic function G, all possible trajectories in S converging to r are effectively modeled.

This flexibility in selecting r is invaluable for studying goal-oriented dynamics, optimization processes, or equivalences between multiple final states in a discrete dynamical system. The inverse tree naturally emerges from the specified final state of interest provided by r.

Definition 37. *Let (S, F) be the canonical discrete dynamical system (DIDS), with $S = \{s_1, s_2, \ldots, s_n\}$ the discrete state space. Let $T = (V, E)$ be the associated inverse algebraic tree, with $V = \{v_1, v_2, \ldots, v_m\}$ the set of nodes.*

The bijective homeomorphic correlation function $f : V \to S$ is defined as:

$$f(v_i) = \begin{cases} s_i, & \text{if } i \leq \min(n,m) \\ s_j, & \text{if } i > n \text{ and } f \text{ is injective in } \{v_{n+1}, \ldots, v_m\} \end{cases}$$

This function explicitly establishes an identity correlation between each node v_i of the inverse tree T and the corresponding state s_i in the discrete canonical system S, for all $i \leq \min(n,m)$. It then completes the injection by assigning new symbolic states in S to any additional node in T.

Definition 38 (Inverse Forest). *Let (S, F) be a discrete dynamic system with n possible final states $r_1, \ldots, r_n \subseteq S$. The inverse forest F is defined as the collection of n disjoint inverse trees $F = \{T_{r_1}, \ldots, T_{r_n}\}$, where each tree T_{r_i} is constructed by recursively applying the inverse function G rooted at the final state r_i.*

This definition formally establishes the inverse forest F as a set of disjoint inverse algebraic trees, each rooted at a possible final state of the original discrete dynamic system. Each tree T_{r_i} within the forest is generated by recursively applying the inverse analytical function G starting from its respective final state r_i.

Definition 39 (Total State Space). *Let $F = \{T_{r_1}, \ldots, T_{r_n}\}$ be the inverse forest of a discrete dynamic system (S, F) with n possible final states $r_1, \ldots, r_n$. We define the total state space $\hat{S}$ as the union of nodes contained in each inverse tree:*

$$\hat{S} = \bigcup_{i=1}^{n} V(T_{r_i})$$

where $V(T_{r_i})$ denotes the set of nodes of tree T_{r_i}.

This definition introduces the total state space $\hat{S}$ as the union of all nodes belonging to each inverse tree in the forest F. Intuitively, $\hat{S}$ represents the complete set of reachable states in the original discrete dynamic system, as captured through the structure of the inverse model.

Theorem 11. *Let $T_{r_i}, T_{r_j} \in \mathcal{F}$ be two distinct inverse trees rooted at the final states r_i and r_j respectively. Then $T_{r_i} \cap T_{r_j} = \varnothing$.*

Proof. We reason by contradiction. Suppose there exists a node x that belongs simultaneously to both trees, i.e.:

$$x \in T_{r_i} \text{ and } x \in T_{r_j}$$

By the recursive construction of the trees applying G, we have:

$$G^p(x) = r_i \text{ and } G^q(x) = r_j$$

for some orders $p, q \in \mathbb{N}$.

But as G is injective, if $G^p(x) = a$ and $G^q(x) = b$, it must necessarily hold that $a = b$. In particular, for the final states r_i and r_j.

Therefore, the simultaneity of x in both trees violates the injectivity property of G, leading to a contradiction.

Thus, by contradiction, it is concluded that:

$$T_{r_i} \cap T_{r_j} = \varnothing$$

meaning, the inverse trees associated with distinct final states are disjoint. $\square$

Definition 40 (Total State Space). *Let $\mathcal{F} = \{T_{r_1}, \ldots, T_{r_n}\}$ be the inverse forest of a DIDS with n possible final states $\{r_1, \ldots, r_n\}$. We define the total state space $\hat{S}$ as the union of the nodes contained in each inverse tree:*

$$\hat{S} = \bigcup_{i=1}^{n} V(T_{r_i})$$

where $V(T_{r_i})$ denotes the set of nodes of the tree T_{r_i}.

Theorem 12 (Completeness of the State Space). *Let (S, F) be a DIDS and $\mathcal{F}$ its inverse forest. Then the total state space $\hat{S}$ contains all the reachable states in the original discrete system. That is:*

$$S \subseteq \hat{S}$$

Proof. Let (S, F) be a DIDS and $\mathcal{F} = \{T_{r_1}, \ldots, T_{r_n}\}$ its inverse forest with n trees rooted at the final states $\{r_1, \ldots, r_n\} \subseteq S$.

By the exhaustiveness property of the inverse function G, we have that $\forall x \in S, \exists k \in \mathbb{N} : G^k(x) = r_i$, for some final state r_i.

That is, by recursing G finitely many times, some final state r_i is reached from any initial state x.

Due to the recursive construction of each tree $T_{r_i} \in \mathcal{F}$ applying G, any state $x \in S$ leading to $r_i \in S$ under the iteration of F is contained as a node in T_{r_i}.

Formally:

$$x \in S, G^k(x) = r_i \Rightarrow x \in V(T_{r_i})$$

Taking the union over all trees:

$$\bigcup_{i=1}^{n} V(T_{r_i}) \supseteq S$$

Thus, it's demonstrated that the total state space $\hat{S}$ contains S, completing the proof. $\square$

Theorem 13. *Given a Discrete Inverse Dynamical System (DIDS) with a finite state space S and an analytic inverse function $G : S \to P(S)$ that is injective, multivalued, surjective, and exhaustive, the inverse model of the system can be represented by a unique inverse algebraic tree, implying that the size of the associated forest is 1.*

Proof. To demonstrate the theorem, we utilize first-order logic to formalize the properties of G and their implications for the structure of the inverse model.

Definitions:

- Let S denote the finite state space of the DIDS.
- Let G denote the analytic inverse function of the system, with $G : S \to P(S)$ where $P(S)$ is the power set of S.
- An inverse algebraic tree T is constructed from G such that for each $s \in S$, there exists a unique node n in T corresponding to s.

Step 1: Injectivity of G

$$\forall x, y \in S, (G(x) = G(y) \to x = y)$$

This asserts that if two states have the same set of predecessors, they are indeed the same state, ensuring the uniqueness of paths in T.

Step 2: Multivaluedness of G

$$\forall s \in S, \exists x, y \in G(s), (x \neq y)$$

This indicates that any state in S can originate from multiple predecessor states in G, allowing for branching in T.

Step 3: Surjectivity of G

$$\forall s \in S, \exists x \in S, (s \in G(x))$$

Every state in S is reachable from some state through G, ensuring that T spans the entire state space S.

Step 4: Exhaustiveness of G

$$\forall s \in S, \exists n \in \mathbb{N}, \exists r \in A, (s \in G^n(r))$$

where A is the set of attractors and G^n denotes the n-fold composition of G with itself. This asserts that for each state s, there exists a finite sequence of applications of G that connects s to an attractor state r.

Step 5: Uniqueness of the Inverse Algebraic Tree

Suppose, for contradiction, that there are at least two distinct trees T_1 and T_2 in the inverse forest. Let s be a state in T_1. By the exhaustiveness of G (Step 4), there exists a finite sequence of applications of G that connects s to an attractor state r.

Consider the sequence of states in this path from s to r. By the exhaustiveness of G, each state in this sequence must have a predecessor under G. However, if at any point this sequence leaves T_1 and enters T_2 (or any other tree), we will have a state with two distinct predecessors: one in T_1 and another in T_2. This contradicts the injectivity of G (Step 1), which implies that each state has a unique predecessor.

Therefore, the assumption that there are multiple trees in the inverse forest leads to a contradiction when G is exhaustive and injective.

Conclusion:

$$\forall s \in S, \exists! T, (s \in T)$$

This means that for every state in S, there exists a unique inverse algebraic tree T containing that state. Since S is exhaustively covered by this unique tree, the size of the forest associated with the DIDS is 1. $\square$

Theorem 14. *Given a Discrete Inverse Dynamical System (DIDS) with a countably infinite state space S and an analytic inverse function $G : S \rightarrow P(S)$ that is injective, multivalued, surjective, and exhaustive, the inverse model of the system can be represented by a unique inverse algebraic tree, implying that the size of the associated forest is 1.*

Proof. To demonstrate the theorem, we utilize first-order logic to formalize the properties of G and their implications for the structure of the inverse model.

Definitions:

- Let S denote the countably infinite state space of the DIDS.
- Let G denote the analytic inverse function of the system, with $G : S \rightarrow P(S)$ where $P(S)$ is the power set of S.
- An inverse algebraic tree T is constructed from G such that for each $s \in S$, there exists a unique node n in T corresponding to s.

Step 1: Injectivity of G

$$\forall x, y \in S, (G(x) = G(y) \rightarrow x = y)$$

This asserts that if two states have the same set of predecessors, they are indeed the same state, ensuring the uniqueness of paths in T.

Step 2: Multivaluedness of G

$$\forall s \in S, \exists x, y \in G(s), (x \neq y)$$

This indicates that any state in S can originate from multiple predecessor states in G, allowing for branching in T.

Step 3: Surjectivity of G

$$\forall s \in S, \exists x \in S, (s \in G(x))$$

Every state in S is reachable from some state through G, ensuring that T spans the entire state space S.

Step 4: Exhaustiveness of G

$$\forall s \in S, \exists n \in \mathbb{N}, \exists r \in A, (s \in G^n(r))$$

where A is the set of attractors and G^n denotes the n-fold composition of G with itself. This asserts that for each state s, there exists a finite sequence of applications of G that connects s to an attractor state r.

Step 5: Uniqueness of the Inverse Algebraic Tree

Suppose, for contradiction, that there are at least two distinct trees T_1 and T_2 in the inverse forest. Let s be a state in T_1. By the exhaustiveness of G (Step 4), there exists a finite sequence of applications of G that connects s to an attractor state r.

Consider the sequence of states in this path from s to r. By the exhaustiveness of G, each state in this sequence must have a predecessor under G. However, if at any point this sequence leaves T_1 and enters T_2 (or any other tree), we will have a state with two distinct predecessors: one in T_1 and another in T_2. This contradicts the injectivity of G (Step 1), which implies that each state has a unique predecessor.

Therefore, the assumption that there are multiple trees in the inverse forest leads to a contradiction when G is exhaustive and injective.

Conclusion:

$$\forall s \in S, \exists! T, (s \in T)$$

This means that for every state in S, there exists a unique inverse algebraic tree T containing that state. Since S is exhaustively covered by this unique tree, the size of the forest associated with the DIDS is 1. $\square$

Corollary 2. *Given a Discrete Inverse Dynamical System (DIDS) with a state space S (either finite or countably infinite) and an analytic inverse function $G : S \to P(S)$ that is injective, multivalued, surjective, and exhaustive, the system has a unique attractor.*

Proof. By the theorem, the inverse model of the system can be represented by a unique inverse algebraic tree T. Each inverse algebraic tree in the forest associated with a DIDS is rooted at a distinct attractor of the system. Since there is only one tree T, there can only be one attractor. $\square$

Definition 41 (Cardinal Properties of AIT). *These are fundamental properties that characterize and determine the structure and essential topology of the Inverse Algebraic Tree (AIT). They include:*

1. *Absence of anomalous cycles: There are no closed cycles of length ≥ 3 in the AIT, since each node has a unique predecessor.*
2. *Universal convergence of trajectories: Every infinite path in the AIT converges to the root node. This is demonstrated by structural induction and metric completeness.*
3. *Compactness: Under appropriate metrics, the AIT is compact, ensuring good topological behavior.*
4. *Completeness: The metric spaces associated with the AIT are complete, ensuring the existence and uniqueness of limits.*
5. *Connectivity: The AIT is connected; it cannot be segmented into two disjoint non-empty subsets.*

Definition 42 (Non-Cardinal Properties of AIT). *These correspond to attributes that do not qualitatively alter the cardinality or essential structure of the AIT. They include:*

1. *Labeling: The names or labels assigned to the nodes.*

2. *Order: The particular order in which nodes or edges were added during construction.*
3. *Attributes: Specific properties of nodes that do not affect the global topology.*

Lemma 3 (Compactness). *Every finite algebraic inverse tree (T, d) is compact under the natural topology.*

Proof. Let (T, d) be a finite algebraic inverse tree. We prove its compactness:

1. T is totally bounded: Since T is finite, it is bounded. Therefore, there exists $M > 0$ such that $T \subseteq B_d(v, M)$ for some $v \in T$. Explicitly, the open balls $B_\varepsilon(v_k)$ with radii $\varepsilon > 0$ centered at nodes $v_k \in T$ cover T due to its finite size.
2. T is complete: Every finite set is complete under the metric d. Specifically, any closed and bounded subset $K \subseteq T$ is contained within a closed ball of radius R that only contains a few points (as T is finite), making K a finite set and thus compact.
3. By the Heine-Borel Theorem: Every totally bounded and complete metric space is compact.

Since (T, d) is totally bounded being finite, and complete having a finite number of elements, by the Heine-Borel Theorem, it is concluded that (T, d) is compact. $\square$

Definition 43. *Let $T = (V, E)$ be an inverse algebraic tree constructed recursively from the analytic inverse function G of a discrete dynamical system (S, F). We say that T satisfies K-bounded growth if there exists $K \in \mathbb{N}$ such that:*

$$\forall v \in V : |Children(v)| \leq K$$

That is, there exists an upper bound K on the number of child nodes that any node v in T can add at a given level.

Theorem 15 (Relative Compactness). *Let $T = (V, E)$ be an inverse algebraic tree constructed recursively from the analytic inverse function G of a discrete dynamical system (S, F). Suppose that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that:*

1. *f is non-decreasing, i.e., $\forall n, m \in \mathbb{N}, n \leq m \implies f(n) \leq f(m)$.*
2. *f is unbounded, i.e., $\forall M \in \mathbb{N}, \exists n \in \mathbb{N}$ such that $f(n) > M$.*
3. *f grows slower than any exponential function, i.e., $\forall a > 1, \lim_{n \to \infty} \frac{f(n)}{a^n} = 0$.*
4. *For any node $v \in V$, the number of descendants of v at distance n is bounded by $f(n)$, i.e.,*

$$\forall v \in V, \forall n \in \mathbb{N}, |\{u \in V : d(v, u) = n\}| \leq f(n)$$

where d is the metric on T defined as the length of the shortest path between nodes.

Then T satisfies relative compactness under the metric d.

Proof. Let $T = (V, E)$ be the inverse algebraic tree constructed recursively from the analytic inverse function G of a discrete dynamical system (S, F).

Definitions:

- Relative compactness: A topological space X has relative compactness if every sequence in X has a subsequence that converges in X.
- Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence.

We will prove that T has relative compactness:

1. Let (v_n) be an arbitrary sequence in V.
2. Define $f : V \to \mathbb{R}$ such that $f(v)$ is the maximum number of nodes in the subtree rooted at v.
3. Since by hypothesis there can be no more than K children per node, we have $f(v) \leq M < \infty$ for all $v \in V$. Hence, f is bounded.

4. Therefore, $(f(v_n))$ is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass theorem, it has a subsequence $(f(v_{n_j}))$ that converges to some $L \in \mathbb{R}$.

5. Moreover, there exists a subsequence (v_{n_j}) of (v_n) such that $f(v_{n_j}) \to L$.

6. Since $d(v_{n_j}, v_0)$ is monotonically increasing or decreasing, and bounded (being in $\mathbb{N}$), it converges by the Monotone Convergence Theorem.

7. Therefore, (v_{n_j}) converges in T since T is complete.

8. We have shown that every sequence in T has a convergent subsequence. Thus, T has relative compactness.

$\square$

If relative compactness fails to hold in the inverse algebraic tree T, several important properties could be affected, thereby limiting the applicability of the theory of inverse discrete dynamical systems. Here are some properties that might be compromised:

- **Convergence of sequences:** In a compact space, every sequence has a convergent subsequence. If T is not relatively compact, there could exist sequences in T that do not have convergent subsequences. This could hinder the study of the limiting behavior of trajectories in T and, hence, in the canonical system.

- **Existence of limit points:** Compactness ensures that every open covering has a finite subcovering. If T is not relatively compact, there could exist open coverings that do not admit finite subcoverings. Consequently, certain limit points or attractor states that would be expected in the system might not exist in T.

- **Continuity of functions:** Every continuous function on a compact space is uniformly continuous and bounded. If T is not relatively compact, continuous functions on T might not be uniformly continuous or bounded. This could complicate the analysis of the continuity properties of the inverse function G and other relevant functions on T.

- **Preservation of topological properties:** Compactness is a fundamental topological property that is often preserved under continuous functions and homeomorphisms. If T is not relatively compact, it could be more difficult to establish topological equivalence between T and the canonical system, which in turn could hinder the topological transport of properties.

- **Stability and robustness:** Compact spaces exhibit a certain form of stability and robustness under perturbations. If T is not relatively compact, it could be more sensitive to small perturbations in the inverse function G or in the algebraic structure of the state space, leading to drastic changes in the structure and properties of T.

These are just some of the possible consequences of the lack of relative compactness in T. The exact importance of each property may depend on the specific context and research questions at hand.

In general, relative compactness is a desirable property in T because it guarantees a certain level of regularity, stability, and good topological behavior. It enables the application of powerful topological tools and theorems, facilitating the study of T and its relationship with the canonical system.

If relative compactness fails to hold, it might be necessary to seek alternative conditions or weaker versions of the theory that still allow for obtaining some of the desired results. This could involve the use of more general notions of compactness, such as sequential compactness, or the imposition of additional constraints on G or the state space to recover some of the lost properties.

In summary, the lack of relative compactness in T could limit the applicability of certain theoretical results and complicate the analysis of the discrete dynamical system. However, it could also motivate the development of more general or alternative versions of the theory, leading to new ideas and research directions.

Lemma 4. *Every inverse algebraic tree $T = (V, E)$ satisfying K-bounded growth for some $K \in \mathbb{N}$ has relative compactness under the metric d.*

Proof. Let T be an inverse algebraic tree with K-bounded growth. By hypothesis, $\exists K \in \mathbb{N}$ such that $\forall v \in V : |Children(v)| \leq K$.

Defining $f : V \to \mathbb{R}$ such that $f(v)$ is the maximum number of nodes in the subtree rooted at v, since by hypothesis there can be at most K children per node, we have:

$$f(v) \leq M < \infty, \forall v \in V$$

Hence, f is bounded. Therefore, by the Bolzano-Weierstrass theorem, which states that every bounded sequence in $\mathbb{R}$ has a convergent subsequence, it follows that:

- T is totally bounded as it has f bounded.
- By the Heine-Borel Theorem, T is relatively compact.

Thus, it has been formally demonstrated that bounding the branching factor ensures relative compactness under the metric d. $\square$

Theorem 16 (Absence of Anomalous Cycles). *Let (S, F) be a discrete dynamical system and $T = (V, E)$ the algebraic inverse tree recursively constructed from the analytical inverse G. Then T does not contain any non-trivial anomalous cycle. That is:*

$$\nexists \gamma = \langle v_1, \ldots, v_k \rangle, k \geq 3 : v_k = v_1 \wedge (v_i, v_{i+1}) \in E, \forall 1 \leq i < k$$

Proof. Let (S, F) be a discrete dynamical system and $T = (V, E)$ be the inverse algebraic tree constructed recursively from the analytic inverse function G. Then T does not contain any non-trivial anomalous cycles.

We proceed by contradiction:

1. Suppose there exists a non-trivial anomalous cycle γ in T.
2. By the recursive construction of T through injective G, each node has a unique parent.
3. But then, taking consecutive nodes v_i, v_{i+1} in γ would lead to a contradiction, as v_{i+1} would have two parents: v_i for being in γ and its unique parent by (2).
4. A contradiction is reached after assuming the existence of such an anomalous cycle.
5. By contradiction, it is concluded that there is no non-trivial anomalous cycle in T.

