Article

Not peer-reviewed version

A Note on Invo-Regular Unital Rings

S K Pandey *

Posted Date: 8 November 2023

doi: 10.20944/preprints202310.0485.v4

Keywords: Boolean ring; unit regular ring; invo-regular ring; tripotent ring, quasi invo-regular ring, strongly invo-regular ring, weakly tripotent ring

Article

A Note on Invo-Regular Unital Rings

S. K. Pandey

Faculty of Science, Technology and Forensic, SPUP, Jodhpur-342304, India; E-mail:skpandey12@gmail.com

Abstract: In this paper we provide some important and significant observations on invo-regular rings appeared in Ann. Univ. Mariae Curie-Sklodowska Sect. A Mathematica (2018). In addition we exhibit that strongly invo-regular rings, invo-regular rings and quasi invo-regular rings all coincide with the well known and well characterized tripotent rings and all these rings further coincide with the well known unit regular rings.

Keywords: Boolean ring; unit regular ring; invo-regular ring; quasi invo-regular ring; strongly invo-regular ring; weakly tripotent ring; tripotent ring

MSC 2020: 16U40; 16E50

1. Introduction

In this paper each ring is a unital and associative ring and following [1] we assume that the identity element of a ring is different from the zero element. A ring R is called invo-regular if for each $a \in R$ there exists $b \in Inv(R)$ such that $a = aba$ [1–3]. Here $Inv(R)$ is the set of all involutions. One may note that an element b of R satisfying $b^2 = 1$ is called an involution [1–3] and the notion of invo-regular rings is a generalization of the well known notion of unit regular rings [4–6]. Further a ring R is called Boolean if for each $a \in R$, we have the identity $a^2 = a$ [7]. A ring R is called tripotent if for each $a \in R$, we have the identity $a^3 = a$ and a ring R is called weakly tripotent if for each $a \in R$, we have the identity $a^3 = a$ or $(1-a)^3 = 1-a$ [7,8]. A ring R is called strongly invo-regular ring if $a^2 = au$ for each $a \in R$ and some $u \in R$ with $u^2 = 1$ [10]. Similarly as per [2] a ring R is said to be a quasi invo-regular ring if for for each $a \in R$ there exists $b \in R$ such that $a = aba$, where $b^2 = 1$ or $(1-b)^2 = 1$.

In this paper we take an opportunity to exhibit that strongly invo-regular rings, invo-regular rings and quasi invo-regular rings all coincide with the well known and well characterized tripotent rings all these are nothing but unit regular rings. We also give relation between weakly tripotent rings and these rings. In addition we provide counterexample for the following results appeared in [1] and we provide corrected version of these results.

It should be emphasized that as per the existing literature ([1] Proposition 2.5) a ring R is invo-regular iff $R \cong R_1 \times R_2$, here R_1 is an invo-regular ring of characteristic two and R_2 is an invo-regular ring of characteristic three.

However we prove that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two. In addition we exhibit that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then R_1 need not be a non-zero Boolean ring. However it was asserted in ([1] Proof of Theorem 2.6) that if R is an invo-regular ring then $R \cong R_1 \times R_2$ and R_1 is a ring of characteristic two which must be a Boolean ring.

We now provide our observations and results in the next section.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 November 2023 doi:10.20944/preprints202310.0485.v4

2. Some Important Observations and Results

Proposition 2.1. *If R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two.*

Proof. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}$.

Clearly R is a commutative ring of characteristic three under addition and multiplication of matrices modulo three. We have $Inv(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\}$. It is easy to check that R is an invo-regular unital ring. Now we have the following cases.

Case I: $R \cong R \times \{0\}$. One may note that R is not a ring of characteristic two.

Case II: $R \cong \{0\} \times R$. It is clear that $\{0\}$ is not a ring of characteristic two.

Case III: $R \cong R_1 \times R_2$. Here $R_1 = 0$, $R_2 = Z_3 \times Z_3$. We note that the characteristic of R_1 is not two.

Further we emphasize that if the characteristic of R_1 is two, then the order of R must be even. But the order of R is nine. Thus we see that in the above example the characteristic of R_1 can never be two even though R is an invo-regular ring.

Proposition 2.2. *If R is an invo-regular ring such that $R \cong R_1 \times R_2$, then R_1 need not be a non-zero Boolean ring.*

Proof. Let R is an invo-regular ring and $R \cong R_1 \times R_2$. Clearly the characteristic of R_1 need not be two (we refer Proposition1). But it is well known that a non-zero Boolean ring must have characteristic two, hence R_1 need not be a non-zero Boolean ring.

Proposition 2.3. *A weakly tripotent ring is a strongly invo-regular ring iff it is a tripotent ring.*

Proof. Let R is a weakly tripotent and strongly invo-regular ring. Then R is a subdirect product of copies of the field of order two and the field of order three [10]. Hence by [9] R is tripotent. Conversely let R is tripotent. Then clearly it is weakly tripotent and by [9] it is a subdirect product of copies of the field of order two and the field of order three. Therefore by [10] it is a strongly invo-regular ring.

Corollary 2.4. *Every strongly invo-regular ring is a tripotent ring. The converse is also true.*

Corollary 2.5. *There does not exist a noncommutative strongly invo-regular ring.*

Proof. Every tripotent ring is commutative [7]. Therefore it follows from Corollary 2.4 that every strongly invo-regular ring is commutative. Hence there does not exist a noncommutative strongly invo-regular ring.

