Preprints.org

Article

Not peer-reviewed version

Linear Entropy in the Atom-Field Interaction at Finite Temperature

Raul Juarez-Amaro and Hector Moya-Cessa *

Posted Date: 7 October 2023

doi: 10.20944/preprints202310.0404.v1

Keywords: dispersive regime; von neumann entropy; linear entropy

Article

Linear Entropy in the Atom-Field Interaction at Finite Temperature

R. Juárez-Amaro [1] and H. M. Moya-Cessa [2,*]

1 Universidad Tecnológica de la Mixteca, Apdo. Postal 71, Huajuapan de León, Oax., 69000, Mexico; rjamaro@hotmail.com
2 Instituto Nacional de Astrofísica Óptica y Electrónica, Calle Luis Enrique Erro No. 1 Santa María Tonantzintla, Puebla, 72840, Mexico
* Correspondence: hmmc@inaoep.mx

Abstract: We study the atom-field interaction at finite temperature and in the dispersive regime. We show that the master equation for this sytem may be solved with the use of superoperator techniques. We calculate the linear entropy in case the field is initially in a coherent state and the atom in a superposition of its ground and excited states.

Keywords: dispersive regime; von neumann entropy; linear entropy

1. Introduction

One of the mots important quantities to measure entanglement between two subsystems [1,2] is the von Neumann entropy [3]. It defines, among other things and very clearly, when two subsystems are entangled. In order to calculate the von Neumann entropy for one of the subsystems usually it is used the Araki-Lieb inequality [4,5], that allows, when initially the two sub-systems are found in pure states, to relate the two sub-systems entropies after they interact. To achieve this it is necessary to perform a partial trace on the system, such that we remain with only one subsystem described by its density matrix which is, in the most general case, a so-called "statistical mixture" state. It is more difficult the calculation of the entropy when one studies statistical mixtures [5,6] because it is a (logarithmic) function of the density matrix and the calculation may become extremely complicated [7,8].

One of the interactions where entanglement is usually studied is the resonant atom field interaction [9] where it is well known that the atomic inversion shows collapses and revivals of the Rabi oscillations [10–12]. Such revivals may be considered as a first measurement of the field as it gives light on the nature of the photon distribution of the cavity field: For instance, in the case that an initial squeezed state is considered, the atomic inversion shows so-called ringing revivals [10,11]. Entropy, together with the atomic inversion, may be used to obtain information about the non-classicality of a given state [6]. For example, if at an specific time, the atomic inversion is in the collapse region and the entropy of a field initially prepared in an squeezed state is close to zero it is well-known that a superposition of squeezed states may be generated [11].

However, if a superposition of coherent states, the so-called Schrödinger cat states, is considered as initial state, the revival of oscillations occurs sooner [12] than that of a single coherent state [13]. But, if a mixture of coherent states is initially considered, the revival of Rabi oscillations appear as in the case of a single coherent state. This indicates that the atomic inversion, together with the degree of mixedness of states may give information about the initial state used in a given interaction [5,14,15]. Such mixedness may be measured via the von Neumann entropy.

The von Neumann entropy is given by the expression

$$S = -Tr\{\rho \ln \rho\}, \tag{1}$$

where ρ, that defines the state of the system, is the density matrix and Tr means the trace over the system's degrees of freedom. Calculation of this quantity, however, may become a difficult task as

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2023 doi:10.20944/preprints202310.0404.v1

mentioned before. A more convenient quantity to calculate because of its simplicity that however contains the same information as the von Neumann entropy is the linear entropy [16], defined as

$$S_L = 1 - Tr\{\rho^2\}. \tag{2}$$

The eigenbasis of the density matrix may be used to show that

$$Tr\{\rho^2\} = \sum_n \rho_n^2 \leq \sum_n \rho_n = 1, \tag{3}$$

where the equality holds for pure states only. Therefore S_L uniquely discriminates between pure and mixed states, just as the von Neumann entropy does.

In the atom-field interaction there are effects such as decoherence that play a strong role in the purity of the states of quantum systems [17]. Effects of environment strongly damage entanglement and the nonclassicality of states of light [18]. Entanglement may be even damaged by an environment where gravitational fluctuations are considered [19,20]. Moreover, studies to endure quantum coherence by using photon indistinguishability in noisy quantum networks have been performed [21]. When an environment is taken into account, however, complete information about an initial quantum state may be still recovered by using reconstruction techniques [22]. Such reconstruction technique are realized using quasiprobability distribution functions [23] that may be even generated in classical physics [24]. The damage becomes worst in the case that the environment is at finite temperature as the bath injects photons into the cavity destroying faster the possible coherences of the quantized field.

In this contribution we study this case and show how to calculate the linear entropy in the dispersive interaction between a two-level atom and a quantized field when we consider losses at temperature $T > 0$. In the next Section we present the master equation that describes this system and show how to solve it, then in Section 3 we calculate the field linear entropy and do some numerical analysis in Section 4. Section 5 is left for conclusions.