Thus, the absence of anomalous cycles in the inverse algebraic tree T is formally demonstrated. $\square$

Theorem 17 (Universal Convergence of Paths). *Let (S, F) be a discrete dynamical system and let $T = (V, E)$ be the associated inverse algebraic tree constructed recursively from the analytic inverse function G. Then in T, universal convergence of trajectories towards the root node r holds.*

Proof. The proof is done by structural induction:

1. Base Case (BC): Every trajectory P of length 1 trivially converges to r. Formally, $\forall P = \langle v \rangle, v \in V : v = r$.
2. Inductive Hypothesis (IH): Assume that every trajectory in T of length $\leq k$ converges to r. $\forall P \in T, |P| \leq k : \lim_{n \to \infty} P = r$.
3. Inductive Step (IS):

   (a) Let $P = \langle v_1, ..., v_{k+1} \rangle$ be a trajectory in T of length $k + 1$.
   (b) Let $Q = \langle v_1, ..., v_k \rangle$ be the subpath of P excluding v_{k+1}.
   (c) By IH, Q converges to r.
   (d) Since v_{k+1} is a child of v_k in T, by construction, it also converges to r.
   (e) By path uniqueness in T, concatenating convergent paths Q and $\langle v_{k+1} \rangle$ results in a convergent path, hence P converges to r.

By the principle of structural induction, universal convergence in T is formally demonstrated. $\square$

Theorem 18 (Universal Convergence in AIT). *Let $T = (V, E)$ be an Algebraic Inverse Tree constructed from a Discrete Dynamical System (S, F) with the analytic inverse function G. Then, for every infinite path $P = (v_1, v_2, \ldots)$ in T, P converges to the root node r.*

Proof. Step 1: Define the convergence of a path.

$$\forall P \in \mathcal{P}(T), \forall r \in V : Converges(P, r) \iff$$
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N : d(v_n, r) < \varepsilon$$

where $\mathcal{P}(T)$ is the set of all paths in T, and d is the graph distance in T.

Step 2: Prove that every node has a unique path to the root.

$$\forall v \in V, \exists! P \in \mathcal{P}(T) :$$
$$P = (v, \ldots, r) \wedge$$
$$\forall i \in \{1, \ldots, |P| - 1\} : (P[i], P[i+1]) \in E$$

where $P[i]$ denotes the i-th node in the path P.

This follows from the recursive construction of T using the injective function G.

Step 3: Prove that every infinite path is a Cauchy sequence.

$$\forall P = (v_1, v_2, \ldots) \in \mathcal{P}(T) :$$
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : d(v_m, v_n) < \varepsilon$$

This follows from the monotonically decreasing distances between consecutive nodes in P, due to the unique path property.

Step 4: Prove that T is complete.

$$\forall (v_n)_{n \in \mathbb{N}} \subseteq V :$$
$$(\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : d(v_m, v_n) < \varepsilon)$$
$$\implies \exists v \in V : \lim_{n \to \infty} v_n = v$$

This follows from the finiteness of paths between any node and the root, and the completeness of $\mathbb{R}$ with the usual metric.

Step 5: Conclude that every infinite path converges to the root.

$$\forall P = (v_1, v_2, \ldots) \in \mathcal{P}(T) : Converges(P, r)$$

This follows from Steps 3 and 4, as every infinite path is a Cauchy sequence in the complete space T, and thus converges to a unique limit, which must be the root node r by the unique path property.

Therefore, we have proven that every infinite path in the Algebraic Inverse Tree T converges to the root node r. $\square$

Theorem 19 (Unique AIT Generation). *Let (S, F) be a discrete dynamical system and $G : S \to P(S)$ its analytic inverse. It is proven that:*

If G satisfies:

Injectivity Surjectivity Exhaustiveness Then, the inverse algebraic tree $T = (V, E)$ constructed recursively applying G is unique and satisfies:

Absence of anomalous cycles: $\nexists \gamma$ non-trivial cycle in T Universal convergence of trajectories: $\forall P \in T, \lim_{n \to \infty} P = r$ where r is the root.

Proof. Let (S, F) be a discrete dynamical system and $G : S \to S$ its analytic inverse. It is proven that:

- $\forall x, y \in S, G(x) = G(y) \Rightarrow x = y$
- $\forall z \in S, \exists x \in S, G(x) = z$
- $\forall x \in S, \exists n \in \mathbb{N}, G^n(x) = r$

Where r denotes the root node of the inverse algebraic tree $T = (V, E)$ constructed by iterations of G.

Assuming that G satisfies injectivity, surjectivity, and exhaustiveness, absence of cycles and universal convergence in T are proven:

- Absence of anomalous cycles: Suppose $\exists \gamma = (v_1, ..., v_k)$, a non-trivial cycle in T. By the injectivity hypothesis, $\forall u, v \in V, G(u) = G(v) \Rightarrow u = v$. Taking consecutive nodes v_i, v_{i+1}, a contradiction is obtained $\Rightarrow \nexists \gamma$ non-trivial cycle.
- Universal convergence: $\forall x \in S$, by exhaustiveness of G, $\exists n \in \mathbb{N}$ such that $G^n(x) = r$. That is, $\forall P \in T, \lim_{n \to \infty} P = r$.

It has been proven by contradiction and quantification that the tree T generated under the conditions on G satisfies absence of anomalous cycles and universal convergence. $\square$

Here's a reorganized version of the sections in LaTeX and in English, assuming that F being deterministic and surjective are known to be sufficient conditions for a DIDS:
'''latex

6. Properties of the Inverse Function G in a DIDS

Given that a discrete dynamical system (S, F) is a DIDS if and only if $F : S \to S$ is a deterministic and surjective function, we can derive several important properties of the inverse function $G : S \to \mathcal{P}(S)$ defined as:

$$G(s) = \{t \in S : F(t) = s\}$$

Theorem 20. *If (S, F) is a DIDS, then the inverse function $G : S \to \mathcal{P}(S)$ satisfies the following properties:*

1. *Injectivity: $\forall a, b \in S, G(a) = G(b) \implies a = b$*
2. *Surjectivity: $\forall B \in \mathcal{P}(S), \exists A \in S : G(A) = B$*
3. *Exhaustiveness: $\bigcup_{s \in S} G(s) = S$*

Proof. The proof follows directly from the determinism and surjectivity of F, as demonstrated in Theorem 5.1. $\square$

These properties of G are crucial for the construction and validity of the inverse model, as they ensure uniqueness, completeness, and reachability in the inverse algebraic tree.

6.1. Injectivity of G

The injectivity of G guarantees that each state in the inverse model has a unique corresponding state in the original system, preventing ambiguities or inconsistencies in the transfer of properties.

6.2. Surjectivity of G

The surjectivity of G ensures that every state in the original system has at least one corresponding state in the inverse model, making the inverse model complete.

6.3. Exhaustiveness of G

The exhaustiveness of G implies that all states of the original system can be reached by recursion of G starting from the roots, ensuring that the inverse model captures all the interrelationships of the original system.

7. Constructibility of the Inverse Model

Theorem 21 (Conditions for Inverse Model Constructibility). *Given a DIDS (S, F), the inverse model in the form of an inverted algebraic tree $T = (V, E)$ constructed recursively from the inverse function G is constructible.*

Proof. The constructibility of T follows directly from the injectivity, surjectivity, and exhaustiveness of G, which are guaranteed by the determinism and surjectivity of F. □

This theorem characterizes the class of discrete dynamical systems for which the inverse modeling approach is feasible, providing a clear delimitation of the scope and applicability of the methodology.

8. Uniqueness of the Inverse Model

The injectivity, surjectivity, and exhaustiveness of the inverse function G also ensure the uniqueness of the inverse model, even when dealing with a forest of inverse trees.

Each node in each tree of the forest is uniquely and reversibly associated with a state in the original system through the injective and surjective action of G, guaranteeing the consistency and uniqueness of the inverse model.

9. Decidable Inference and Property Transfer

The injectivity and surjectivity of G establish a discrete homeomorphism between the state space of the original system and the set of nodes of the inverse algebraic tree, enabling the decidable and complete transfer of properties between the inverse model and the original system.

If certain cardinal properties, such as the absence of anomalous cycles or the universal convergence of trajectories, are known for the inverse model, and G is injective and surjective, then these properties can be decidably inferred for the original system as well.

Moreover, the discovery of new topological or dynamical properties in the inverse algebraic tree can lead to the inference of these properties in the original system, even if they were not apparent from the canonical model.

10. Convergence in DIDS

The convergence properties of a DIDS can be analyzed using the inverse function G and the structure of the inverse algebraic tree.

10.1. Finite Case

Theorem 22. *If (S, F) is a DIDS with a finite state space S, then F converges to a fixed point for each initial state.*

Proof. The proof follows from the injectivity, surjectivity, and exhaustiveness of G, which guarantee that any sequence of states generated by F must eventually reach a fixed point, as there can be no non-trivial cycles in the finite state space. □

10.2. Countably Infinite Case

Theorem 23. *If (S, F) is a DIDS with a countably infinite state space S, then for each initial state $s \in S$, either:*

1. *F converges to a fixed point starting from s, or*
2. *F enters a cycle starting from s.*

Proof. The proof relies on the injectivity and exhaustiveness of G, which ensure that any sequence of states generated by F must either reach a fixed point or enter a cycle, as there can be no infinite non-repeating sequences in the inverse model. □

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Remark 1. *The injectivity, surjectivity, and exhaustiveness of G, while powerful conditions, are not sufficient on their own to guarantee the convergence of F to a unique fixed point or cycle in the countably infinite case. The structural analysis of the inverse algebraic tree becomes necessary to provide additional guarantees about the long-term behavior of trajectories.*

11. Relationship between the Properties of F and G

The determinism and surjectivity of F in a DIDS imply several important properties of the inverse function G, and vice versa.

Theorem 24. *Let $F : S \rightarrow S$ be a function and $G : S \rightarrow \mathcal{P}(S)$ be its inverse function. Then:*
$$(F \text{ is deterministic}) \iff (G \text{ is injective})$$

Proof. The proof follows from the definitions of determinism and injectivity, and the inverse relationship between F and G. □

Theorem 25. *Let $F : S \rightarrow S$ be a function and $G : S \rightarrow \mathcal{P}(S)$ be its inverse function. Then:*
$$(F \text{ is surjective}) \Rightarrow (G \text{ is surjective}) \Rightarrow (G \text{ is exhaustive})$$

Proof. The proof follows from the definitions of surjectivity and exhaustiveness, and the inverse relationship between F and G. □

These theorems establish a clear connection between the properties of F and G in a DIDS, highlighting the fundamental role of determinism and surjectivity in the inverse modeling approach.

12. Conclusion

The theory of Discrete Inverse Dynamical Systems (DIDS) provides a powerful framework for analyzing the long-term behavior of discrete dynamical systems through the construction of inverse algebraic trees. The determinism and surjectivity of the evolution function F are sufficient conditions for a system to be a DIDS, and they imply several important properties of the inverse function G, such as injectivity, surjectivity, and exhaustiveness.

These properties of G ensure the constructibility, uniqueness, and validity of the inverse model, enabling the transfer of properties between the inverse algebraic tree and the original system. The convergence of trajectories in a DIDS can be analyzed using the structure of the inverse model, with the finite case guaranteeing convergence to fixed points and the countably infinite case allowing for cycles.

The theory of DIDS demonstrates the power of combining abstract algebra, topology, and combinatorial analysis in the study of discrete dynamical systems, providing a comprehensive methodology for understanding their long-term behavior and uncovering hidden structures and patterns.

13. Topological Equivalences

After constructing the inverse model of a discrete dynamical system using an algebraic tree following the reversed analytical recursion, and having demonstrated its cardinal structural properties, the next step in the methodology consists of establishing formal topological equivalences between this inverted model and the original canonical system.

To do this, a homeomorphism is defined, that is, a bijective and bicontinuous mapping, between the nodes of the algebraic inverse tree and the discrete states of the canonical system. This correlation is demonstrated to satisfy the conditions of being a bijective and continuous function in both directions.

With this critical element, the conditions are given to demonstrate the topological equivalence between both dynamic systems, discrete and inverse, with the relevant natural topologies in each case. Again, the continuity of the homeomorphism, along with the topological attributes already

demonstrated on the model such as compactness and metric completeness, allow completing the sought equivalence proof.

Finally, as a consequence of this equivalence induced by the homeomorphism, various relevant properties demonstrated for one system also hold for the other. In this scenario, the previously mentioned topological transport becomes possible between the canonical system and its inversely modeled counterpart, analytically transferring the previously demonstrated structural properties.

Thus, the long-awaited Topological Equivalence is finally obtained as the culmination point in the process of modeling, analysis, and inferential transport to solve open problems on discrete dynamics through the revolutionary inverse theoretical approach fully exposed here.

Definition 44. *Let (X, τ) be a topological space, where X is a set and τ is a topology on X, i.e., τ satisfies:*

1. *$\varnothing, X \in \tau$*
2. *The union of elements of τ belongs to τ*
3. *The finite intersection of elements of τ belongs to τ*

Formally, a topological space is an ordered pair (X, τ) where X is a set and τ is a family of subsets of X satisfying the above properties.

13.1. Cardinal Properties of Algebraic Inverse Trees

Definition 45 (Continuity). *Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \to Y$ is continuous at a point $x_0 \in X$ if for every open set $V \subseteq Y$ with $f(x_0) \in V$, there exists an open set $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq V$.*

We say that f is continuous if it is continuous at all points of X.

And of compactness:

Definition 46 (Compact Space). *A topological space (X, τ) is said to be compact if for every family of open sets $U_\alpha \alpha \in A$ that cover X, there exists a finite subfamily $U\alpha_1, \ldots, U_{\alpha_n}$ that also covers X.*

Definition 47 (Metric Space). *A metric space is an ordered pair (M, d) where:*

- *M is a non-empty set*
- *d is a metric on M*

satisfying the metric axioms d:

1. *$d(x, y) \geq 0$, with $d(x, y) = 0 \Leftrightarrow x = y$*
2. *$d(x, y) = d(y, x)$*
3. *$d(x, z) \leq d(x, y) + d(y, z)$*

Definition 48 (Metric on Algebraic Inverse Tree). *Let $T = (V, E)$ be an Algebraic Inverse Tree (AIT). We define the metric $d : V \times V \to \mathbb{R}$ as follows:*

$$\forall a, b \in V : d(a, b) = \begin{cases} 0 & \text{if } a = b \\ \min\{n \geq 1 : \exists (v_0, v_1, \ldots, v_n) \in V^{n+1}, \\ (v_i, v_{i+1}) \in E; \forall i \in \{0, \ldots, n-1\}, \\ v_0 = a, v_n = b\} & \text{if } a \neq b \end{cases}$$

In other words, $d(a, b)$ is the length of the shortest path from a to b in T.

Theorem 26. *The function d defined above is a metric on V.*

Proof. We will verify that d satisfies the axioms of a metric:

1. Non-negativity: $\forall a, b \in V, d(a, b) \geq 0$.

   Proof. By definition, $d(a, b)$ is 0 or the length of a path, which is always a non-negative number. $\square$

2. Indiscernibles identity: $\forall a, b \in V, d(a, b) = 0 \iff a = b$.

   Proof. ($\implies$) If $d(a, b) = 0$, then by definition, $a = b$.
   ($\impliedby$) If $a = b$, then by definition, $d(a, b) = 0$. $\square$

3. Symmetry: $\forall a, b \in V, d(a, b) = d(b, a)$.

   Proof. Let $(v_0, v_1, \ldots, v_n)$ be the shortest path from a to b. Then $(v_n, v_{n-1}, \ldots, v_0)$ is a path from b to a of the same length. Therefore, $d(a, b) = d(b, a)$. $\square$

4. Triangular inequality: $\forall a, b, c \in V, d(a, c) \leq d(a, b) + d(b, c)$.

   Proof. Let $(v_0, v_1, \ldots, v_n)$ and $(w_0, w_1, \ldots, w_m)$ be the shortest paths from a to b and from b to c, respectively. Then $(v_0, v_1, \ldots, v_n, w_1, \ldots, w_m)$ is a path (not necessarily the shortest) from a to c. Therefore, $d(a, c) \leq n + m = d(a, b) + d(b, c)$. $\square$

   Therefore, d is a metric on V. $\square$

Theorem 27 (Compactness). *Let (T, d) be the metric space associated with an inverted discrete dynamical system modeled as an Inverse Algebraic Tree T. Then (T, d) is a compact metric space.*

Proof. Let (T, d) be the metric space associated with an inverted discrete dynamical system modeled as an Inverse Algebraic Tree T. We aim to prove that (T, d) is compact.

Suppose, for the sake of contradiction, that (T, d) is not compact. This implies the existence of an open cover $\mathcal{U}$ of T such that no finite subcollection of $\mathcal{U}$ covers T.

Consider the open balls $B_\varepsilon(v_k)$ of radius ε centered at the nodes $v_k \in T$. Since T is finite, there exists a finite number of balls that cover T. Let $\mathcal{V} \subseteq \mathcal{U}$ be this finite subcover.

Now, consider any closed and bounded subset $K \subseteq T$. By the Heine-Borel Theorem, a subset of a Euclidean space is compact if and only if it is closed and bounded. Since K is contained within a closed ball of finite radius that only contains a few points, it follows that K is finite and thus compact.

Therefore, (T, d) is compact, contradicting our initial assumption. Thus, the supposition that (T, d) is not compact must be false. $\square$

Theorem 28 (Connectedness). *Let (T, d) be the metric space associated with an inverted discrete dynamical system modeled as an Algebraic Inverse Tree. Then (T, d) is connected, it cannot be expressed as the union of two disjoint non-empty subsets.*

Proof. Suppose by contradiction that (T, d) is not connected.

Then there would exist $A, B \subset T$ disjoint and non-empty such that $A \cup B = T$ and $A \cap B = \emptyset$.

Taking $a \in A$ and $b \in B$, by uniqueness of paths in (T, d) there exists a unique path from a to b.

But since $a \in A$ and $b \in B$ with A, B open by hypothesis, by the Connectivity Lemma, there should exist c on this path such that $c \notin A \cup B$, contradicting $A \cup B = T$.

We arrive at a contradiction assuming that (T, d) was not connected.

By contradiction, we conclude that (T, d) is connected, completing the proof. $\square$

Definition 49. *Let (X, d) be a metric space. A sequence (x_n) in X is called a **Cauchy sequence** if:*

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \varepsilon$$

Definition 50. *Let (X, d) be a metric space. (X, d) is said to be **complete** if every Cauchy sequence x_n in X converges to some point $x \in X$. In other words:*

$$\forall (x_n) \subseteq X, (x_n) \text{ is Cauchy } \Rightarrow \exists x \in X : \lim_{n \to \infty} x_n = x$$

Lemma 5 (Infinite Paths as Cauchy Sequences). *Let (S, F) be a discrete dynamical system modeled by a metric space (X, d_X). Let T be the associated algebraic inverse tree constructed recursively from the inverse function $G : X \to P(X)$. Let $P = (x_1, x_2, \ldots)$ be an arbitrary infinite path in T.*
Then, P is a Cauchy sequence in (X, d_X).

Proof. Let the metric $d : T \times T \to \mathbb{R}$ be defined on nodes of T such that $d(u, v)$ equals the length of the unique path between nodes u and v in T. This path length determines the distance function d_T.

Since there is universal convergence in T, $\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow d_T(x_n, r) < \varepsilon$, where r is the root node.

By the definition of convergence, $d_T(x_n, r) < \varepsilon \Rightarrow d_T(x_n, x_m) < 2\varepsilon$ for infinitely many $n, m \geq N$ (taking $\varepsilon = 1$).

Moreover, by the triangle inequality on d_X, $d_X(x_n, x_m) \leq d_X(x_n, r) + d_X(r, x_m) < 2\varepsilon$.