Proposition 2.6. *Strongly invo-regular rings, invo-regular rings and quasi-invo regular rings all coincide with tripotent rings.*

Proof. Let R is a tripotent ring. Then R is a subdirect product of copies of the field of order two and the field of order three. The converse is also valid (we refer [9]). Now if R is strongly invo-regular ring then R is a subdirect product of copies of the field of order two and the field of order three (we refer [10]). Hence R is a tripotent ring. Similarly if R is a quasi invo-regular ring then it is a invo-regular ring (we refer [2]) and if R is a quasi invo-regular, then R is a subdirect product of copies of the field of order two and the field of order three (we refer [2]). Hence strongly invo-regular rings, invo-regular rings and quasi invo-regular rings coincide with the well known notion of tripotent rings.

Corollary 2.7. *Strongly invo-regular rings, invo-regular rings, quasi-invo regular rings and tripotent rings all coincide with unit regular rings.*

Proof. It is clear from the definition of invo-regular rings that invo-regular rings are unit regular rings. Further by Proposition 2.6, strongly invo-regular rings, invo-regular rings and quasi-invo regular rings all coincide with tripotent rings. Hence Strongly invo-regular rings, invo-regular rings, quasi-invo regular rings and tripotent rings all coincide with unit regular rings.

Corollary 2.8. *The converse of Proposition 2.7 is not valid.*

Note. 2.9. *By Proposition 2.6, strongly invo-regular rings, invo-regular rings and quasi-invo regular rings all coincide with tripotent rings. Therefore it is a fact that all these rings are the same. One may find it interesting to note that if we consider a ring R which is not a quasi-invo-regular (and hence none of these rings), then a quasi-invo-regular element of R need not be an invo-regular element or a tripotent element. For example, $R = Z_5$ is not a quasi invo regular ring and hence not an invo-regular ring or a tripotent ring. But $3 \in Z_5$ is a quasi invo-regular element which is neither invo-regular nor tripotent.*

Now we shall provide the corrected version of Proposition 2.5 [1].

Proposition 2.10. *A ring R is invo-regular iff $R \cong R_1 \times R_2$, here $R_1 = 0$ or R_1 is an invo-regular ring of characteristic two and $R_2 = 0$ or R_2 is an invo-regular ring of characteristic three.*

Proof. Let R is an invo-regular ring. Let $b \in R$ is an involution. Then we have $b^2 = 1$. Without the loss of generality we take $2 = 4b$ in R. This gives $12 = 0$. Using The Chinese Remainder Theorem R decomposes as the direct product of two invo-regular rings: $R \cong R_1 \times R_2$, where $R_1 \cong \frac{R}{4R}$ and $R_2 \cong \frac{R}{3R}$. Clearly $4 = 0$ in R_1 and $3 = 0$ in R_2. But $4 = 0$ in R_1 implies that $2 = 0$ in R_1. Hence we have $2 = 0$ in R_1 and $3 = 0$ in R_2. It follows that if $R = 2R$, then $R_1 = 0$ and if $R = 3R$, then $R_2 = 0$.

Conversely let $R \cong R_1 \times R_2$, where $R_1 = 0$ or R_1 is an invo-regular ring of characteristic two and $R_2 = 0$ or R_2 is an invo-regular ring of characteristic three. We discuss the following cases.

Case I: If R_1 and R_2 both are zero, then R is clearly invo-regular.

Case II: If $R_1 = 0$ and R_2 is an invo-regular ring of characteristic three, then R is clearly invo-regular.

Case III: If R_1 is an invo-regular ring of characteristic two and $R_2 = 0$, then R is clearly invo-regular.

Case IV: Let R_1 is an invo-regular ring of characteristic two and R_2 is an invo-regular ring of characteristic three. Let $(u,v) \in R$. Then we have $(u,v) = (u,v)(b,c)(u,v)$. Here $(b,c)^2 = (1,1)$ and $u = ubu$, $v = vcv$ for some $b \in R_1$ with $b^2 = 1$ and $c \in R_2$ with $c^2 = 1$. Hence R is an invo-regular ring.

Note 2.11. *The homomorphic images of an invo-regular ring is invo-regular [1]. One may also note that the homomorphic images of a tripotent ring is tripotent and hence the homomorphic images of an invo-regular ring is invo-regular.*

 doi:10.20944/preprints202310.0485.v4

Corollary 2.12. *If* R *is an invo-regular ring such that* $R \cong R_1 \times R_2$*, then* R_1 *is a Boolean ring (of characteristic one or two).*

Statement and Declaration: The author declares that there is no competing interest.

References

1. P. V. Danchev, Invo-regular unital rings, Ann. Univ. Mariae Curie-Sklodowska Sect. A Mathematica, 72 (2018) 45-53.
2. P. V. Danchev, Quasi Invo-Regular rings, Journal of Prime Research in Mathematics, 15 (2019) 1-4.
3. P. V. Danchev, LocallyInvo-Regular Rings, Azerbijan Journal of Mathematics, 11 (2021) 28-44.
4. V. P. Camillo, D. Khurana, A characterization of unit regular rings, Commun. Algebra 29 (2001), 2293-2295.
5. P. P. Nelson, J. Ster, Connections between unit-regularity, regularity, cleanness and strongly cleanness of elements and rings, Trans. Amer. Math. Soc. 370 (2018), 1759-1782.
6. T. Y. Lam, W. Murray, Unit regular elements in corner rings, Bull. Hong Kong Math. Soc. 1 (1997), 61-65.
7. Z. Ying. T. Kosan. Y. Zhou, Rings in which every element is a sum of two tripotents. Canad. Math. Bull., 59 (3), 2016, 661-672.
8. Breaz, S., Cimpean, A., Weakly tripotent rings, Bull. Korean Math. Soc., 55 (4) (2018), 1179-1187.
9. H. Tominaga, On anti-inverse rings, Publications De L' Institute Mathematique, 33 (47), 1983, pp 225.
10. P. V. Danchev, On weakly clean and weakly exchange rings having the strong property, Publ. Inst. Math. Beograd 101 (2017), 135–142.