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A Note on Invo-Regular Unital Rings

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Abstract: In this paper we provide some important and significant observations on invo-regular rings. This work improves some of the exiting results on invo-regular rings appeared in Ann. Univ. Mariae Curie-Sklodowska Sect. A Mathematica (2018).

Keywords: Boolean ring; unit regular ring; invo-regular ring; involution

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1. Introduction

In this paper each ring is a unital and associative ring and following [1] we assume that the identity element of a ring is different from the zero element. A ring R is called invo-regular if for each $a \in R$ there exists $b \in Inv(R)$ such that $a = aba$ [1–3]. Here $Inv(R)$ is the set of all involutions. One may note that an element b of R satisfying $b^2 = 1$ is called an involution [1–3] and the notion of invo-regular rings is a generalization of the well known notion of unit regular rings [4–6].

It should be emphasized that as per the existing literature [1, Proposition 2.5] a ring R is invo-regular iff $R \cong R_1 \times R_2$, here R_1 is an invo-regular ring of characteristic two and R_2 is an invo-regular ring of characteristic three.

However we prove that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two. In addition we exhibit that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then R_1 need not be Boolean. However it was asserted in [1, Proof of Theorem 2.6] that if R is an invo-regular ring then $R \cong R_1 \times R_2$ and R_1 is a ring of characteristic two which must be a Boolean ring. One may note that a ring R is called Boolean if for each $a \in R$, we have the identity $a^2 = a$ [7]. A ring R is called tripotent if for each $a \in R$, we have the identity $a^3 = a$ and a ring R is called weakly tripotent if for each $a \in R$, we have the identity $a^3 = a$ or $(1-a)^3 = 1-a$ [7-8].

2. Some Important Observations

Proposition 2.1: If R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two.

Proof. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}$.

Clearly R is a commutative ring of characteristic three under addition and multiplication of matrices modulo three. We have

$Inv(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\}$. It is easy to check that R is an invo-regular unital ring. Now we have the following cases.

Case I: $R \cong R \times \{0\}$. One may note that R is not a ring of characteristic two.

Case II: $R \cong \{0\} \times R$. It is clear that $\{0\}$ is not a ring of characteristic two.

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Case III: $R \cong R_1 \times R_2$. Here $R_1 = Z_3 = R_2$. We note that the characteristic of $R_1 = Z_3$ is not two.

Further we emphasize that if the characteristic of R_1 is two, then the order of R must be even. But the order of R is nine. Thus we see that in the above example the characteristic of R_1 can never be two even though R is an invo-regular ring.

Proposition 2.2: If R is an invo-regular ring such that $R \cong R_1 \times R_2$, then R_1 need not be a non-zero Boolean ring.

Proof. Let R is an invo-regular ring and $R \cong R_1 \times R_2$. Clearly the characteristic of R_1 need not be two (we refer Proposition1). But it is well known that a non-zero Boolean ring must have characteristic two, hence R_1 need not be a non-zero Boolean ring.

Proposition 2.3: A weakly tripotent ring is an invo-regular ring iff it is a tripotent ring.

Proof. Let R is a weakly tripotent invo-regular ring. Then R is a subdirect product of copies of field of order two and the field of order three [1]. Hence by [9] R is tripotent. Conversely let R is tripotent. Then clearly it is weakly tripotent and by [9] it is a subdirect product of copies of the field of order two and the field of order three. Therefore by [1] it is an invo-regular ring.

Corollary2. 4: Every invo-regular ring is a tripotent ring. The converse is also true.

Corollary2. 5: There does not exist a noncommutative invo-regular ring.

Proof. Every tripotent ring is commutative [7]. Therefore it follows from Corollary 2.4 that every invo-regular ring is commutative. Hence there does not exist a noncommutative invo-regular ring.

Statement and Declaration: The author declares that there is no competing interest.

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