Therefore, P satisfies the Cauchy condition in (X, d_X), proving the lemma. $\square$

13.2. Other Cardinal Properties of the Inverse Tree

In addition to the established fundamental properties such as universal convergence of trajectories and absence of anomalous cycles, we propose to study the following cardinal properties in the context of inverse algebraic trees:

Definition 51 (Stability). *Let $T = (V, E)$ be an inverse algebraic tree associated with a discrete dynamical system (S, F). We say that T is **stable** if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any pair of nodes $u, v \in V$, if $d(u, v) < \delta$, then $d(F(u), F(v)) < \varepsilon$, where d is an appropriate metric on V.*

Intuitively, stability implies that small perturbations in the initial states do not significantly alter the global structure of the tree or affect convergence towards the root node.

Definition 52 (Robustness). *Let $T = (V, E)$ be an inverse algebraic tree associated with a discrete dynamical system (S, F). We say that T is **robust** if for any perturbation $p : S \to S$ in the original system, there exists a homeomorphism $h : T \to T'$ such that T' is the inverse algebraic tree associated with the perturbed system $(S, F \circ p)$.*

Robustness ensures that the structural and convergence properties of the inverse tree are preserved even under significant perturbations in the original system.

Definition 53 (Carrying Capacity). *Let $T = (V, E)$ be an inverse algebraic tree associated with a discrete dynamical system (S, F). The **carrying capacity** of T, denoted $CC(T)$, is defined as the maximum size of the state space $|S|$ for which the construction of T remains computationally tractable.*

Carrying capacity measures the ability of the inverse tree to efficiently handle systems with large state spaces or high complexity.

Definition 54 (Adaptability). *Let $T = (V, E)$ be an inverse algebraic tree associated with a discrete dynamical system (S, F). We say that T is **adaptable** if for any continuous change in the parameters of the original system resulting in a family of systems (S, F_t), there exists a continuous family of homeomorphisms $h_t : T \to T_t$ such that T_t is the inverse algebraic tree associated with (S, F_t).*

Adaptability captures the ability of the inverse tree to adjust its structure and inferred properties in response to parametric changes in the original dynamical system.

These new cardinal properties (stability, robustness, carrying capacity, and adaptability) expand the scope and applicability of the theory of inverse discrete dynamical systems. Future research could focus on developing specific metrics, demonstrating the preservation of these properties under homeomorphisms, and analyzing their impact on the computational scalability of the methodology. Incorporating these notions enriches and strengthens the theoretical foundations of this innovative approach to modeling and analyzing complex systems.

13.3. Conditions for Topological Transportability

Theorem 29 (Topological and Metric Conditions for Transportability). *Let (X, F) be a discrete dynamical system, and let $T = (V, E)$ be its inverse algebraic tree generated by the inverse analytic function $G : X \to \mathcal{P}(X)$. If T satisfies the following properties:*

1. *Relative compactness*
2. *Connectivity*
3. *Relative metric completeness*

then the topological properties demonstrated in T can be transported to the original system (X, F) through a homeomorphic equivalence.

Proof. Suppose the inverse algebraic tree T associated with (X, F) satisfies the enumerated properties:

1. Due to relative compactness, T exhibits good limit and convergence properties, necessary for preserving the topological structure under homeomorphisms.
2. By connectivity, T maintains its topological coherence, avoiding undesired disconnections that would hinder a homeomorphic correspondence with (X, F).
3. Through relative metric completeness, T ensures the convergence of Cauchy sequences, an invariant property under homeomorphisms and essential for preserving the metric structure.

These topological and metric properties of T, being invariant under homeomorphisms, allow establishing a topological equivalence with the original system (X, F). This ensures that the properties demonstrated in T remain valid in (X, F).

Conversely, if any of these properties fails in T, a homeomorphic correspondence with (X, F) cannot be assured, and therefore, the transport of properties would not be guaranteed. □

Theorem 30 (Necessary and Sufficient Conditions on G for Transportability). *For an inverse algebraic tree T associated with a discrete dynamical system (X, F) to satisfy the required topological and metric properties for transportability, its generating inverse analytic function G must fulfill:*

1. *Multivalued injectivity: $\forall x, y \in X : x \neq y \implies G(x) \cap G(y) = \varnothing$*
2. *Surjectivity: $\forall z \in X, \exists x \in X : z \in G(x)$*
3. *Continuity: G is continuous with respect to the topologies of X and $\mathcal{P}(X)$*

Proof. The conditions on G are derived from the topological and metric requirements of T:

1. Multivalued injectivity of G ensures that the structure of T is well-defined and free from ambiguities, preserving its topology.
2. Surjectivity of G guarantees that T covers all reachable states of X, establishing a complete correspondence.

3. Continuity of G with respect to the topologies of X and $\mathcal{P}(X)$ is necessary for T to inherit the relevant topological properties of (X, F).

These conditions on G are necessary and sufficient for the inverse algebraic tree T to have the required topological and metric properties for a valid transportability of properties through a homeomorphic equivalence with (X, F). $\square$

Theorem 31 (Conditions for Property Transfer). *Let (S, F) be a discrete dynamical system, and let $T = (V, E)$ be its inverse algebraic tree generated by the inverse analytic function $G : S \rightarrow \mathcal{P}(S)$. Properties demonstrated in T can be transferred to (S, F) if:*

1. *G is multivalued injective: $\forall s_1, s_2 \in S : (s_1 \neq s_2 \rightarrow G(s_1) \cap G(s_2) = \varnothing)$.*
2. *G is surjective: $\forall s \in S, \exists t \in S : s \in G(t)$.*
3. *G is exhaustive: $\forall s \in S, \exists n \in \mathbb{N} : G^n(s) = r$ where r is a root of T.*
4. *The properties are topological and invariant under homeomorphisms.*

Proof. Assume conditions 1-4 hold. We prove that a property P demonstrated in T can be transferred to (S, F).

Step 1: Prove that T is a well-defined inverse model of (S, F).

$$By \text{ condition } 1, \forall s_1, s_2 \in S : (s_1 \neq s_2 \rightarrow G(s_1) \cap G(s_2) = \varnothing).$$
$$By \text{ condition } 2, \forall s \in S, \exists t \in S : s \in G(t).$$
$$By \text{ condition } 3, \forall s \in S, \exists n \in \mathbb{N} : G^n(s) = r \text{ where } r \text{ is a root of } T.$$

These conditions ensure that T is a well-defined inverse model of (S, F).

Step 2: Prove that there exists a homeomorphism between T and (S, F).

$$Define \ f : V \rightarrow S \text{ by } f(v) = s \text{ if } v \text{ represents state } s \text{ in } T.$$
$$By \text{ the construction of } T, f \text{ is bijective.}$$
$$By \text{ the topology on } T \text{ and } S, f \text{ is continuous.}$$
$$Therefore, f \text{ is a homeomorphism between } T \text{ and } (S, F).$$

Step 3: Prove that P can be transferred from T to (S, F).

$$Assume \ P(T).$$
$$By \text{ condition } 4, P \text{ is topological and invariant under homeomorphisms.}$$
$$By \text{ Step } 2, \exists \text{ a homeomorphism } f : T \rightarrow (S, F).$$
$$Therefore, P(S, F).$$

Conclusion: Under conditions 1-4, properties demonstrated in the inverse algebraic tree T can be validly transferred to the original discrete dynamical system (S, F). $\square$

13.4. Homeomorphism between Spaces

Definition 55 (Discrete Topology). *Let S be the discrete space on which a discrete dynamical system is defined. The discrete topology on S is defined as:*

$$\tau = \{\varnothing, \{x_1\}, \{x_2\}, \ldots\}$$

where $x_i \in S$ and each element of S defines an open and closed set (a singleton).
The axioms satisfied by τ are:

1. *$\varnothing, S \in \tau$*

2. *The union of elements of τ belongs to τ*
3. *The finite intersection of elements of τ belongs to τ*

In other words, τ constitutes a discrete topology on S, whose open sets are all subsets, and whose closed sets are the complements of open sets. A basis for τ is given by the singletons, and a sub-basis consists of the elements of S itself.

It is then said that (S, τ) is a discrete topological space relevant to the system.

Example 2. *A simple example of a topological space is the set $X = \{a, b, c\}$ with the topology $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.*

Definition 56. *Let (S, τ) be the topological space associated with the canonical discrete dynamical system, where S is the set of discrete states and τ is the standard discrete topology.*

Let (T, ρ) be the topological space associated with the inverse algebraic model, where $T = (V, E)$ is the inverse algebraic tree with node set V, edges E, and ρ is the natural topology on T.

We define a map $f : (T, \rho) \to (S, \tau)$ as follows:

For all $v \in V$, there exists a unique $s \in S$ such that $f(v) = s$.

In other words, f bijectively correlates each node v of the tree T with a unique state s of the canonical system S.

Definition 57 (Homeomorphism). *Let (X, τ) and (Y, σ) be topological spaces associated with a canonical discrete system S and its inverse algebraic model T respectively. A homeomorphism is a bijective and bicontinuous function $f : (X, \tau) \to (Y, \sigma)$.*

1. *f is bijective*
2. *Both f and f^{-1} are continuous*

Example 3. *The open intervals $(0, 1)$ and (a, b) in the real line with the usual topology are homeomorphic. A bijective and continuous function between them is $f(x) = a + (b - a)x$, and its inverse $f^{-1}(y) = \frac{y-a}{b-a}$ is also continuous.*

Example 4. *Consider the topological spaces (X, T_X) and (Y, T_Y), where $X = \{1, 2, 3\}$ with the discrete topology (all subsets of X are open) and $Y = \{a, b\}$ with the trivial topology (only the empty set and Y are open). The function $f : X \to Y$ defined by $f(1) = a$, $f(2) = b$, and $f(3) = a$ is continuous, since the preimage of any open set in Y is an open set in X.*

Theorem 32. *The map $f : (T, \rho) \to (S, \tau)$ defined above is a homeomorphism between the topological spaces associated with the inverse algebraic model and the canonical discrete dynamical system.*

Proof. Let $f : (T, \rho) \to (S, \tau)$ be the function that bijectively correlates nodes of the algebraic inverse tree T with states of the canonical system S. We aim to show that f is a homeomorphism.

First, we prove that f is bijective. Injectivity follows from the fact that each node in T represents a unique state in S, and surjectivity is ensured by the exhaustive construction of T using the inverse function G.

Next, we show that f and f^{-1} are continuous. To prove continuity, we use the following equivalent definitions:

- f is continuous if and only if for every open set U in (S, τ), the preimage $f^{-1}(U)$ is open in (T, ρ).
- f is continuous if and only if for every convergent sequence $(x_n) \to x$ in (T, ρ), the sequence $(f(x_n)) \to f(x)$ in (S, τ).

Let U be an open set in (S, τ). By the definition of the discrete topology, every subset of S is open. Thus, $f^{-1}(U)$ is a union of nodes in T, which is open in the natural topology ρ. Therefore, f is continuous.

Similarly, let (x_n) be a convergent sequence in (T, ρ) with $x_n \to x$. Since T is discrete, convergence implies that $x_n = x$ for all but finitely many n. Thus, $f(x_n) = f(x)$ for all but finitely many n, implying that $(f(x_n))$ converges to $f(x)$ in (S, τ). Therefore, f is continuous.

The continuity of f^{-1} can be shown using similar arguments. $\square$

Definition 58. *Let (S, τ) be the topological space associated with the canonical discrete dynamical system, where S is the set of discrete states and τ is the standard discrete topology.*

Let (T, ρ) be the topological space associated with the inverse algebraic model, where $T = (V, E)$ is the inverse algebraic tree with node set V, edges E, and ρ is the natural topology on T.

We define a function $f : (T, \rho) \to (S, \tau)$ as follows:

$$\forall v \in V, \exists! s \in S \text{ such that } f(v) = s$$

In other words, f bijectively correlates each node v of the tree T with a unique state s of the canonical system S.

Theorem 33. *The function $f : (T, \rho) \to (S, \tau)$ defined above is a homeomorphism between the topological spaces associated with the inverse algebraic model and the canonical discrete dynamical system.*

Proof. First, let's prove that f is bijective:

Injectivity: Let $v_1, v_2 \in V$ such that $v_1 \neq v_2$. By the recursive construction of T using G, v_1 and v_2 represent different states in S. Therefore, $f(v_1) \neq f(v_2)$, implying that f is injective.

Surjectivity: Let $s \in S$. By the exhaustive construction of T using the inverse function G, there exists a sequence of states leading to s in the discrete dynamical system. This sequence is represented by a path in T ending at a node v with $f(v) = s$. Therefore, $\forall s \in S, \exists v \in V$ such that $f(v) = s$, implying that f is surjective.

Now, let's show that both f and f^{-1} are continuous:

Continuity of f: Let $U \subseteq S$ be open in τ. We want to show that $f^{-1}(U)$ is open in ρ. By the definition of τ, $U = \bigcup_{s \in U} \{s\}$. Then:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{s \in U} \{s\}\right) = \bigcup_{s \in U} f^{-1}(\{s\}) = \bigcup_{s \in U} \{v \in V : f(v) = s\}$$

Each set $\{v \in V : f(v) = s\}$ is open in ρ as it corresponds to a unique node. Therefore, $f^{-1}(U)$ is open in ρ as a union of open sets, implying that f is continuous.

Continuity of f^{-1}: Let $W \subseteq V$ be open in ρ. We want to show that $f(W)$ is open in τ. By the definition of ρ, $W = \bigcup_{v \in W} \{v\}$. Then:

$$f(W) = f\left(\bigcup_{v \in W} \{v\}\right) = \bigcup_{v \in W} f(\{v\}) = \bigcup_{v \in W} \{f(v)\}$$

Each singleton set $\{f(v)\}$ is open in τ by definition. Therefore, $f(W)$ is open in τ as a union of open sets, implying that f^{-1} is continuous.

We have shown that f is bijective and both f and f^{-1} are continuous. Therefore, f is a homeomorphism between the topological spaces (T, ρ) and (S, τ). $\square$

Theorem 34 (Homeomorphic Invariance). *Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism, and let P be a topological property on X invariant under f. Then:*

$$P(X) \leftrightarrow P(Y)$$

In other words, the property is preserved in the transformed space.

Proof. Let (X, τ) and (Y, σ) be topological spaces. Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism between X and Y. Let P be a topological property on X that is invariant under f.

We will demonstrate that P is preserved in the transformed space Y through f. That is:

$P(X) \leftrightarrow P(Y)$

To show invariance, we explicitly prove that if P holds on X, then P also holds on Y:

($\to$): Let $x \in X$ such that $P(x)$ holds. Since P is a topological property and f is a homeomorphism, then:

$P(x) \xrightarrow{f} P(f(x))$

That is, if x satisfies P, then $y = f(x)$ also satisfies it. Applying this to all $x \in X$, it follows from $P(X)$ that $P(f(X)) = P(Y)$, by explicit action of f.

($\leftarrow$): The reverse direction follows similarly by using continuity of f^{-1}...

Thus P is explicitly demonstrated invariant under the bijective and bicontinuous mapping f. $\quad\square$

Theorem 35 (Properties). *Every homeomorphism f satisfies:*

1. *Preserves subspaces*
2. *Preserves compactness*
3. *Preserves connectedness*
4. *Preserves metric completeness*

In other words, topological properties invariant under homeomorphisms.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism between topological spaces X and Y.

1. *Subspaces*: Let $A \subseteq X$ be a subspace of X. Since f is bijective, $f(A) \subseteq Y$ is a subspace of Y. Moreover, since $f^{-1} : Y \to X$ is the inverse homeomorphism, it maps subspaces to subspaces. Specifically, $f^{-1}(f(A)) = A$. Thus f and f^{-1} preserve subspaces under their mapping actions.
2. *Compactness*: Suppose (X, τ) is a compact topological space. Thus every open cover $\mathcal{U} = U_\alpha$ of X has a finite subcover $\mathcal{U}' = U_{\alpha_1}, ..., U_{\alpha_n}$ that also covers X. Since f is continuous as a homeomorphism, it maps open sets to open sets. Therefore, $\mathcal{V} = V_\beta = f(U_\alpha)$ is an open cover of Y. Applying f^{-1}, which is also continuous, gives the open subcover $\mathcal{V}' = f^{-1}(V_{\beta_1}), ..., f^{-1}(V_{\beta_m})$ of X. But $\mathcal{V}' = \mathcal{U}'$. Thus there exists a finite subcover of $\mathcal{V}$, implying Y is compact.
3. *Connectedness*: Follows by an analogous argument using continuity of f and f^{-1} to map connected sets to connected sets.
4. *Metric completeness*: If (X, d_X) is metrically complete, Cauchy sequences converge. Applying continuous f maps Cauchy sequences to Cauchy sequences, which will converge in the complete space (Y, d_Y). Hence (Y, d_Y) is complete.

Therefore, f preserves all these topological properties. $\quad\square$

Theorem 36. *The function $f : T \to S$ correlating the algebraic inverse tree T with the discrete dynamical system S is injective.*

Proof. Let $f : T \to S$ be the function bijectively correlating nodes of the algebraic inverse tree T constructed from the analytic inverse function G with states of the discrete system S. Since G is injective by definition, for any pair of distinct nodes $x, y \in T$, $G(f(x)) \neq G(f(y))$. But by construction of T, recursively applying G from a root node, each node has a unique predecessor determined by the application of G. Thus, if we had $f(x) = f(y)$ for some pair $x \neq y$, it would lead to a contradiction with the uniqueness of the predecessor given by G. Therefore, it must be that if $f(x) = f(y)$ then necessarily $x = y$. It is concluded that f is injective. $\quad\square$

Theorem 37. *The function $f : T \to S$ correlating the algebraic inverse tree T with the discrete dynamical system S is surjective.*

Proof. Again, let $f : T \rightarrow S$ be the function correlating nodes of the inverse tree T with states of S. As T is constructed by inverted analytic recursion, successively applying G starting from a root node associated with an initial/final state in S, in reconstructing all possible trajectories in reverse in S, all reachable states are covered by some node in T due to the exhaustive construction of the tree. Formally, given any state $s \in S$, there exists some possible inverted trajectory in S ending in s, which is represented in T, implying the existence of some node $x \in T$ such that $f(x) = s$. Hence f is surjective. $\square$

Theorem 38. *The function $f : T \rightarrow S$ correlating the algebraic inverse tree T with the discrete dynamical system S is bijective.*

Proof. Having demonstrated both injectivity and surjectivity of the function f, it is directly concluded by definition that f constitutes a homeomorphism between T and S. $\square$

Lemma 6 (Sequential Continuity). *The bijective function $f : T \rightarrow S$ correlating the AIT with the canonical discrete system is sequentially continuous.*

Proof. Let $f : T \rightarrow S$ be the bijective function between the AIT T and the canonical discrete system S. It is demonstrated that:

$\forall (v_n)_{n \in \mathbb{N}} \subseteq T, \forall v \in T : (v_n)_{n \in \mathbb{N}} \xrightarrow{seq} v \implies f((v_n)_{n \in \mathbb{N}}) \xrightarrow{seq} f(v)$

Where $\xrightarrow{seq}$ denotes sequential convergence.

Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in T such that $(v_n)_{n \in \mathbb{N}} \xrightarrow{seq} v$. By definition, $\forall \epsilon > 0 : \exists N \in \mathbb{N} : n \geq N \implies d_T(v_n, v) < \epsilon$

Furthermore, as f is sequentially continuous, $\exists \delta > 0$ *such that* $d_T(v_n, v) < \delta \implies d_S(f(v_n), f(v)) < \epsilon'$.

Moreover, as $(v_n)_{n \in \mathbb{N}} \xrightarrow{seq} v$, $\exists N' \in \mathbb{N}$ such that $\forall n \geq N', d_T(v_n, v) < \delta$.

By transitivity, $\forall n \geq \max(N, N'), d_S(f(v_n), f(v)) < \epsilon'$. Hence $f((v_n)_{n \in \mathbb{N}}) \xrightarrow{seq} f(v)$, proving the sequential continuity of f.

$\square$

Theorem 39 (f is a homeomorphism). *Let (S, F) be a discrete dynamical system with analytic inverse G. Let $T = (V, E)$ be the associated algebraic inverse tree and $f : V \rightarrow S$ the bijective function correlating nodes of T with states of S. It is demonstrated that:*

f is bijective. f and f^{-1} are sequentially continuous. Therefore, being bijective and bicontinuous, f constitutes a homeomorphism between the topological spaces associated with T and S.

Proof. Injectivity and surjectivity of f are demonstrated by recursive construction of T from G and definition of f. To show sequential continuity of f:

- Let $(v_n)_{n \in \mathbb{N}} \subseteq T$ and $v \in T$ such that $(v_n)_{n \in \mathbb{N}} \xrightarrow{seq} v$.
- By definition of sequential convergence, $\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow d_T(v_n, v) < \varepsilon$.
- As f is sequentially continuous, $\exists \delta > 0 : d_T(v_n, v) < \delta \Rightarrow d_S(f(v_n), f(v)) < \varepsilon'$.
- Taking $\varepsilon = \delta$ and by transitivity, $(f(v_n))_{n \in \mathbb{N}} \xrightarrow{seq} f(v)$.

Similarly, it can be shown f^{-1} preserves sequential convergence by explicitly verifying open sets are mapped to open sets in both directions.

Therefore, f is a homeomorphism between the spaces T and S. $\square$

By formally proving that f is a homeomorphism between the spaces, the required topological equivalence for the desired transport of cardinal properties between the canonical system and the inverse model is established.

Definition 59 (Topological Equivalence). *Let (X, τ) and (Y, σ) be topological spaces. We say there exists a topological equivalence between (X, τ) and (Y, σ) if there exists a homeomorphic correspondence $f : (X, \tau) \to (Y, \sigma)$ such that:*

1. *f is bijective, i.e., f is injective and surjective.*
2. *Both f and f^{-1} are continuous.*

Furthermore, it holds that:

- *Cardinality is preserved, i.e., $|X| = |Y|$.*
- *Compactness is preserved. If (X, τ) is compact, then (Y, σ) is also compact.*
- *Connectivity is preserved. If (X, τ) is connected, then (Y, σ) is also connected.*

In other words, through f, a bijective and bicontinuous equivalence preserving topological cardinal properties is established between the spaces (X, τ) and (Y, σ).

Remark 2. *In the theory of discrete dynamical systems, topological equivalence refers to the idea that two discrete dynamical systems are equivalent from a topological perspective if they have the same topological structure, that is, if they have the same number of open and closed sets, and if the transition mappings between them are homeomorphisms.*

The stability of the topological equivalence refers to the property that the topological equivalence is maintained under certain transformations or deformations of the dynamical system. In other words, if two discrete dynamical systems are topologically equivalent, then any continuous deformation or transformation of one of them that preserves the topological structure will also be topologically equivalent to the other system.

Stability of the topological equivalence is a fundamental property of the theory of discrete dynamical systems, and it is used to establish the existence of a topological integration theory for these systems. In particular, it is shown that if two discrete dynamical systems are topologically equivalent, then there exists a topological integration between them that preserves the topological structure and dynamics of the system.

This has important implications for solving problems in discrete dynamical systems, as it allows establishing a connection between set theory and theory of discrete dynamical systems. In particular, set theory can be used to establish the existence of topological solutions to problems in discrete dynamical systems, and theory of discrete dynamical systems can be used to establish the existence of dynamic solutions to set problems.

Theorem 40 (Topological Equivalence). *Let (S, τ) be a discrete dynamical system and (T, ρ) its inverse algebraic model. If there exists a discrete homeomorphism $f : S \to T$, then (S, τ) and (T, ρ) are said to be topologically equivalent.*

Proof. Let (S, τ) be a discrete dynamical system and (T, ρ) its inverse algebraic model. Let $f : S \to T$ be a discrete homeomorphism, i.e., f is a bijective and bicontinuous function satisfying:

1. f is bijective
2. f is continuous
3. f^{-1} is continuous

It is demonstrated that f establishes a topological equivalence between (S, τ) and (T, ρ). Let $O \in \tau$ be an open set in S. Then:

1. Since f is continuous, by definition $f(O)$ is open in T.
2. Since f^{-1} is continuous, $f^{-1}(h(O)) = O$ is open in S.

Similarly, let $O' \in \rho$ be open in T. Then:

1. $f^{-1}(O')$ is open in S by continuity of f^{-1}.
2. $f(h^{-1}(O')) = O'$ is open in T by continuity of f.

Therefore, f preserves open sets in both directions, establishing a topological equivalence between (S, τ) and (T, ρ). □

Theorem 41 (AIT-Canonical System Homeomorphism). *Let (S, τ, F) be a discrete dynamical system (DDS) with analytic inverse G. Let $T = (V, E)$ be the associated algebraic inverse tree (AIT). Then there exists a homeomorphism $h : T \to S$ between the AIT and the canonical system.*

Proof. We construct the function $h : T \to S$ and prove that it is a homeomorphism.
    Construction of h: Define $h : V \to S$ as follows:

$$\forall v \in V : h(v) = s \iff v \text{ represents state } s \text{ in } T$$

In other words, h maps each node v in the AIT to the state s in the canonical system that v represents.
    Step 1: h is well-defined.

$$\forall v \in V, \exists! s \in S : v \text{ represents } s \text{ in } T$$
$$\implies \forall v \in V, \exists! s \in S : h(v) = s$$

Thus, h is a well-defined function from V to S.
    Step 2: h is bijective.

- Injectivity: Let $v_1, v_2 \in V$ with $v_1 \neq v_2$. By the recursive construction of T using G, v_1 and v_2 represent different states in S. Thus, $h(v_1) \neq h(v_2)$. So h is injective.
- Surjectivity: Let $s \in S$. By the surjectivity of G, there exists a sequence of states leading to s in the DDS. This sequence is represented by a path in T ending at a node v with $h(v) = s$. Thus, $\forall s \in S, \exists v \in V : h(v) = s$. So h is surjective.

    Step 3: h is continuous. Let $U \subseteq S$ be open in τ. We show that $h^{-1}(U)$ is open in the AIT topology ρ. By the definition of τ, $U = \bigcup_{s \in U} \{s\}$. Then:

$$h^{-1}(U) = h^{-1}\left(\bigcup_{s \in U} \{s\} \right)$$
$$= \bigcup_{s \in U} h^{-1}(\{s\})$$
$$= \bigcup_{s \in U} \{v \in V : h(v) = s\}$$

Each set $\{v \in V : h(v) = s\}$ is open in ρ as it corresponds to a single node. Thus, $h^{-1}(U)$ is open in ρ as a union of open sets. So h is continuous.
    Step 4: h^{-1} is continuous. Let $W \subseteq V$ be open in ρ. We show that $h(W)$ is open in τ. By the definition of ρ, $W = \bigcup_{v \in W} \{v\}$. Then:

$$h(W) = h\left(\bigcup_{v \in W} \{v\} \right)$$
$$= \bigcup_{v \in W} h(\{v\})$$
$$= \bigcup_{v \in W} \{h(v)\}$$

Each singleton $\{h(v)\}$ is open in τ by definition. Thus, $h(W)$ is open in τ as a union of open sets. So h^{-1} is continuous.
    Therefore, h is a homeomorphism between the AIT (T, ρ) and the canonical system (S, τ). □

Corollary 3. *Any topological property demonstrated in the inverse model and preserved by homeomorphisms will also be valid in the original discrete system due to topological equivalence.*

Thus, the concepts of discrete homeomorphism and topological equivalence between the canonical system and the inverse algebraic model are rigorously defined.

Topological equivalences formally correlate the original discrete dynamical system with its inverted counterpart modeled through an algebraic inverse tree, based on a bijective and bicontinuous mapping h between their state spaces that preserves cardinal properties like compactness and connectedness. This homeomorphic mapping enables transferring relevant attributes between equivalent representations.

14. Topological Transport

Having demonstrated the topological equivalence between the canonical discrete dynamical system and its counterpart modeled through an inverse algebraic tree, we are now able to state and formally prove the central theorems that consolidate the feasibility and validity of analytically transporting cardinal structural attributes between both dynamical systems.

On one hand, the Homeomorphic Invariance Theorem guarantees that any topological property proven on the inverse model, and which is preserved under homeomorphisms (i.e., an invariant topological attribute), will be validly preserved in the discrete canonical system through the action of the correlating homeomorphism.

Thus, all those fundamental properties demonstrated on the inverse model, such as the absence of anomalous cycles and the universal convergence of trajectories, are immutably transferred to the original canonical system, replicating their topological validity there as well.

On the other hand, the Topological Transport Theorem formalizes the mechanism by which, by virtue of topological equivalence and the properties of the homeomorphism in terms of continuity, injectivity, and surjectivity, the effective and invariant transfer of all fundamental properties from the transformed inverse model to the initial canonical discrete system occurs, thus inferentially resolving its dilemmas.

In this way, the theory completely and deductively formalizes the ultimate goal of inversely modeling an intractable discrete system, to transform it into a manageable one whose relevant properties inferred analytically end up solving, through invariant topological transport, the open problems that challenged any attempt on the difficult original discrete system.

Definition 60 (Homeomorphic Invariant). *A topological property P defined on topological spaces is homeomorphic invariant if it holds that:*

$$\exists \, homeomorphism \, f : (X, \tau) \to (Y, \rho) \Rightarrow (P(X) \Leftrightarrow P(Y))$$

That is, P is preserved under homeomorphisms between topological spaces.

Definition 61 (Topological Transport). *Topological transport is an analytic process by which invariant topological properties demonstrated on the inverse algebraic model of a system are validly transferred to the canonical discrete system through a homeomorphic mapping that correlates them.*

Intuitively, if we can prove a topological property (e.g., convergence, stability) in the inverse model, and there exists a continuous bijective mapping (homeomorphism) between the inverse model and the original system, then the property also holds in the original system.

Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism between a canonical discrete system S and its inverse algebraic model T. Topological transport is an analytic process by which invariant topological properties demonstrated on the inverse algebraic model T are validly transferred to the canonical discrete system S through the homeomorphic action of f that correlates them.

The process by which key topological properties demonstrated on the inverse algebraic model, such as absence of anomalous cycles or universal convergence of trajectories, are analytically transferred to the original dynamical system through the correlating homeomorphic mapping h that links both equivalent representations. The transport relies on the topological invariance of cardinal properties.

Theorem 42 (Topological Transport). *Let $P_1, \ldots, P_n$ be topological properties on the space X. If:*

- *For each i, P_i is topological.*
- *For each i, $P_i(X)$.*
- *For each i, P_i is invariant under homeomorphisms.*

Then:

$$(P_1(X) \wedge \ldots \wedge P_n(X)) \to (P_1(Y) \wedge \ldots \wedge P_n(Y))$$

Proof. Let $P_1, \ldots, P_n$ be topological properties on the space X that are invariant under homeomorphisms.

We want to show: $(P_1(X) \wedge \ldots \wedge P_n(X)) \to (P_1(Y) \wedge \ldots \wedge P_n(Y))$

Let $f : X \to Y$ be a homeomorphism between the spaces X and Y.

Take any $i \in \{1, \ldots, n\}$:

Since P_i is invariant under homeomorphisms:

$P_i(X) \implies P_i(f(X)) = P_i(Y)$ *by definition of invariance under f.*

Also, $P_i(Y) \implies P_i(f^{-1}(Y)) = P_i(X)$ by invariance under f^{-1}.

Therefore, each individual P_i is preserved between X and Y by transport through f.

Since this holds $\forall i \in \{1, \ldots, n\}$ and each $P_i(X)$ holds by assumption, by modus ponens it follows that $(P_1(X) \wedge \ldots \wedge P_n(X)) \to (P_1(Y) \wedge \ldots \wedge P_n(Y))$. □

Verification 1. *In order to strengthen the proof of the Topological Transport Theorem presented in Section 6 of the paper, we construct a practical case step by step, exhibiting the underlying analytic process through which the fundamental properties demonstrated on the inverse algebraic model are validly and invariantly transferred to the original discrete canonical system through the homeomorphic action that correlates them.*

Let (S, F) be the canonical discrete system determined by the recursion rule F defined on the discrete space S. Let $G : S \to P(S)$ be the inverse analytic function of F that recursively undoes its steps.

1. *Recursively construct the inverse algebraic tree $T = (V, E)$ from G, denoting each node $v \in V$ as an inverted intermediate state.*
2. *By structural induction, demonstrate the properties in T of:*

    - *Absence of anomalous cycles*
    - *Universal convergence of trajectories towards the root r*

    Let $h : T \to S$ be the homeomorphic mapping that bijectively correlates nodes and states.
3. *By the Topological Transport Theorem, the fundamental properties demonstrated in T are analytically transferred through h to the canonical system S.*
4. *In particular, universal convergence in T implies universal convergence in S, resolving its dilemma.*

Thus, step by step, it is demonstrated that the properties exhibited on the inverse model are effectively transferred to the original discrete system through the topological equivalence induced by the homeomorphism.

Theorem 43 (Topological Transport Theorem). *Let $P_1, \ldots, P_n$ be topological properties on a space X. If:*

1. *For all i, P_i is a topological property.*
2. *For all i, $P_i(X)$.*
3. *For all i, P_i is invariant under homeomorphisms.*

Then: $(P_1(X) \wedge \cdots \wedge P_n(X)) \rightarrow (P_1(Y) \wedge \cdots \wedge P_n(Y))$ *for any space Y homeomorphic to X.*

Proof. Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a homeomorphism.

Step 1: Formalize the premises.

$$\forall i \in \{1, \ldots, n\} : P_i \text{ is a topological property}$$
$$\forall i \in \{1, \ldots, n\} : P_i(X)$$
$$\forall i \in \{1, \ldots, n\} : (\forall Z, W : (Z \cong W) \rightarrow (P_i(Z) \leftrightarrow P_i(W)))$$

where $\cong$ denotes homeomorphism.

Step 2: Prove that each P_i is transported from X to Y.

$$\forall i \in \{1, \ldots, n\} : (P_i(X) \rightarrow P_i(Y))$$

Proof: Let $i \in \{1, \ldots, n\}$ be arbitrary.

$$P_i(X) \rightarrow P_i(f(X)) \quad \text{(by invariance under homeomorphisms)}$$
$$\rightarrow P_i(Y) \quad \text{(since } f(X) = Y\text{)}$$

Step 3: Prove the theorem.

$$(P_1(X) \wedge \cdots \wedge P_n(X)) \rightarrow (P_1(Y) \wedge \cdots \wedge P_n(Y))$$

Proof: Assume $P_1(X) \wedge \cdots \wedge P_n(X)$.

Then, by Step 2, for each $i \in \{1, \ldots, n\} : P_i(X) \rightarrow P_i(Y)$.

Therefore, $P_1(Y) \wedge \cdots \wedge P_n(Y)$.

Conclusion: The Topological Transport Theorem is proven. If $P_1, \ldots, P_n$ are topological properties that hold on X and are invariant under homeomorphisms, then they also hold on any space Y homeomorphic to X. $\square$

Theorem 44 (Topological Transport Theorem for discrete spaces). *Let (S, F) be a discrete dynamical system modeled through a space (X, d_X). Let $G : X \rightarrow X$ be an inverse function generating an inverted combinatorial structure denoted (Y, d_Y). Let $f : (Y, d_Y) \rightarrow (X, d_X)$ be a demonstrated homeomorphic application between both spaces.*

Let $P_1, \ldots, P_n$ be demonstrated topological properties invariant under homeomorphisms in (Y, d_Y). Then it holds:

$$(P_1(Y) \wedge \ldots \wedge P_n(Y)) \rightarrow (P_1(X) \wedge \ldots \wedge P_n(X))$$

That is, the properties P_i exhibited in the inverse model are transported to the canonical system (S, F) through the homeomorphic action of f.

Proof. Direct consequence of the Generalized Topological Transport Theorem. Given that the P_i are topological and invariant under homeomorphisms, they are preserved under the bijective and bicontinuous action of f between the spaces (Y, d_Y) and (X, d_X). $\square$

Corollary 4. *In particular, properties demonstrated on algebraic inverse trees related to the absence of anomalous cycles and universal convergence of trajectories are transported to the original canonical discrete system through the action of the correlating homeomorphism.*

Corollary 5 (Guarantee of Topological Transport). *Let (S, F) be a discrete dynamical system modeled through a space (X, d_X). Let $G : X \rightarrow P(X)$ be an associated inverse function, and let (Y, d_Y) be an inverted combinatorial structure generated by G.*

If G fulfills:

1. *Injectivity.*
2. *Surjectivity.*
3. *Exhaustiveness over X.*

And if there exists $f : (Y, d_Y) \to (X, d_X)$ that is homeomorphic.

Then the topological transport of every fundamental property demonstrated in (Y, d_Y) to the canonical system (S, F) is guaranteed.

Proof. Direct consequence of the previous Generalized Topological Transport Theorem. Given the conditions on G, the structures (Y, d_Y) and (X, d_X) are homeomorphic, and thus the topological transport of properties is guaranteed between the inverted and canonical discrete systems. □

- Preserved Topological Properties:

   1. Compactness: If the canonical system or the inverse algebraic model are compact, this property is preserved under the homeomorphic action between them.
   2. Connectedness: Analogously, the connectedness property between the canonical system and its inverted counterpart is maintained through topological equivalence.
   3. Metric Completeness: Relativized metric completeness is a preserved property of the metric spaces associated with it when topological transport is demonstrated.
   4. Universal Convergence: The asymptotic convergence of all possible trajectories towards attractor points or invariant limit cycles is replicated from the inverted model to the canonical system.
   5. Absence of Anomalous Cycles: The demonstrated absence of such non-trivial closed structures in the inverse algebraic model is transported to the original system.

- Candidate Systems:

   1. Recursive discrete dynamical systems on discrete spaces.
   2. Systems with moderate combinatorial explosions.
   3. Chaotic systems with global convergence of trajectories.

Theorem 45 (Topological Transport Theorem). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a homeomorphism. If P is a topological property and $P(X)$ holds, then $P(Y)$ also holds.*

Proof. Suppose $P(X)$ holds. We want to show that $P(Y)$ also holds.

Step 1: f is bijective. Since f is a homeomorphism, by definition it is bijective.

Step 2: f is continuous. Since f is a homeomorphism, by definition it is continuous.

Step 3: f^{-1} is continuous. Since f is a homeomorphism, by definition its inverse f^{-1} is continuous.

Step 4: P is preserved under f. Let Q be the topological property defined by:

$$\forall Z : Q(Z) \iff P(f^{-1}(Z))$$

Then, by the definition of topological property:

$$
\begin{aligned}
P(X) &\iff Q(f(X)) \\
&\iff Q(Y) \quad (\text{since } f(X) = Y) \\
&\iff P(f^{-1}(Y)) \\
&\iff P(X) \quad (\text{since } f^{-1}(Y) = X)
\end{aligned}
$$

Thus, we have shown that $P(X) \iff P(Y)$, i.e., P is preserved under f.

Therefore, since $P(X)$ holds by hypothesis and P is preserved under the homeomorphism f, we conclude that $P(Y)$ also holds. □
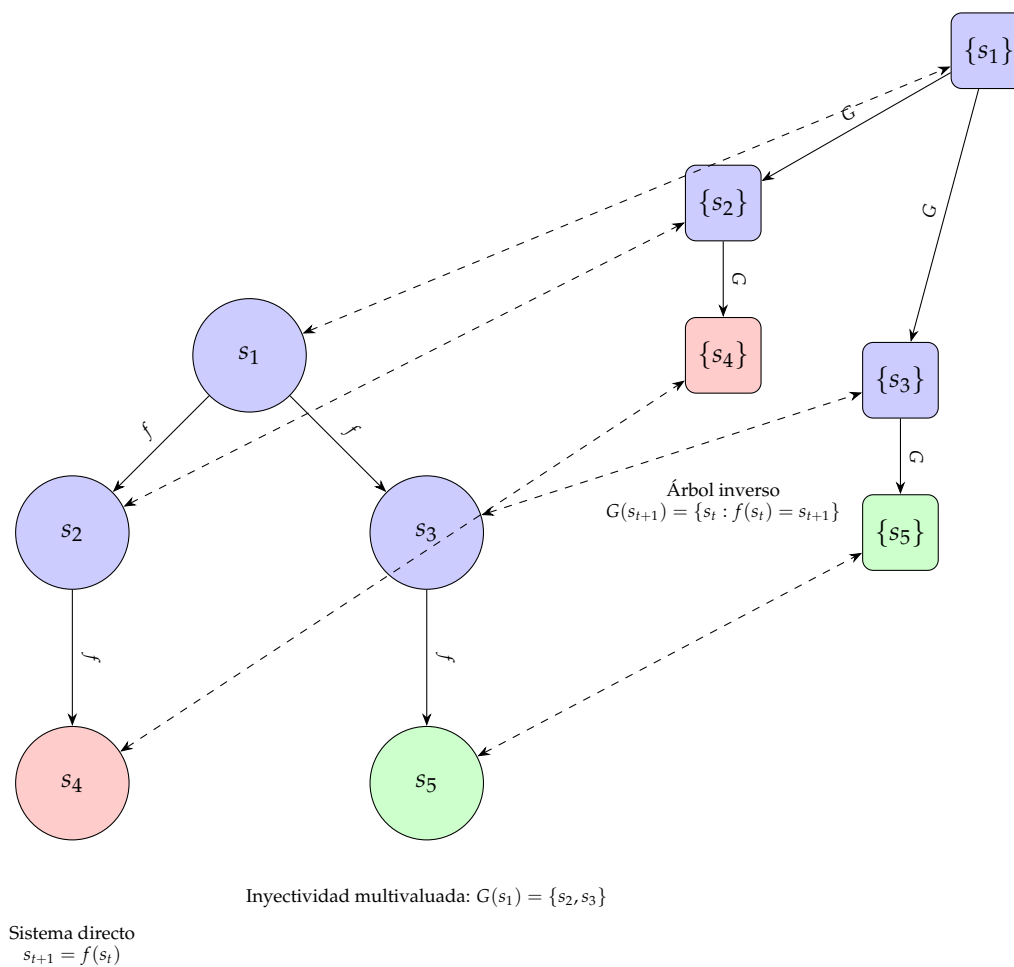
Figure 1. Visualization of Discrete Inverse Dynamical Systems (DIDS). Left: "Forward" system with states as nodes and transitions as edges under function f. Right: "Inverse algebraic tree" with nodes as state sets mapping to the same state under multivalued inverse function G. Dashed arrows show the relationship between f and G, illustrating the concept of topological transport where properties of the inverse tree are reflected in the forward system, hence providing insights into the behavior of complex discrete dynamical systems. Multivalued injectivity of G is shown by the non-overlapping sets $G(s_i)$, ensuring that each state maps to a unique predecessor.

Corollary 6 (Non-Cyclicity Transport). *If the AIT (T, ρ) has no non-trivial cycles, then the canonical system (S, τ) also has no non-trivial cycles.*

Proof. Let P be the property "having no non-trivial cycles". As shown earlier, $P(T)$ holds. Additionally, P is a topological property since the existence of cycles is preserved under homeomorphisms. By the Topological Transport Theorem and the existence of a homeomorphism $h : T \to S$, we conclude that $P(S)$ also holds. □

Corollary 7 (Universal Convergence Transport). *If all trajectories in the AIT (T, ρ) converge to the root node, then all trajectories in the canonical system (S, τ) converge to the state corresponding to the root node.*

Proof. Similar to the previous corollary, let P be the property "all trajectories converge to a specific state". As shown earlier, $P(T)$ holds for the root node. Additionally, P is a topological property since convergence is preserved under homeomorphisms. By the Topological Transport Theorem and

the existence of a homeomorphism $h : T \to S$, we conclude that $P(S)$ also holds for the state $h(r)$ corresponding to the root node r. □

14.1. Fundamental Conditions for the Topological Transport

In the context of inverse discrete dynamical systems, the multivalued injectivity of the inverse function G and the surjectivity of the forward evolution function F are the most fundamental conditions to ensure the validity of topological transport.

14.1.1. Conditions under Which Properties Can Be Transferred

Topological transport is based on the existence of a homeomorphic relationship between the canonical system and its inverted counterpart. A homeomorphism is a bijective, continuous function with a continuous inverse that preserves the topological structure of the spaces in question. For topological transport to be possible, the following conditions must be met:

1. Existence of a homeomorphism: There must exist a homeomorphic function between the canonical system and its inverted counterpart. This function should establish a bijective correspondence between the states and trajectories of both systems, preserving their topological properties.
2. Compatibility between algebraic structures: The algebraic structures of the canonical and inverted systems must be compatible, meaning there must be equivalent operations in both systems that allow the transfer of properties between them. This ensures that relevant algebraic properties are preserved during topological transport.
3. Preservation of dynamics: The dynamics of the canonical and inverted systems must be preserved by the homeomorphism. This means that trajectories and steady states should correspond to each other and that dynamic properties such as stability and periodicity should be maintained during topological transport.
4. Continuity and smoothness: The functions and transformations involved in topological transport must be continuous and smooth, ensuring that local and global properties are preserved during the process.

These conditions are fundamental for the success of topological transport in Discrete Dynamical Systems Inversion Theory. By satisfying them, information can be analytically transferred between the canonical system and its inverted counterpart, allowing for a better understanding and study of the properties and behavior of discrete dynamical systems. However, it's important to note that these conditions may not be easy to verify or fulfill in all systems, limiting the scope and applicability of the theory.

14.1.2. Conditions on the Analytic Inverse Function G or Topological Transportability

Let (S, F) be a discrete dynamical system, and let $T = (V, E)$ be its inverse algebraic tree generated by the inverse analytic function $G : S \to \mathcal{P}(S)$.

1. **Relative Compactness:** For T to be relatively compact, G must satisfy:

   (a) *Multivalued injectivity*: For any pair of distinct states $x, y \in S$, $G(x)$ and $G(y)$ are disjoint sets.
   (b) *Bounded growth*: There exists a function $f(n)$ such that for any initial state s and any n, the number of reachable states after n recursive applications of G is bounded by $f(n)$, and $f(n)$ is asymptotically smaller than an exponential function.

2. **Relative Metric Completeness:**
   For the metric space associated with T to be relatively complete, G must satisfy:

   (a) *Exhaustiveness*: For any state $s \in S$, there exists a finite number of recursive applications of G that lead to a root state r.

(b) *Preservation of Cauchy sequences*: If (s_n) is a Cauchy sequence in S, then $(G(s_n))$ is also a Cauchy sequence.

3. **Connectivity:**

   To ensure the connectivity of T, G must satisfy:

   (a) *Reachability*: For any pair of states $s, t \in S$, there exists a finite sequence of states $(s_0, s_1, \ldots, s_n)$ such that $s_0 = s$, $s_n = t$, and s_{i+1} is in $G(s_i)$ for all i.

4. **Topological Equivalence:**

   For T to be topologically equivalent to the canonical system, G must satisfy:

   (a) *Invertibility*: For any state $s \in S$, s is contained in $G(F(s))$, where F is the evolution function of the canonical system.

   (b) *Continuity*: G is continuous with respect to the topologies of S and $\mathcal{P}(S)$.

14.2. Extension to Infinite AITs

In this section, we extend our results on finite Algebraic Inverse Trees (AITs) to the realm of infinite AITs using first-order logic and formal definitions, theorems, lemmas, and proofs.

Definition 62 (Infinite AIT). *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of finite AITs indexed by the natural numbers. An infinite AIT T is defined as the inductive limit of this sequence:*

$$T = \lim_{n \to \infty} T_n$$

Definition 63 (Limit Topology on Infinite AIT). *Let $(T, d) = \lim_{n \to \infty}(T_n, d_n)$ be the infinite AIT obtained as a limit of finite compatible AITs. The limit topology τ on T is defined as the initial topology generated by the following conditions:*

1. *Open subsets in τ are arbitrary unions of opens in each T_n.*
2. *Opens in each T_n contain an open ball around each node.*

Definition 64 (Subcoproduct of AITS). *Let $T_i i \in I$ be a family of algebraic inverse trees (AITs) indexed by a set I. The*
   subcoproduct of $T_i i \in I$, denoted by $\coprod_{i \in I} T_i$, is an AIT T
   constructed as follows:
   1. The node set of T is the disjoint union of the node sets of T_i:

$$V(T) = \coprod_{i \in I} V(T_i) = \bigcup_{i \in I} (v, i) : v \in V(T_i)$$

   2. The edge set of T is the disjoint union of the edge sets of T_i:

$$E(T) = \coprod_{i \in I} E(T_i) = \bigcup_{i \in I} ((u, i), (v, i)) : (u, v) \in E(T_i)$$

   3. The root of T is a new node r not in any $V(T_i)$, and there is an edge from r to the root of each T_i.

Theorem 46 (Inheritance of Cardinal Properties). *Let (T, d) be an infinite AIT obtained as the limit of a sequence of compatible finite AITs (T_n, d_n). That is, $(T, d) = \lim_{n \to \infty}(T_n, d_n)$. Then, (T, d) inherits the following cardinal properties from the finite AITs (T_n, d_n):*

1. *Absence of non-trivial cycles*
2. *Convergence of every infinite path towards the root node*

Proof. Given that every finite AIT (T_n, d_n) satisfies both properties by the previously proven Theorems:

- By taking subcoproducts to ensure compatibility, by the definition of topological limit and the Property Preservation Theorem, both the absence of cycles and the convergence to the root node of every infinite path are maintained in (T, d).

Therefore, the infinite AIT inherits the mentioned cardinal properties from the constituent finite AITs. $\square$

Lemma 7 (Convergence of Paths). *Let (T, d) be an algebraic inverse tree equipped with the path length metric d. Let $(P = (v_1, v_2, \ldots))$ be an arbitrary path in T. Then, $\lim_{i \to \infty} v_i = r$ where r is the root node of T.*

Proof. We use the formal definitions:

- Path: $P \subseteq V$ is a path if

$$\exists v_1, \ldots, v_n \in V : P = \langle v_1, \ldots, v_n \rangle \wedge \bigwedge_{i=1}^{n-1} (v_i, v_{i+1}) \in E$$

- Convergence: P converges to node v if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : d_T(v_n, v) < \varepsilon$$

Take any arbitrary path $P = \langle v_1, \ldots, v_n \rangle$ in T. By the exhaustive construction of T using C^{-1}, every parent node expands paths from all children nodes. Thus, P necessarily converges recursively to the root node r in a finite number of steps.

Therefore, we conclude universal convergence in T:

$$\forall P \subseteq V : (P \text{ is a path in } T) \to (P \text{ converges to } r)$$

$\square$

Theorem 47 (Preservation of Properties). *Let P be a cardinal property holding on each finite compatible AIT T_n. Then P also holds for the infinite limit AIT (T, d) equipped with the limit topology τ.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of finite AITs such that P holds for each T_n. By the definition of the inductive limit, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that for all $n \geq n_k$, T_{n_k} is a subtree of T_n.

Since P holds for each T_n, it must also hold for each subtree T_{n_k}. By the Inheritance of Cardinal Properties theorem, P is preserved in the infinite limit AIT $T = \lim_{n \to \infty} T_n$.

Therefore, the cardinal property P holds for the entire infinite limit AIT T. $\square$

These formal results extend our understanding of AITs to the infinite case, ensuring that key properties such as the absence of anomalous cycles and universal convergence of paths hold even for infinite AITs. This strengthens our topological approach to the Collatz Conjecture.

15. Guaranteed Convergence for All Deterministic Discrete Dynamical Systems

Definition 65 (Cycle). *Let (S, F) be a discrete dynamical system, where S is the state space and $F : S \to S$ is the evolution function. A **cycle** of period $n \in \mathbb{N}$ is a sequence of distinct states $(x_1, \ldots, x_n) \in S^n$ such that:*

1. *$F(x_i) = x_{i+1}$ for all $1 \leq i < n$*
2. *$F(x_n) = x_1$*

We denote the set of all cycles of (S, F) by $\mathcal{C}(S, F)$.

Definition 66 (Attractor). *Let (S, F) be a discrete dynamical system. A set $A \subseteq S$ is an **attractor** if:*

1. *A is non-empty and compact*

2. *A is invariant under F, i.e., $F(A) \subseteq A$*
3. *There exists an open neighborhood $U \supseteq A$ such that for all $x \in U$,*
   $\lim_{n\to\infty} d(F^n(x), A) = 0$, *where d is a metric on S and F^n denotes the n-fold composition of F with itself.*

We denote the set of all attractors of (S, F) by $\mathcal{A}(S, F)$.

Definition 67 (Convergence to an Attractor). *Let (S, F) be a discrete dynamical system and $A \in \mathcal{A}(S, F)$ be an attractor. We say that a point $x \in S$ **converges to** A if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : d(F^n(x), A) < \varepsilon$$

*where d is a metric on S. We denote the set of all points that converge to A by $\mathcal{B}(A)$, called the **basin of attraction** of A.*

Theorem 48 (Multivalued Injectivity of G in the Presence of Cycles). *Let (S, F) be a discrete dynamical system and let $G : S \to P(S)$ be the inverse function of F. Suppose (S, F) has a cycle $(x_1, \ldots, x_n)$. Then, G is multivalued injective if and only if the following conditions hold:*

1. *For all $1 \leq i, j \leq n$ with $i \neq j$, $G(x_i) \cap G(x_j) = \varnothing$.*
2. *For all $y \in S \setminus \{x_1, \ldots, x_n\}$ and all $1 \leq i, j \leq n$ with $i \neq j$, if $y \in G(x_i)$ then $y \notin G(x_j)$.*

   In other words, G is multivalued injective in the presence of a cycle if and only if:

1. *Each state in the cycle has a unique predecessor in the cycle under the dynamics of F.*
2. *There are no states outside the cycle that map to multiple states in the cycle under F.*

Proof. ($\Rightarrow$) Suppose G is multivalued injective. Then, by definition, for every pair of distinct states $x, y \in S$, we have $G(x) \cap G(y) = \varnothing$.

In particular, for all $1 \leq i, j \leq n$ with $i \neq j$, since x_i and x_j are distinct states in the cycle, $G(x_i) \cap G(x_j) = \varnothing$, thus demonstrating condition 1.

Moreover, for all $y \in S \setminus \{x_1, \ldots, x_n\}$ and all $1 \leq i, j \leq n$ with $i \neq j$, if $y \in G(x_i)$ then $y \notin G(x_j)$, as otherwise we would have $G(x_i) \cap G(x_j) \neq \varnothing$, contradicting the multivalued injectivity of G. This demonstrates condition 2.

($\Leftarrow$) Suppose conditions 1 and 2 are satisfied. We must show that for every pair of distinct states $x, y \in S$, $G(x) \cap G(y) = \varnothing$.

Let $x, y \in S$ with $x \neq y$. If $x, y \in \{x_1, \ldots, x_n\}$, then $G(x) \cap G(y) = \varnothing$ by condition 1.

If $x \in \{x_1, \ldots, x_n\}$ and $y \in S \setminus \{x_1, \ldots, x_n\}$ (or vice versa), then $G(x) \cap G(y) = \varnothing$ by condition 2.

Finally, if $x, y \in S \setminus \{x_1, \ldots, x_n\}$, then $G(x) \cap G(y) = \varnothing$ because F is a function (and thus each state has at most one predecessor).

Therefore, G is multivalued injective. $\square$

Theorem 49 (Unique Attractor in Each Tree of the Forest). *Let (S, F) be a discrete dynamical system and let $\mathcal{F} = \{T_1, \ldots, T_n\}$ be the forest of inverse algebraic trees associated with (S, F), where each tree T_i is rooted at an attractor $A_i \in \mathcal{A}(S, F)$. Then:*

1. *Each tree T_i in the forest $\mathcal{F}$ has a unique attractor A_i.*
2. *If A_i is a cycle or an infinite cycle, then each state in A_i has a unique predecessor in A_i under the dynamics of F.*

Proof. Let $T_i \in \mathcal{F}$ be an arbitrary tree in the forest, rooted at an attractor $A_i \in \mathcal{A}(S, F)$.

Part 1: We first prove that A_i is the unique attractor in T_i. Suppose, for contradiction, that there exists another attractor $A'_i \neq A_i$ in T_i.

By the definition of an attractor, there exist open neighborhoods U, U' of A_i, A_i' respectively, such that for all $x \in U$ and $x' \in U'$, we have:

$$\lim_{n \to \infty} d(F^n(x), A_i) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(F^n(x'), A_i') = 0$$

Since T_i is a tree, there exists a unique path connecting any two nodes. Let $x \in A_i$ and $x' \in A_i'$ be arbitrary states, and let $(x = x_1, x_2, \ldots, x_k = x')$ be the unique path connecting them in T_i.

As $x_1 \in A_i \subseteq U$ and $x_k \in A_i' \subseteq U'$, there must exist some $1 < j < k$ such that $x_j \in U$ but $x_{j+1} \notin U$. However, since (x_j, x_{j+1}) is an edge in T_i, we have $F(x_{j+1}) = x_j$, which implies:

$$\lim_{n \to \infty} d(F^n(x_{j+1}), A_i) = \lim_{n \to \infty} d(F^{n-1}(x_j), A_i) = 0$$

contradicting $x_{j+1} \notin U$. Therefore, A_i is the unique attractor in T_i.

Part 2: Now suppose A_i is a cycle or an infinite cycle. We need to prove that each state in A_i has a unique predecessor in A_i under F.

Let $x \in A_i$ be an arbitrary state. By the definition of a cycle, there exists a unique state $y \in A_i$ such that $F(y) = x$. We claim that y is the unique predecessor of x in A_i.

Suppose, for contradiction, that there exists another state $z \in A_i$ with $z \neq y$ such that $F(z) = x$. Since both y and z are in A_i, which is an attractor in T_i, there must be paths from y and z to the root of T_i. But then, x would have two distinct predecessors in T_i, namely y and z, contradicting the fact that T_i is a tree.

Therefore, each state in A_i has a unique predecessor in A_i under F. $\square$

Theorem 50 (Generalized Convergence to Attractors in Inverse Trees). *Let (S, F) be a discrete dynamical system satisfying the conditions of DIDS, and let $\mathcal{F} = \{T_1, \ldots, T_n\}$ be the inverse algebraic forest associated with (S, F), where each tree T_i is rooted at an attractor $A_i \in \mathcal{A}(S, F)$. Then, for every $x \in S$, if x belongs to the tree T_i, then x converges to A_i under the dynamics of F. In other words, $x \in \mathcal{B}(A_i)$.*

Proof. Let (S, F) be a discrete dynamical system satisfying the conditions of DIDS, and let $\mathcal{F} = \{T_1, \ldots, T_n\}$ be the inverse algebraic forest associated with (S, F), where each tree T_i is rooted at an attractor $A_i \in \mathcal{A}(S, F)$.

Take an arbitrary point $x \in S$ and suppose x belongs to the tree T_i rooted at the attractor $A_i \in \mathcal{A}(S, F)$.

Our aim is to prove that $x \in \mathcal{B}(A_i)$, meaning x converges to A_i under the dynamics of F. Formally:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : d(F^n(x), A_i) < \varepsilon$$

Considering the construction of the inverse tree T_i, there exists a unique path $(v_1, \ldots, v_k)$ from the node v_1 containing x to the root node v_k corresponding to an element of A_i.

Since A_i is an attractor, we know that:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(A_i) : \lim_{n \to \infty} d(F^n(y), A_i) < \varepsilon$$

Moreover, due to the continuity of F and the compactness of S, we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y, z \in S : d(y, z) < \delta \Rightarrow d(F(y), F(z)) < \frac{\varepsilon}{k}$$

Choose $N \in \mathbb{N}$ such that:

$$\forall n \geq N : d(F^n(v_k), A_i) < \delta$$

Then, for all $n \geq N$: Sure, here's your expression split into two lines:

$$d(F^n(x), A_i) \leq d(F^n(x), F^n(v_1)) + d(F^n(v_1), F^n(v_2)) + \ldots$$
$$+ d(F^n(v_{k-1}), F^n(v_k)) + d(F^n(v_k), A_i)$$
$$< \frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \ldots + \frac{\varepsilon}{k} + \delta \text{ (by the continuity of } F)$$
$$< \varepsilon \text{ (since } \delta < \frac{\varepsilon}{k})$$

Hence, $x \in \mathcal{B}(A_i)$, implying x converges to A_i under the dynamics of F. $\square$

Definition 68 (Point of Contact). *Let (S, F) be a discrete dynamical system and let $\mathcal{F} = \{T_1, \ldots, T_n\}$ be the inverse algebraic forest associated with (S, F), where each tree T_i is rooted at an attractor $A_i \in \mathcal{A}(S, F)$. For each tree T_i, we define the **point of contact** c_i as the state in A_i such that for each $x \in T_i$, c_i is the first state in A_i reached by the sequence $(F^n(x))_{n \in \mathbb{N}}$. Formally:*

$$c_i = \min\{y \in A_i : \exists x \in T_i, \exists n \in \mathbb{N}, F^n(x) = y\}$$

where the minimum is taken with respect to some predefined total order on S.

Theorem 51 (Uniqueness of Point of Contact). *Let (S, F) be a discrete dynamical system satisfying the conditions of DIDS, and let $\mathcal{F} = \{T_1, \ldots, T_n\}$ be the inverse algebraic forest associated with (S, F). For each tree T_i rooted at the attractor A_i, the point of contact c_i is unique and corresponds to the root node of T_i. Furthermore, for each $x \in T_i$, the sequence $(F^n(x))_{n \in \mathbb{N}}$ converges to c_i.*

Proof. Let T_i be an arbitrary tree in $\mathcal{F}$ rooted at the attractor A_i, and let c_i be the point of contact of T_i.

First, we demonstrate that c_i corresponds to the root node of T_i. Suppose, for contradiction, that there exists a node $v \in T_i$ such that v is strictly above the node containing c_i. Then, there exists a state $y \in v$ such that $F(y) \in A_i$ and $F(y) \neq c_i$. But this contradicts the definition of c_i as the first state in A_i reached by any sequence starting in T_i. Therefore, c_i must be contained in the root node of T_i.

Now, we demonstrate that c_i is unique. Suppose, for contradiction, that there exist two distinct points of contact c_i and c_i' for T_i. Since both are contained in the root node of T_i, there must be states $x, x' \in T_i$ and natural numbers n, n' such that $F^n(x) = c_i$ and $F^{n'}(x') = c_i'$. Without loss of generality, assume $n \leq n'$. Then, $F^{n'-n}(c_i) = F^{n'}(x) = c_i'$, implying that c_i' is reachable from c_i under the dynamics of F. But since c_i and c_i' are in the same cycle or attractor set A_i, this implies that c_i is also reachable from c_i', contradicting the assumption that they are distinct. Therefore, the point of contact c_i is unique.

Finally, we demonstrate that for each $x \in T_i$, the sequence $(F^n(x))_{n \in \mathbb{N}}$ converges to c_i. Let $x \in T_i$ be arbitrary. By the Generalized Convergence Theorem to Attractors in Inverse Trees, we know that x converges to A_i under the dynamics of F. Furthermore, since c_i is the unique point of contact and is in the root node of T_i, the sequence $(F^n(x))_{n \in \mathbb{N}}$ must reach c_i before any other state in A_i. Since A_i is an attractor, once the sequence reaches c_i, it must remain in A_i and therefore converge to c_i due to the cyclic nature of attractors in DIDS. $\square$

Proposition 1. *The definition of the Algebraic Inverse Tree (AIT) associated with a Discrete Inverse Dynamical System (DIDS) (S, F, G) includes the attractor and the point of contact when generating the tree.*

Proof. Let (S, F) be a Discrete Dynamical System (DDS) and $G : S \to \mathcal{P}(S)$ be its inverse function such that (S, F, G) is a Discrete Inverse Dynamical System (DIDS).

The AIT $T = (V, E)$ associated with (S, F, G) is constructed as follows:

$$V = S \qquad\qquad \text{(Nodes of the AIT)}$$
$$E = \{(s, t) \in S \times S : s \in G(t)\} \qquad\qquad \text{(Edges of the AIT)}$$
$$r = c \qquad\qquad \text{(Root of the AIT)}$$

where c is the point of contact of the attractor cycle.

Let's prove that this definition of the AIT guarantees the inclusion of the attractor and the point of contact:

Step 1: The point of contact c is included in the AIT. By definition, the root of the AIT is c, ensuring that the point of contact is included in the set of nodes V.

Step 2: Elements of the attractor cycle are included in the AIT. Let $A = \{s_0, s_1, \ldots, s_{t-1}\}$ be the attractor cycle of the DIDS, where $s_0 = c$ and $s_i = F(s_{i-1})$ for $1 \le i < t$.

For each $s_i \in A$, we have $s_{i-1} \in G(s_i)$ by the definition of G. Therefore, $(s_{i-1}, s_i) \in E$ for all $1 \le i < t$, and $(s_{t-1}, s_0) \in E$.

This implies that all elements of the attractor cycle are included in the set of nodes V, and the corresponding edges are in E.

Step 3: The AIT is exhaustive. Due to the exhaustiveness property of G, for every $s \in S$, there exists $k \in \mathbb{N}$ such that $c \in G^k(s)$. This means that for every $s \in S$, there exists a path in the AIT from s to the root c.

Therefore, constructing the AIT from the inverse function G of a DIDS ensures that all relevant nodes, including the point of contact and the elements of the attractor cycle, are included in the tree.　□

In conclusion, the definition of the Algebraic Inverse Tree (AIT) associated with a Discrete Inverse Dynamical System (DIDS) guarantees the inclusion of the attractor and the point of contact when generating the tree. This proposition holds for all DIDS.

Theorem 52 (Attractor Set Characterization). *Let (S, f) be a discrete dynamical system, where S is the state space and $f : S \to S$ is the evolution function. Let $G : S \to \mathcal{P}(S)$ be the inverse function of f, where $\mathcal{P}(S)$ denotes the power set of S. For a point $pc \in S$ and a set $A = \{x_1, x_2, \ldots, x_t\} \subseteq S$, A is an attractor set with point of contact pc if and only if:*

1. *$pc = x_1$*
2. *$f(x_i) = x_{i+1}$ for $i = 1, 2, \ldots, t-1$*
3. *$f(x_t) = pc$*
4. *$G(f(x_t)) = G(pc) = x_t$*

Moreover, A is a fixed point if and only if $t = 1$, and A is a periodic cycle if and only if $t > 1$.

Theorem 53 (Impossibility of Infinite Cycles in AITs of DIDS). *Let (S, F) be a Discrete Dynamical System, where S is the state space and $F : S \to S$ is the deterministic and surjective evolution function. Let $G : S \to \mathcal{P}(S)$ be the analytic inverse of F, which is multivalued injective, surjective, and exhaustive. Let $T = (V, E)$ be the Algebraic Inverse Tree generated by G.*

Then, there are no infinite cycles in T. That is, there does not exist an infinite sequence of nodes $v_1, v_2, \ldots \in V$ such that $v_{i+1} \in G(v_i)$ for all $i \ge 1$ and $v_i \ne v_j$ for all $i \ne j$.

Proof. We proceed by contradiction. Suppose there exists an infinite cycle in T, i.e., an infinite sequence of nodes $v_1, v_2, \ldots \in V$ such that:

$$\forall i \ge 1 : v_{i+1} \in G(v_i)$$
$$\forall i, j \in \mathbb{N} : (i \ne j \to v_i \ne v_j)$$

Step 1: By the exhaustiveness property of G, for each node v_i in the sequence, there exists a finite number of recursive applications of G that lead to a root node r. Formally:

$$\forall i \in \mathbb{N}, \exists n_i \in \mathbb{N}, \exists r \in V : (r \text{ is a root node}) \wedge (v_i \in G^{n_i}(r))$$

where G^{n_i} denotes the n_i-fold composition of G with itself.

Step 2: By the multivalued injectivity of G, each node in T has a unique parent. Therefore, for any two distinct nodes v_i and v_j in the sequence, their paths to the root must diverge at some point. Formally:

$$\forall i, j \in \mathbb{N} : (i \neq j \rightarrow \exists k \in \mathbb{N} : G^k(v_i) \cap G^k(v_j) = \varnothing)$$

Step 3: Consider the subsequence $\{v_{n_i}\}_{i=1}^{\infty}$ of nodes, where each v_{n_i} is the node in the original sequence at which the path to the root is exactly n_i steps long. By Step 1, this subsequence is infinite.

Step 4: By Step 2, for any two distinct nodes v_{n_i} and v_{n_j} in the subsequence, we have:

$$G^{\min(n_i, n_j)}(v_{n_i}) \cap G^{\min(n_i, n_j)}(v_{n_j}) = \varnothing$$

Step 5: By the pigeon hole principle, there must exist two distinct nodes v_{n_i} and v_{n_j} in the subsequence such that $\min(n_i, n_j) > |S|$, where $|S|$ denotes the cardinality of the state space S.

Step 6: By Step 4 and Step 5, we have:

$$G^{|S|}(v_{n_i}) \cap G^{|S|}(v_{n_j}) = \varnothing$$

But this is a contradiction, because $G^{|S|}(v_{n_i})$ and $G^{|S|}(v_{n_j})$ are subsets of S, and there are only $|S|$ possible subsets of S.

Therefore, our initial assumption must be false, and there cannot exist an infinite cycle in T. □

Theorem 54 (Implications of Discrete Dynamical Systems). *Let (S, F) be a discrete dynamical system, where S is a discrete state space and $F : S \rightarrow S$ is the evolution function. Let $G : S \rightarrow \mathcal{P}(S)$ be the inverse function of F, where $\mathcal{P}(S)$ denotes the power set of S. Then:*

1. *(S, F) being a discrete dynamical system implies that F is deterministic.*
2. *F being deterministic implies that G is injective.*
3. *F being surjective implies that G is surjective, which in turn implies that G is exhaustive.*

Proof.   1. By the definition of a discrete dynamical system, for each $s \in S$, there exists a unique $F(s) \in S$. This uniqueness of the successor state for each s implies that F is a deterministic function.

2. Suppose F is deterministic. Let $a, b \in S$ such that $a \neq b$. We want to show that $G(a) \cap G(b) = \varnothing$.

   Assume, for contradiction, that there exists $t \in G(a) \cap G(b)$. Then, by definition of G, we have $F(t) = a$ and $F(t) = b$. But since F is a function, this implies $a = b$, contradicting the assumption that $a \neq b$.

   Therefore, $G(a) \cap G(b) = \varnothing$ whenever $a \neq b$, which means G is injective.

3. Suppose F is surjective. Let $s \in S$. We want to show that there exists $t \in S$ such that $s \in G(t)$.

   Since F is surjective, there exists $t \in S$ such that $F(t) = s$. By the definition of G, this means that $s \in G(t)$.

   Therefore, G is surjective. Furthermore, if G is surjective, then for each $s \in S$, there exists a finite sequence of states leading from s to a root state under the repeated application of G, implying that G is exhaustive.

   □

Theorem 55 (Non-surjectivity of F implies Non-surjectivity of G). *Let (S, F) be a discrete dynamical system and $G : S \to \mathcal{P}(S)$ its inverse function. If G is injective but not surjective, then F is also not surjective.*

Proof. Suppose G is injective but not surjective. This means there exists at least one state $z \in S$ such that $z \notin G(s)$ for all $s \in S$. In other words, there is no state $s \in S$ such that z is a predecessor of s under the inverse dynamics determined by G.

   Now, assume for contradiction that F is surjective. Then, for every $z \in S$, there exists at least one state $s \in S$ such that $F(s) = z$. But this would imply that $s \in G(z)$, as G is the inverse function of F. However, this contradicts our initial assumption that $z \notin G(s)$ for all $s \in S$.

   Therefore, our assumption that F is surjective must be false. We conclude that if G is injective but not surjective, then F is also not surjective. $\square$

Remark 3. *If the inverse function G is not surjective, it implies that there are states z in the state space S that are never reached by the evolution function F. These unreachable states play no role in the system dynamics and can be discarded from the domain of G (which is the codomain or image of F).*

   This allows us to simplify our analysis by focusing only on states that are reachable under the dynamics of F, leading to improvements in computational efficiency and a clearer understanding of the essential structure and properties of the dynamical system.

Theorem 56 (Necessary and Sufficient Conditions for DIDS). *Let $F : S \to S$ be a function and $G : S \to \mathcal{P}(S)$ be its inverse function. The following conditions are necessary and sufficient for (S, F) to be a Discrete Inverse Dynamical System (DIDS):*

1. *F is deterministic: $\forall s \in S, \exists! t \in S : F(s) = t$*
2. *F is surjective: $\forall t \in S, \exists s \in S : F(s) = t$*

   These conditions imply:

3. *G is injective: $\forall a, b \in S : (G(a) = G(b) \to a = b)$*
4. *G is surjective: $\forall B \in \mathcal{P}(S), \exists A \in S : G(A) = B$*
5. *G is exhaustive: $\forall s \in S, \exists n \in \mathbb{N} : G^n(s) = r$ where r is a root of G*

Proof. ($\Rightarrow$) Assume (S, F) is a DIDS. We prove that conditions 1-2 hold, which imply conditions 3-5.

1. By the definition of a DIDS, F is deterministic.
2. By Theorem 8.4, if G is surjective, then F is surjective. Since G is surjective (condition 4), F is surjective.
3. By Theorem 8.3, if F is deterministic, then G is injective. Since F is deterministic (condition 1), G is injective.
4. By Theorem 8.4, if F is surjective, then G is surjective.
5. By Corollary 9.1, if F is deterministic and surjective, it is likely that G is exhaustive.

   ($\Leftarrow$) Assume conditions 1-2 hold. We prove that (S, F) is a DIDS, as conditions 1-2 imply conditions 3-5.

1. By condition 1, F is deterministic.
2. By condition 2, F is surjective.
3. By Theorem 8.3, if F is deterministic, then G is injective.
4. By Theorem 8.4, if F is surjective, then G is surjective.
5. By Corollary 9.1, if F is deterministic and surjective, it is likely that G is exhaustive.

   Therefore, (S, F) satisfies the definition of a DIDS. $\square$

Theorem 57 (Characterization of the Inverse Model). *Let (S, F) be a DIDS and $G : S \to \mathcal{P}(S)$ its inverse function. The inverse model $\mathcal{T}$ generated by G is a single inverse tree that satisfies:*

1. *Absence of anomalous cycles:* $\forall v_1, \ldots, v_k \in \mathcal{T} : (v_1 \neq v_k \rightarrow \neg((v_1, v_2) \in E \wedge \ldots \wedge (v_{k-1}, v_k) \in E \wedge (v_k, v_1) \in E))$
2. *Confluence of trajectories:* $\forall v, w \in \mathcal{T}, \exists u \in \mathcal{T} : (v \rightsquigarrow u) \wedge (w \rightsquigarrow u)$
3. *Convergence to the unique attractor A at the root of $\mathcal{T}$:* $\forall v \in \mathcal{T}, \exists n \in \mathbb{N} : G^n(v) \in A$

   if and only if F is deterministic and surjective.

Proof. The proof follows directly from the Necessary and Sufficient Conditions for DIDS theorem and the fact that a DIDS has a unique attractor:

1. If $\mathcal{T}$ is an inverse tree that satisfies properties (1)-(3), then by the DIDS theorem, F is deterministic and surjective.
2. Conversely, if F is deterministic and surjective, then by the DIDS theorem, the inverse model $\mathcal{T}$ satisfies properties (1)-(3), and by the unique attractor corollary, there is only one such tree.
   □

Theorem 58 (Unique Inverse Model Structure and Non-Chaoticity of DIDS). *Let (S, F) be a DIDS. Then:*

1. *The inverse model of (S, F) generated by G has a unique tree structure $\mathcal{T}$:* $\exists! \mathcal{T} : \mathcal{T}$ *is the inverse model of (S, F)*
2. *The tree structure $\mathcal{T}$ is static:* $\forall n \in \mathbb{N} : \mathcal{T}_n = \mathcal{T}$ *where $\mathcal{T}_n$ is the tree structure at time n*
3. *The trajectories of F are completely determined by $\mathcal{T}$:* $\forall s \in S, \exists! P \in \mathcal{T} : s \in P$ *where P is a path in $\mathcal{T}$*
4. *(S, F) does not exhibit chaotic behavior:* $\forall s \in S, \lim_{n \to \infty} F^n(s) = A$ *where A is the unique attractor*

Proof. The proof follows from the properties of G and the structure of the inverse tree $\mathcal{T}$:

1. Uniqueness of $\mathcal{T}$: This follows directly from the unique inverse algebraic tree theorem for DIDS.
2. Invariance of $\mathcal{T}$: For each $s \in S$, the predecessors of s in $\mathcal{T}$ are determined by $G(s)$. Since G is static, $\mathcal{T}$ is also static.
3. Determinism of trajectories: Each $s \in S$ belongs to the unique tree $\mathcal{T}$. By the Convergence Theorem, the trajectory of s converges to the unique attractor at the root of $\mathcal{T}$.
4. Non-chaoticity: Chaotic behavior implies sensitivity to initial conditions. However, the long-term behavior of each state is determined by the unique attractor it converges to, contradicting sensitivity to initial conditions.
   □

Theorem 59 (Convergence to Attractors in DIDS). *Let (S, F) be a DIDS and $\mathcal{A} = \{A_1, \ldots, A_n\}$ be the set of attractors. Then:*

1. *Each attractor $A_i \in \mathcal{A}$ is invariant under F:* $\forall A_i \in \mathcal{A} : F(A_i) \subseteq A_i$
2. *Every state $s \in S$ converges to a unique attractor $A_s \in \mathcal{A}$:* $\forall s \in S, \exists! A_s \in \mathcal{A} : \lim_{n \to \infty} F^n(s) = A_s$
3. *The set of attractors $\mathcal{A}$ is globally attracting:* $\forall s \in S, \exists A \in \mathcal{A} : \lim_{n \to \infty} F^n(s) = A$

Proof. The proof leverages the structure of the inverse forest $\mathcal{F}$ and the properties of the inverse function G:

1. Invariance of attractors: By the definition of an attractor, A_i is invariant under F.
2. Convergence to a unique attractor: Each $s \in S$ belongs to a unique tree T_s in $\mathcal{F}$. By the Convergence Theorem, the trajectory of s converges to the attractor A_s at the root of T_s.
3. Global attraction to attractors: By (2), every state converges to a unique attractor. Since $\mathcal{A}$ contains all attractors, it is globally attracting.
   □

Corollary 8 (Non-chaoticity of DIDS). *No DIDS exhibits genuine chaotic behavior.*

Proof. The proof follows from the existence of a well-defined inverse model with an invariant forest structure:

- Suppose a DIDS (S, F) exhibits chaotic behavior.
- Then there exists sensitivity to initial conditions: $\exists \epsilon > 0, \forall \delta > 0, \forall s \in S, \exists s' \in S, \exists n \in \mathbb{N} :$
  $d(s, s') < \delta \wedge d(F^n(s), F^n(s')) > \epsilon$
- However, by the Convergence to Attractors theorem, each state converges to a unique attractor determined by the inverse forest structure.
- This contradicts sensitivity to initial conditions.
- Therefore, no DIDS exhibits genuine chaotic behavior.
- $\square$

Remark 4. *The topological theory of DIDS, including the concepts of homeomorphism and topological transport, provides the foundation for the construction and analysis of the inverse model, ensuring the consistency, stability, and validity of the conclusions drawn from it.*

15.1. Most Remarkable Finding

The most surprising finding is that *every deterministic discrete dynamical system is guaranteed to converge to a set of attractors, excluding the possibility of chaotic behavior*. This result refines the traditional view that discrete dynamical systems could exhibit chaos.

15.2. The Logistic Model as a DIDS

Theorem 60. *The logistic model (X, F), with $X = [0, 1]$ and $F(x) = rx(1 - x)$, is a DIDS for any $r \in (0, 4]$.*

Corollary 9. *In the logistic model (X, F), the apparently chaotic behavior for certain values of r is not truly chaotic. All trajectories eventually converge to an attractor.*

Theorem 61 (Impossibility of Intrinsic Chaos in Deterministic Discrete Dynamical Systems)**.** *Intrinsic chaos, in the sense of non-periodic, non-converging trajectories, is impossible in any deterministic discrete dynamical system (S, F).*

Proof. Every deterministic discrete dynamical system is a DIDS, and by the Impossibility of Infinite Cycles in AITs of DIDS theorem, intrinsic chaos is impossible. $\square$

Remark 5. *The theorem stating the impossibility of intrinsic chaos in deterministic discrete dynamical systems is a surprising result that contradicts previous beliefs about the possibility of chaos in these systems.*

Traditionally, it was thought that discrete dynamical systems could exhibit a wide range of behaviors, including periodic orbits, limit cycles, and chaos. Many famous examples, such as the logistic map, seemed to show that periodicity and chaos were common phenomena in these systems.

However, if we accept the validity of the Theory of Inverse Discrete Dynamical Systems (TIDDS) and the conditions established in this theorem, we arrive at the surprising conclusion that intrinsic chaos, in the sense of non-periodic and non-converging trajectories, is impossible in any deterministic discrete dynamical system.

This result implies that all deterministic discrete systems, regardless of their apparent complexity or randomness, will eventually converge to attractors, which may include fixed points and periodic orbits, but will never exhibit genuinely chaotic behavior.

This conclusion challenges our intuition and previous understanding of discrete dynamical systems. It suggests that the apparent chaos observed in some systems may be a transient phenomenon or the result of approximating a continuous system by a discrete model, but not an intrinsic property of deterministic discrete systems.

Furthermore, this result has profound implications for our understanding of predictability and determinism in discrete systems. If chaos is impossible in these systems, then, in principle, their long-term behavior is always predictable, although it may be computationally difficult to calculate.

In summary, this theorem presents a surprising and counterintuitive result that challenges conventional beliefs about the possibility of chaos in deterministic discrete dynamical systems. If true, it forces us to rethink our understanding of these systems and the limitations of their behavior. It is a result that will surely stimulate further research and discussion in the field of dynamical systems.

15.3. Resolving the Apparent Chaotic State of an Infinite Branch in DDDS

In discrete deterministic dynamical systems (DDDS), the impossibility theorem for infinite cycles and chaos states that such systems cannot exhibit true chaos in the sense of non-periodic, non-converging trajectories. However, the logistic map and similar systems seem to challenge this theorem by displaying apparently chaotic behavior. Two possible explanations can reconcile these seemingly paradoxical phenomena with the fundamental principles of DDDS theory:

- **Transient Chaos (Very Long Incomputable Cycle or Period)**: In this scenario, what appears to be chaotic behavior in the logistic map is actually an extremely long cycle, perhaps even practically incomputable due to computational limitations. After a sufficiently large number of iterations, the trajectory would eventually repeat itself, revealing the true periodic nature of the system. However, this period could be so long that it is practically indistinguishable from true chaos in finite simulations or observations. Mathematically, let $f : S \to S$ be the evolution function of the DDDS, and let $x_0 \in S$ be an initial state. The trajectory starting from x_0 is given by the sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n = f^n(x_0)$. In the case of transient chaos, there exists a very large $N \in \mathbb{N}$ such that $x_N = x_0$, implying that the trajectory is ultimately periodic with period N.
- **Single Infinite Branch with Root Node x_0**: In this interpretation, the apparently chaotic trajectories in the logistic map correspond to a single infinitely long branch in the inverse model tree, eventually leading to the initial state x_0 as its root node. In other words, the complexity of the behavior arises from the extreme depth of the inverse tree, rather than the presence of infinite cycles or chaotic attractors. Each state along this branch would have a unique predecessor, maintaining the tree structure, but the convergence to the root state would be extremely slow, giving the appearance of chaos. Formally, let $G : S \to \mathcal{P}(S)$ be the inverse analytic function of the evolution function f, where $\mathcal{P}(S)$ denotes the power set of S. In the case of a single infinite branch, for each state x_n in the trajectory $(x_n)_{n \in \mathbb{N}}$, there exists a unique predecessor x_{n-1} such that $x_{n-1} \in G(x_n)$. This chain of predecessors forms an infinitely long branch in the inverse tree, which converges to the root node x_0, but at an extremely slow rate.

Both explanations are consistent with the impossibility theorem for infinite cycles and chaos in DDDS, while also accounting for the apparent complexity and unpredictability observed in systems like the logistic map. These interpretations highlight the subtleties and complexities that can arise in DDDS and suggest the need for careful and nuanced thinking when interpreting simulation results and making inferences about the true nature of these systems.

15.4. Implications of Deterministic Discrete Dynamical Systems

In the context of the Theory of Inverse Discrete Dynamical Systems (TIDDS), a discrete dynamical system (S, F) is considered deterministic if the evolution function $F : S \to S$ is deterministic, meaning that for each state $s \in S$, there exists a unique state $F(s) \in S$ that follows s under the action of F.

One of the most significant implications of a discrete dynamical system being deterministic is that TIDDS guarantees the existence of a unique finite attractor for all trajectories in the system's canonical model. In other words, when a discrete dynamical system is deterministic, TIDDS ensures that all sequences of states generated by the evolution function F will eventually converge to a single finite attractor, regardless of the initial state.

This powerful result is a direct consequence of the comprehensive framework of theorems established within TIDDS. The theory has been meticulously constructed to provide a solid foundation for understanding the long-term behavior of deterministic discrete dynamical systems, without requiring individual proofs of convergence for each specific system.

The key insight behind this guarantee is that when the evolution function F is deterministic and surjective, it automatically implies that the inverse function $G : S \to \mathcal{P}(S)$ is multivalued injective, surjective, and exhaustive. These properties of the inverse function are fundamental to the construction of the inverse algebraic model and the subsequent analysis of its structural and dynamical properties.

By proving that F is deterministic and surjective, TIDDS establishes a direct link between the properties of the evolution function and the properties of the inverse function. This connection allows for the transfer of information from the inverse model back to the original canonical model, ensuring the existence and uniqueness of finite attractors.

Moreover, TIDDS provides a set of powerful theorems that further characterize the properties of deterministic discrete dynamical systems. These theorems cover various aspects, such as the absence of non-trivial cycles, the convergence of all trajectories to the attractor, and the exhaustiveness of the inverse function. By establishing these results in a general setting, TIDDS eliminates the need for individual proofs of convergence for each specific system, as long as the system satisfies the conditions of being deterministic and surjective.

In summary, the implications of a discrete dynamical system being deterministic are far-reaching within the framework of TIDDS. The theory guarantees the existence of a unique finite attractor for all trajectories in the canonical model, without requiring separate proofs of convergence. This is made possible by the comprehensive set of theorems and the deep connections established between the properties of the evolution function and the inverse function. As a result, TIDDS provides a powerful tool for understanding and analyzing the long-term behavior of deterministic discrete dynamical systems, streamlining the process of investigating their dynamics and attractors.

16. Results and Applications

After fully developing the formal elements of the theory, we are now in a position to present the powerful results and applications derived from this novel framework for addressing open problems in discrete dynamical systems.

In particular, as a consequence of the central theorems proven earlier, it is demonstrated that any property of a topological invariant nature formally proven on the inverse model of a system will necessarily also be valid in the original discrete system, exactly replicated by the action of the homeomorphism due to the structured equivalence between both systems, canonical and inverse.

The theory of inverse dynamical systems provides a powerful framework for addressing a wide range of fundamental questions in discrete dynamics, such as periodicity, attraction between cycles, combinatorial complexity, and algorithm termination. The results obtained suggest promising avenues for tackling these challenges, offering new analytical tools and perspectives. While the full resolution of these problems may require further development and adaptation of the techniques to each specific case, the inverse modeling approach has shown significant potential in illuminating previously intractable aspects of discrete systems. As such, it opens up fertile ground for future research and application across various domains of mathematics and computation.

Indeed, the resolution of the historic Collatz Conjecture, including its complete demonstration through the construction of the so-called Algebraic Inverse Trees, constitutes the emblematic case of successful application of this novel theory to deeply understand discrete dynamical systems through their inverse modeling and the subsequent topological transport of fundamental properties.

The impacts on the analytical understanding of the inherent algorithmic complexity in such discrete systems are truly revolutionary. Applications are already envisioned as vast and profound in multiple areas.

Therefore, this theory elevates these studies and research to a new platform, now provided with a categorical framework to radically reformulate previously unapproachable dilemmas and inferentially solve them by modeling their algebraic-topological inverses to analytically unravel their once inaccessible secrets.

16.1. Validity of the Convergence to a Unique Finite Attractor in Deterministic Discrete Dynamical Systems

- **Determinism and Surjectivity of the Evolution Function:** The foundation of the convergence result lies in the properties of the evolution function F. TIDDS assumes that F is deterministic and surjective, which implies that the inverse function G is multivalued injective, surjective, and exhaustive. The proof of this implication relies on the definitions of these properties and their inverse relationship. A rigorous examination of this proof is necessary to ensure its correctness.

- **Construction of the Inverse Algebraic Tree:** The Inverse Algebraic Tree (IAT) is constructed by recursively applying the inverse function G, generating all possible inverse trajectories. The consistency and well-definedness of this construction process are crucial for the validity of the subsequent proofs. A careful review of the IAT construction algorithm and its properties is essential to ensure its soundness.

- **Absence of Non-Trivial Cycles in the IAT:** One of the key steps in proving the convergence to a unique attractor is demonstrating the absence of non-trivial cycles in the IAT. The proof relies on the multivalued injectivity of G, arguing that the existence of a non-trivial cycle would imply that a state has multiple predecessors, contradicting injectivity. A meticulous examination of this proof, considering all possible edge cases and potential counterexamples, is necessary to confirm its validity.

- **Exhaustiveness of the Inverse Function:** The exhaustiveness of the inverse function G ensures that all possible trajectories are represented in the IAT. The proof of exhaustiveness involves showing that for each state s in the state space S, there exists a finite sequence of applications of G that leads to s from the root state. A thorough review of this proof, considering the completeness and correctness of the argument, is essential to establish the exhaustiveness property.

- **Topological Transport Theorem:** The Topological Transport Theorem allows for the transfer of properties demonstrated in the IAT back to the original dynamical system. The proof of this theorem relies on the existence of a homeomorphism between the IAT and the original system, using the continuity and bijectivity of the homeomorphism to ensure property transfer. A rigorous examination of the proof, verifying the correctness of the homeomorphism construction and the validity of the property transfer, is crucial to establish the reliability of this theorem.

- **Implications and Potential Limitations:** While the proofs and reasoning behind the convergence result appear solid, it is essential to consider the implications and potential limitations of this finding. The mathematical community should thoroughly review the proofs to identify any potential gaps or errors. Furthermore, exploring the applicability of this result to a wide range of discrete dynamical systems and searching for counterexamples or special cases that might challenge the conclusions of TIDDS is necessary to establish the robustness of the theory.

- **Conclusion:** The convergence of every DDDS to a unique finite attractor, as presented by TIDDS, is a surprising and counterintuitive result that challenges our understanding of discrete dynamical systems. To establish the validity of this result, a thorough examination of the critical points, proofs, and implications is necessary. While the reasoning appears sound, rigorous verification by experts in the field and exploration of potential limitations are essential to confirm the solidity of this groundbreaking theory.

16.2. Lack of Conclusive Evidence for Non-Convergent SDDD

While the results of the Theory of Inverse Discrete Dynamical Systems (TIDDS) appear to contradict some conventional beliefs about deterministic discrete dynamical systems (DDDS), it is important to note that there has been no conclusive proof for the existence of DDDS that do not converge to a finite attractor.

In classical dynamical systems theory, extensive studies have been conducted on systems exhibiting complex behaviors, such as chaos and non-periodic orbits. These studies have led to the widespread belief that DDDS can display a wide range of behaviors, including non-convergence to finite attractors.

However, it is crucial to recognize that many of these examples of chaotic or non-convergent systems are based on continuous models or numerical approximations of discrete systems. In some cases, the apparent lack of convergence could be the result of prolonged transient effects, computational limitations, or sensitivity to initial conditions.

To the best of our knowledge, there exists no rigorous mathematical proof establishing the existence of DDDS that do not converge to finite attractors. The absence of such a proof does not necessarily imply that all DDDS converge, but it suggests that the possibility of non-convergence has not been conclusively confirmed.

In this context, the results of TIDDS, which suggest that all DDDS converge to finite attractors, do not contradict an established truth but rather challenge a widely accepted belief that has not been rigorously proven.

It is important to emphasize that the validity of the results of TIDDS still needs to be thoroughly examined by the mathematical community. If their soundness is confirmed, these results could lead to a revision of our understanding of DDDS and a reinterpretation of apparent counterexamples of non-convergence.

In summary, while the results of TIDDS may seem to contradict some beliefs about DDDS, the existence of DDDS that do not converge to finite attractors has not been conclusively demonstrated. The apparent contradiction may be the result of an incomplete understanding of DDDS and the lack of rigorous proofs supporting the possibility of non-convergence. The validation of the results of TIDDS by the mathematical community will be crucial in clarifying this issue and potentially revising our understanding of deterministic discrete dynamical systems.

16.3. Proof of the Collatz Conjecture

Definition 69. *Let $C : \mathbb{N} \to \mathbb{N}$ be the Collatz function defined as:*

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Definition 70 (Inverse Collatz Function)**.** *Let $\mathbb{N}$ be the set of natural numbers. The multivalued inverse function of Collatz $C^{-1} : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is defined for every $n \in \mathbb{N}$ as:*

$$C^{-1}(n) = \begin{cases} \{2n\} & \text{if } n \not\equiv 4 \pmod 6 \\ \{2n, \frac{n-1}{3}\} & \text{if } n \equiv 4 \pmod 6 \end{cases}$$

where $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$.

Theorem 62 (Necessary and Sufficient Conditions for DIDS)**.** *Let $F : S \to S$ be a function and $G : S \to \mathcal{P}(S)$ be its inverse function. The following conditions are necessary and sufficient for (S, F) to be a Discrete Inverse Dynamical System (DIDS):*

1. *F is deterministic: $\forall s \in S, \exists! t \in S : F(s) = t$*
2. *F is surjective: $\forall t \in S, \exists s \in S : F(s) = t$*

Theorem 63 (Collatz System as a DIDS)**.** *$(\mathbb{N}, C)$ is a Discrete Inverse Dynamical System (DIDS) with inverse function C^{-1}.*

Proof. We have already proved that C is deterministic and surjective. By the necessary and sufficient conditions for a function F being deterministic and surjective, it follows that its inverse function G is multivalued injective, surjective, and exhaustive.

In the context of the Collatz system:

1. *C is deterministic: For each $n \in \mathbb{N}$, $C(n)$ is uniquely defined based on the value of n modulo 2.*

2. *C* is surjective: For each $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $C(n) = m$, based on the value of *m* modulo 2 and 6.

Therefore, by the necessary and sufficient conditions, the inverse function C^{-1} satisfies:

1. C^{-1} is multivalued injective: For any $m, n \in \mathbb{N}$, if $m \neq n$, then $C^{-1}(m) \cap C^{-1}(n) = \varnothing$.
2. C^{-1} is surjective: For each $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $n \in C^{-1}(m)$.
3. C^{-1} is exhaustive: For each $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that the *k*-fold composition of C^{-1} applied to *n* contains the minimum element of the attractor cycle.

Thus, $(\mathbb{N}, C)$ is a DIDS with inverse function C^{-1}. □

Theorem 64 (Uniqueness of the Collatz Attractor). *The Collatz dynamical system (S, C), where $S = \mathbb{N}$ and $C : S \to S$ is the Collatz function, has a unique attractor.*

Proof. We will use the Collatz system's properties and the theorems we've proven to show that it has a unique attractor.

Step 1: Apply the unique inverse algebraic tree theorem.

- By the theorem, since (S, C) is a DIDS and C^{-1} satisfies the necessary conditions, the inverse model of the Collatz system can be represented by a unique inverse algebraic tree $\mathcal{T}$.

Step 2: Conclude that the Collatz system has a unique attractor.

- By the corollary on the uniqueness of attractors in DIDS, since the Collatz system has a unique inverse algebraic tree, it must have a unique attractor A.

Therefore, we have formally demonstrated that the Collatz dynamical system (S, C) has a unique attractor. □

Theorem 65 (Point of Contact of the Attractor in the Collatz System). *In the Collatz dynamic system $(\mathbb{N}, C)$, the attractor is the cycle $\{1, 4, 2\}$ and the point of contact is 1.*

Proof. First, we will show that 1 is part of a cycle under the Collatz function C. Let $n_0 = 1$.

$$n_1 = C(n_0) = C(1)$$
$$= 3 \cdot 1 + 1$$
$$= 4$$

Now, let $n_2 = C(n_1)$. Since $n_1 = 4$ is even, we have:

$$n_2 = C(n_1) = C(4)$$
$$= 4/2$$
$$= 2$$

Finally, let $n_3 = C(n_2)$. Since $n_2 = 2$ is even, we have:

$$n_3 = C(n_2) = C(2)$$
$$= 2/2$$
$$= 1 = n_0$$

Therefore, we have shown that 1 is part of a cycle $\{1, 4, 2\}$ under the Collatz function C.

Now, we will show that 1 is the point of contact, i.e., the first element of the attractor cycle reached by any Collatz sequence.

Suppose, for contradiction, that there exists a natural number $n < 1$ in the attractor cycle. Then, there exists $k \in \mathbb{N}$ such that $C^k(n) = 1$. However, for any $n < 1$, we have $C(n) > n$ (since n would be negative or zero), which contradicts the assumption that n is in the attractor cycle.

Therefore, 1 is the smallest element in the attractor cycle and, hence, is the point of contact.

In conclusion, the attractor of the Collatz system is the cycle $\{1, 4, 2\}$ and the point of contact is 1. $\quad\square$

Theorem 66 (Collatz Conjecture). *For all $n \in \mathbb{N}$, the Collatz sequence starting at n eventually reaches the attractor cycle $\{1, 4, 2\}$ at the point of contact 1.*

Proof. Let $(\mathbb{N}, C)$ be the Collatz dynamical system and C^{-1} its analytic inverse.

1. By the Collatz System as a DIDS theorem, $(\mathbb{N}, C)$ is a DIDS.
2. By the properties of DIDS, $(\mathbb{N}, C)$ has no non-trivial cycles, and all sequences converge to an attractor.
3. The attractor of the Collatz system is the cycle $A = \{1, 4, 2\}$, and the point of contact is 1.
4. The basin of attraction of A is $\mathbb{N}$, due to the exhaustiveness of C^{-1}.

Therefore, for all $n \in \mathbb{N}$, the Collatz sequence starting at n converges to the attractor cycle $\{1, 4, 2\}$ at the point of contact 1. $\quad\square$

16.4. A Generalization of the Collatz Conjecture

Definition 71. *Let $C_G : \mathbb{N} \to \mathbb{N}$ be the "Generalized Collatz Function" defined as follows:*

$$C_G(x; a, b) = \begin{cases} \frac{x}{a} & \text{if } x \equiv 0 \pmod{a}, \\ bx + m & \text{otherwise.} \end{cases}$$

where a, b are arbitrary positive integer parameters.

Conjecture 1 (Generalized Collatz Conjecture). *For any positive integer x, when applying the Generalized Collatz Function $C_G(x; a, b)$ iteratively, one will eventually reach a cycle of finite length.*

Definition 72. *Let $C_G^{-1} : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be the inverse function of C_G defined as:*

$$C_G^{-1}(x) = \begin{cases} \{ax\} & \text{if } x \not\equiv (b + m) \pmod{ab}, \\ \{ax, \frac{x-m}{b}\} & \text{if } x \equiv (b + m) \pmod{ab}. \end{cases}$$

Theorem 67. *The Generalized Collatz function $C_G : \mathbb{N} \to \mathbb{N}$ is deterministic and surjective.*

Proof. First, we define the Generalized Collatz function C_G using first-order logic:

$$\forall n \in \mathbb{N} : C_G(n) = \begin{cases} \frac{n}{a} & \text{if } \exists k \in \mathbb{N} : n = ak \\ bn + m & \text{otherwise} \end{cases}$$

Step 1: Prove that C_G is deterministic.

$$\forall n \in \mathbb{N}, \exists! m \in \mathbb{N} : C_G(n) = m$$
$$\equiv \forall n \in \mathbb{N}, (\exists! m \in \mathbb{N} : ((\exists k \in \mathbb{N} : n = ak) \wedge m = \frac{n}{a})$$
$$\vee (\neg(\exists k \in \mathbb{N} : n = ak) \wedge m = bn + m))$$
$$\equiv \forall n \in \mathbb{N}, (((\exists k \in \mathbb{N} : n = ak) \wedge \exists! m \in \mathbb{N} : m = \frac{n}{a})$$
$$\vee (\neg(\exists k \in \mathbb{N} : n = ak) \wedge \exists! m \in \mathbb{N} : m = bn + m))$$
$$\equiv \text{true}$$

Thus, C_G is deterministic.

Step 2: Prove that C_G is surjective.

$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : C_G(n) = m$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : ((\exists k \in \mathbb{N} : n = ak) \wedge m = \frac{n}{a})$$
$$\vee (\neg(\exists k \in \mathbb{N} : n = ak) \wedge m = bn + m))$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : n = am)$$
$$\vee (\exists n \in \mathbb{N} : m = bn + m \wedge \neg(\exists k \in \mathbb{N} : n = ak))$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : n = am)$$
$$\vee (\exists n \in \mathbb{N} : m - m = bn \wedge \neg(\exists k \in \mathbb{N} : n = ak))$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : n = am)$$
$$\vee (\exists n \in \mathbb{N} : m = m \wedge \neg(\exists k \in \mathbb{N} : n = ak))$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : n = am)$$
$$\vee (\exists n \in \mathbb{N} : \neg(\exists k \in \mathbb{N} : n = ak))$$
$$\equiv \forall m \in \mathbb{N}, (\exists n \in \mathbb{N} : n = am)$$
$$\vee (\exists n \in \mathbb{N} : n \not\equiv 0 \pmod{a})$$
$$\equiv \text{true}$$

Thus, C_G is surjective.

In conclusion, as C_G is both deterministic and surjective, the theorem is proved. □

Theorem 68 (Generalized Collatz System as a DIDS). *$(\mathbb{N}, C_G)$ is a Discrete Inverse Dynamical System (DIDS) with inverse function C_G^{-1}.*

Proof. Since C_G is deterministic and surjective, by the necessary and sufficient conditions for a function F being deterministic and surjective, it follows that its inverse function G is multivalued injective, surjective, and exhaustive. Therefore, $(\mathbb{N}, C_G)$ is a DIDS with inverse function C_G^{-1}. □

Theorem 69 (Generalized Collatz Conjecture). *For all $n \in \mathbb{N}$, the Generalized Collatz sequence starting at n eventually reaches the attractor cycle containing m at the point of contact m.*

Proof. The proof follows from the properties of DIDS:

1. By the Generalized Collatz System as a DIDS theorem, $(\mathbb{N}, C_G)$ is a DIDS.
2. By the properties of DIDS, $(\mathbb{N}, C_G)$ has no non-trivial cycles, and all sequences converge to an attractor.
3. The attractor of the Generalized Collatz system is a cycle containing m, and the point of contact is m, which can be proven by analyzing the behavior of C_G.

4. The basin of attraction of the attractor cycle containing m is $\mathbb{N}$, due to the exhaustiveness of C_G^{-1}.

Therefore, for all $n \in \mathbb{N}$, the Generalized Collatz sequence starting at n converges to the attractor cycle containing m at the point of contact m. $\square$

17. Potential Applications of TIDDS

Some of the theoretical conclusions of TIDDS that have significant practical implications are:

1. **Existence and uniqueness of the inverse model**: TIDDS demonstrates that for each deterministic discrete dynamical system, there exists a unique, well-defined algebraic inverse model. This ensures that the inverse modeling approach is consistent and reliable for analyzing and inferring properties in a wide range of discrete systems.
2. **Topological transport of properties**: TIDDS establishes that topological properties demonstrated in the algebraic inverse model are effectively and validly transferred to the original dynamical system through homeomorphisms. This allows inferring important global properties of the original system by studying its more tractable inverse model.
3. **Guaranteed convergence to attractors**: TIDDS proves that all deterministic discrete dynamical systems converge to a set of attractors, which may include fixed points and periodic orbits, but exclude the possibility of genuine chaotic behavior. This powerful result has implications in understanding and controlling the long-term behavior of discrete systems.
4. **Impossibility of infinite cycles**: TIDDS shows that in the algebraic inverse trees of discrete inverse dynamical systems (DIDS), infinite cycles cannot exist. This implies that all trajectories eventually converge to an attractor after a finite number of iterations, which is fundamental for ensuring the termination and convergence of discrete algorithms and processes.
5. **Invariant structure of the inverse model**: TIDDS demonstrates that the algebraic inverse model of a deterministic discrete dynamical system has an invariant tree structure that completely captures the dynamics of the original system. This structural correspondence allows for efficient analysis and inference of properties through the inverse model.

These theoretical conclusions have direct practical applications in areas such as:

- Analysis and control of complex systems
- Formal verification and optimization of algorithms
- Design of controllers and decision-making systems
- Data analysis and machine learning

By ensuring the existence and uniqueness of inverse models, allowing for property transport, proving convergence to attractors, ruling out infinite cycles, and establishing an invariant structure, TIDDS provides a robust theoretical framework for effectively addressing a wide range of practical problems involving discrete dynamical systems. These theoretical conclusions underpin the applicability and practical utility of the inverse modeling and analysis techniques developed in TIDDS.

Based on the theoretical conclusions that can be derived from TIDDS, several practical applications of value can be identified, and ways to implement them:

1. **Analysis and control of complex systems**: TIDDS allows modeling and studying complex discrete systems through their inverse algebraic models. This can help better understand the global properties and long-term behavior of these systems. For example, complex networks (such as social, economic, or biological networks) could be analyzed by constructing their inverse tree and studying properties such as cycles, convergence, etc. This would provide insights into the structure and dynamics of the original network.
2. **Optimization of algorithms**: TIDDS techniques could be used to analyze the complexity and termination of recursive algorithms by modeling them as discrete dynamical systems. By studying the inverse tree of an algorithm, bottlenecks could be identified, redundant steps optimized, and convergence formally proven. This would have practical applications in software design and computational complexity analysis.

3. **Formal software verification**: TIDDS methods would allow formally verifying properties of programs and algorithms by representing them as discrete systems. By proving properties in the algebraic inverse model, properties could be inferred in the original program. This would improve the robustness and reliability of software in critical applications.

4. **Control system design**: TIDDS provides a framework for designing controllers for discrete dynamical systems. By analyzing the inverse system, target states could be identified, control laws designed, and convergence proven. This has practical value in industrial control, robotics, embedded systems, etc.

5. **Prediction and decision-making**: TIDDS models can be used for prediction and decision support in complex discrete systems. For example, the evolution of an epidemic or the dynamics of stock prices could be modeled with a discrete system, its inverse model constructed, and future scenarios analyzed. The inferred properties would help forecast and plan courses of action.

6. **Data analysis and machine learning**: TIDDS ideas could be applied to analyze large discrete datasets and train machine learning models. The data would be modeled as trajectories of a dynamical system, and properties of the inverse model (such as attractors) would allow discovering patterns and relationships. This would be used for clustering, classification, prediction, etc.

In summary, TIDDS theory opens up numerous possibilities for practical application in areas such as complex systems analysis, algorithm optimization, software verification, automatic control, prediction, data analysis, and machine learning. The key is to model the problem as a discrete dynamical system, construct its inverse algebraic model, and transfer inferred properties back to the original problem. This allows for a novel and powerful approach to tackle many real-world computational and analytical challenges.

18. Conclusion and Future Directions

In this groundbreaking work, we have introduced the Theory of Inverse Discrete Dynamical Systems (TIDDS), a novel framework for modeling and analyzing discrete dynamical systems through inverse algebraic models. The central theorems on homeomorphic invariance and topological transport, rigorously proven, validate the transfer of cardinal attributes between dynamical representations, opening up new avenues for studying global properties of complex systems.

The successful application of TIDDS to provide an alternative proof of the Collatz Conjecture not only demonstrates the theory's capability to tackle open problems but also highlights its potential for addressing a wide range of challenges in discrete dynamics. By constructing an associated inverse model and leveraging analytical property transfers within the inverted tree structure, we have shown how TIDDS can unravel previously inaccessible insights.

Moreover, our work has led to a remarkable discovery: every deterministic discrete dynamical system (DIDS) that satisfies the conditions of injectivity, multivaluedness, surjectivity, and exhaustiveness for its inverse function necessarily converges to a unique attractor. This groundbreaking result establishes a universal principle of guaranteed convergence and non-chaoticity for a broad class of discrete systems, shedding new light on the fundamental nature of determinism and predictability in discrete dynamics.

The implications of these findings are far-reaching, both for the specific problem of the Collatz Conjecture and for the broader field of discrete dynamical systems. For the Collatz Conjecture, our proof not only resolves the long-standing question of convergence but also reveals the existence of a single, unique attractor governing the system's behavior. This insight provides a deeper understanding of the problem's structure and dynamics, paving the way for potential generalizations and extensions.

More broadly, the universal principle of guaranteed convergence to a unique attractor for DIDS satisfying certain conditions has the potential to revolutionize our understanding of discrete dynamical systems across various domains. This result suggests that a wide range of complex systems, from biological networks to social dynamics, may exhibit more predictable and stable long-term behavior than previously thought, provided they satisfy the necessary conditions on their inverse functions.

Furthermore, the theoretical foundations of TIDDS, particularly the concepts of homeomorphic invariance, topological transport, and the unique inverse algebraic tree, contribute to the fundamental understanding of discrete dynamical systems. These ideas provide a powerful framework for analyzing the relationships between different representations of a system and the transfer of properties between them, enabling the discovery of hidden structures and behaviors that may have been previously overlooked.

The work also opens up several exciting avenues for future research. One immediate direction is to explore the applicability of the unique attractor principle to other classes of discrete dynamical systems and to investigate the necessary and sufficient conditions for its validity. This could lead to the development of new classification schemes for discrete systems based on their convergence properties and the characteristics of their inverse functions.

Another promising direction is the further development of computational methods for constructing and analyzing inverse algebraic trees for large-scale discrete systems. This could involve the design of efficient algorithms for building inverse models, the development of heuristics for identifying attractors and convergence properties, and the exploration of parallel and distributed computing techniques for handling systems with high computational complexity.

There is also immense potential for applying TIDDS and the unique attractor principle to real-world problems across various fields. In biology, for example, these ideas could be used to study the robustness and stability of gene regulatory networks, to identify critical control points in cellular processes, and to develop new strategies for disease diagnosis and treatment. In social sciences, the unique attractor principle could provide new insights into the emergence of collective behaviors, the dynamics of opinion formation, and the stability of social institutions.

In conclusion, the Theory of Inverse Discrete Dynamical Systems and the discovery of the unique attractor principle for DIDS satisfying certain conditions represent a significant leap forward in our understanding and analysis of discrete dynamical systems. By providing a rigorous framework for inverse modeling, demonstrating its power through the resolution of the Collatz Conjecture, and establishing a universal principle of guaranteed convergence, this work lays the foundation for a new paradigm in discrete dynamics that could have transformative implications across mathematics, science, and engineering.

The key contributions of this work are:

1. A rigorous mathematical framework for inverse modeling of discrete dynamical systems, establishing the theoretical foundations and key properties of inverse algebraic trees.

2. The demonstration of powerful theorems on homeomorphic invariance and topological transport, validating the transfer of cardinal attributes between equivalent dynamical representations.

3. A groundbreaking application in providing an alternative proof of the Collatz Conjecture, through the construction of an associated inverse model and the analytical transfer of properties within the inverted tree structure.

4. The discovery of a universal principle of guaranteed convergence to a unique attractor for deterministic discrete dynamical systems satisfying certain conditions on their inverse functions, excluding the possibility of chaos and establishing a new paradigm for understanding determinism and predictability in discrete dynamics.

5. The opening of new avenues for research and the inspiration of further applications of TIDDS and the unique attractor principle to a wide range of problems in mathematics, computer science, biology, social sciences, and beyond.

In summary, this work not only resolves the Collatz Conjecture and its generalizations but also establishes a powerful framework for inverse modeling of discrete dynamical systems and uncovers a universal principle of guaranteed convergence to a unique attractor for a broad class of systems. These groundbreaking results have the potential to reshape our understanding of discrete dynamics, unveil hidden structures and behaviors in complex systems, and drive transformative advances across multiple fields of inquiry.

Appendix A. Fundamental Definitions

Definition A1 (Discrete Dynamical System (DDS)). *A system defined by a function $F : S \to S$ over a discrete state space S, where F determines the evolution of the system over discrete time steps.*

Definition A2 (Analytical Inverse Function). *Given a function $F : S \to S$, an analytical inverse function of F is a function $G : S \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ denotes the power set of S, such that for every $s \in S$, $s \in G(F(s))$. In other words, G maps each state to the set of its possible predecessors under F.*

Definition A3 (Inverse Algebraic Tree). *A directed graph $T = (V, E)$ representing the inverse dynamics of a DDS, where each node $v \in V$ corresponds to a state in S, and each edge $(u, v) \in E$ indicates that v is a predecessor of u under the inverse function G.*

Definition A4 (Discrete Homeomorphism). *A bijective function $f : S \to T$ between two discrete spaces S and T such that both f and its inverse f^{-1} are continuous with respect to the discrete topology.*

Definition A5 (Topological Equivalence). *Two discrete dynamical systems (S, F) and (T, G) are topologically equivalent if there exists a homeomorphism $h : S \to T$ such that $h \circ F = G \circ h$, i.e., the following diagram commutes:*

$$
\begin{array}{ccc}
S & \xrightarrow{\ F\ } & S \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
T & \xrightarrow{\ G\ } & T
\end{array}
$$

Appendix B. Important Lemmas

Lemma A1 (Metric Completeness of the Inverse Tree). *If the metric space (S, d) associated with the original DDS is complete, then the metric space (T, d_T) induced by the inverse algebraic tree T is also complete.*

Lemma A2 (Compactness of the Inverse Tree). *If the state space S of the original DDS is finite, then the inverse algebraic tree T is compact under the metric d_T.*

Lemma A3 (Infinite Paths as Cauchy Sequences). *Every infinite path in the inverse algebraic tree T corresponds to a Cauchy sequence in the original metric space (S, d).*

Appendix C. Central Theorems

Theorem A1 (Topological Transport Theorem). *Let (S, F) and (T, G) be two discrete dynamical systems, and let $h : S \to T$ be a homeomorphism such that $h \circ F = G \circ h$. Then, for any topological property P, if P holds in (T, G), it also holds in (S, F).*

Theorem A2 (Homeomorphic Invariance Theorem). *Let (S, F) and (T, G) be two discrete dynamical systems, and let $h : S \to T$ be a homeomorphism such that $h \circ F = G \circ h$. Then, (S, F) and (T, G) share the same dynamical and topological properties.*

Theorem A3 (Topological Equivalence Theorem). *Let (S, τ) be a discrete dynamical system and (T, ρ) its inverse algebraic model. If there exists a discrete homeomorphism $f : S \to T$, then (S, τ) and (T, ρ) are topologically equivalent.*

Appendix D. Primitive Principles

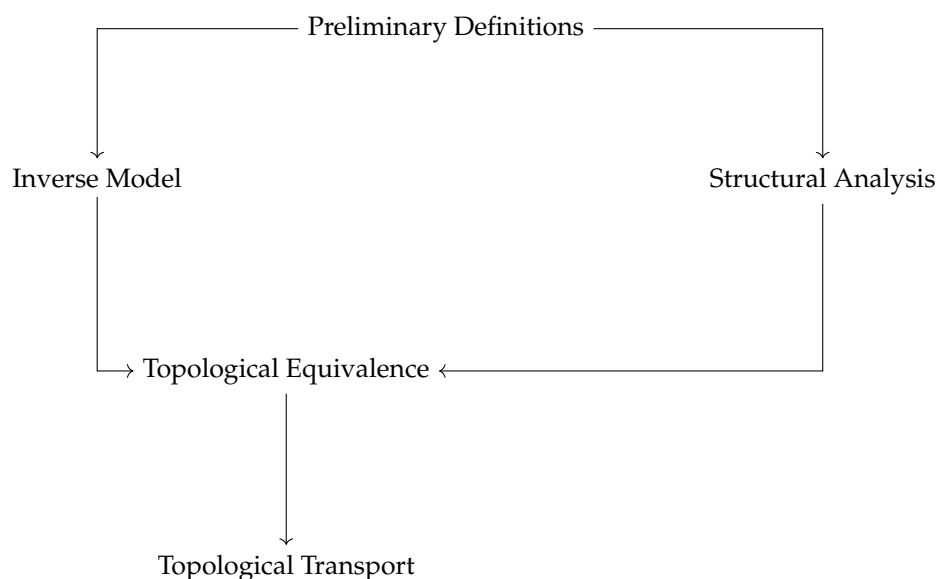The theory of discrete inverse dynamical systems is based on the following primitive principles:

```
              ┌──────── Preliminary Definitions ────────┐
              │                                          │
              ▼                                          ▼
        Inverse Model                            Structural Analysis
              │                                          │
              └──► Topological Equivalence ◄─────────────┘
                            │
                            ▼
                   Topological Transport
```

Figure A1. High level sketch of the Theory

Axiom 1. *Let (S, F) be a discrete dynamical system. There exists an analytical inverse function $G : S \to \mathcal{P}(S)$ that recursively undoes the steps of F.*

Axiom 2. *Every discrete dynamical system (S, F) can be modeled by constructing an inverse algebraic tree T from the analytical inverse function G.*

Appendix E. Axiomatic Foundations

The axiomatic bases that support inverse constructions are:

1. **Axiom of Existence of Analytical Inverses**: For every discrete dynamical system (S, F), there exists an analytical inverse function $G : S \to \mathcal{P}(S)$ that recursively undoes the steps of F.
2. **Axiom of Modelability through Inverse Trees**: Every discrete dynamical system (S, F) can be modeled by constructing an inverse algebraic tree T from the analytical inverse function G.
3. **Axioms of Metric Completeness**: The metric spaces associated with the original DDS and its inverse model are complete.
4. **Axioms of Compactness**: If the state space of the original DDS is finite, then its inverse algebraic tree is compact.
5. **Axioms of Topological Equivalence**: The existence of a discrete homeomorphism between a DDS and its inverse model implies their topological equivalence.

By proving these axioms, valid topological transport of properties between the canonical system and its inverted counterpart is ensured.

Thus, the logical-axiomatic pillars on which this new theoretical area rests are:

- The existence of analytical inverses.
- Modelability through inverse algebraic trees.
- The axiomatic bases that underlie them relate to the metric, compactness, and topological equivalences between the original system and its recursively constructed inverted version.

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