2. Master equation for a real cavity

The master equation for a two- level atom interacting with a quantized electromagnetic field in the dispersive regime, i.e., when the atom and the field stop exchhanging energy because they are sufficiently detuned, at finite temperature is given by [25–27] (we set $\hbar = 1$)

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i\left[\hat{H}_{eff}, \hat{\rho}\right] + k_1\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right)\\ &+ k_2\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right),\end{aligned} \tag{4}$$

where $\hat{H}_{eff} = \chi\hat{n}\hat{\sigma}_z$ [25,26] is the so-called dispersive Hamiltonian, and $k_1 = \frac{C}{2}\bar{n}$, $k_2 = \frac{C}{2}(\bar{n}+1)$ and $\bar{n} = \frac{1}{e^{\left(\frac{v_k}{k_\beta T}\right)} - 1}$. Here C is the decay constant and $\bar{n}$ is the average number of thermal photons. Also, the operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively. By defining $k_{12} = k_1 + k_2$, the above master equation (4) may be rewritten as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= i\chi\hat{\rho}\hat{n}\hat{\sigma}_z - i\chi\hat{n}\hat{\sigma}_z\hat{\rho} + 2k_1\hat{a}^\dagger\hat{\rho}\hat{a}\\ &+2k_2\hat{a}\hat{\rho}\hat{a}^\dagger - k_{12}\hat{\rho}\hat{n} - k_{12}\hat{n}\hat{\rho} - 2k_1\hat{\rho}.\end{aligned} \tag{5}$$

We may define the superoperators (see for instance [28])

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2023 doi:10.20944/preprints202310.0404.v1

$$
\begin{aligned}
\hat{J}_1\hat{\rho} &= 2k_1\hat{a}^{\dagger}\hat{\rho}\hat{a}, \\
\hat{J}_2\hat{\rho} &= 2k_2\hat{a}\hat{\rho}\hat{a}^{\dagger}, \\
\hat{R}\hat{\rho} &= i\chi\hat{\rho}\hat{n}\hat{\sigma}_z - i\chi\hat{\sigma}_z\hat{n}\hat{\rho}, \\
\hat{S}\hat{\rho} &= -k_{12}\hat{\rho}\hat{n} - k_{12}\hat{n}\hat{\rho},
\end{aligned} \tag{6}
$$

to rewrite the master equation in a more compact form, namely

$$
\frac{d\hat{\rho}}{dt} = -2k_1\hat{\rho} + \hat{J}_1\hat{\rho} + \hat{J}_2\hat{\rho} + \hat{R}\hat{\rho} + \hat{S}\hat{\rho}. \tag{7}
$$

This allows us to write a formal solution

$$
\hat{\rho}(t) = e^{-2k_1 t} e^{\left(\hat{J}_1+\hat{J}_2+\hat{R}+\hat{S}\right)t}\hat{\rho}(0), \tag{8}
$$

that may be factorized in the form

$$
\hat{\rho}(t) = e^{f_0(t)} e^{f_1(t)\hat{J}_1} e^{f_2(t)\hat{R}} e^{f_3(t)\hat{S}} e^{f_4(t)\hat{J}_2}\hat{\rho}(0). \tag{9}
$$

We may show that indeed (9) is a solution by deriving $\hat{\rho}(t)$ with respect to time to obtain

$$
\begin{aligned}
\frac{d\hat{\rho}}{dt} = & \left[\frac{df_0(t)}{dt} + \frac{df_1(t)}{dt}\hat{J}_1\right]\hat{\rho}(t) \\
& + \frac{df_2(t)}{dt} e^{f_1(t)\hat{J}_1}\hat{R} e^{-f_1(t)\hat{J}_1}\hat{\rho}(t) \\
& + \frac{df_3(t)}{dt} e^{f_1(t)\hat{J}_1} e^{f_2(t)\hat{R}}\hat{S} \\
& \times e^{-f_2(t)\hat{R}} e^{-f_1(t)\hat{J}_1}\hat{\rho}(t) \\
& + \frac{df_4(t)}{dt} e^{f_1(t)\hat{J}_1} e^{f_2(t)\hat{R}} e^{f_3(t)\hat{S}}\hat{J}_2 \\
& \times e^{-f_3(t)\hat{S}} e^{-f_2(t)\hat{R}} e^{-f_1(t)\hat{J}_1}\hat{\rho}(t).
\end{aligned} \tag{10}
$$

We define the atomic superoperator $\hat{L}\hat{\rho} = i\chi\hat{\sigma}_z\hat{\rho} - i\chi\hat{\rho}\hat{\sigma}_z$ and, by using the relevant commutators given in Appendix A, and using that $e^{w\hat{A}}\hat{B}e^{-w\hat{A}} = \hat{B} + w\left[\hat{A},\hat{B}\right] + \frac{w^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \ldots$ we obtain the terms present in equation, (10) *i.e.,*

$$
e^{f_1(t)\hat{J}_1}\hat{R}e^{-f_1(t)\hat{J}_1}\hat{\rho}(t) = \left[\hat{R} + f_1(t)\hat{L}\hat{J}_1\right]\hat{\rho}(t), \tag{11}
$$

$$
\begin{aligned}
e^{f_1(t)\hat{J}_1} e^{f_2(t)\hat{R}}\hat{S} e^{-f_2(t)\hat{R}} e^{-f_1(t)\hat{J}_1} &= \hat{S}\hat{\rho}(t) \\
& +2k_{12}f_1(t)\hat{J}_1\hat{\rho}(t),
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
e^{f_1\hat{J}_1} e^{f_2\hat{R}} e^{f_3\hat{S}}\hat{J}_2 e^{-f_3\hat{S}} e^{-f_2\hat{R}} e^{-f_1\hat{J}_1} = & [e^{2k_{12}f_3+f_2\hat{L}}\hat{J}_2 \\
& + \tfrac{4k_1k_2}{k_{12}} f_1 e^{2k_{12}f_3+f_2\hat{L}}\hat{S} - 4k_1k_2 f_1 e^{2k_{12}f_3+f_2\hat{L}} \\
& + 4k_1k_2 f_1^2 e^{2k_{12}f_3+f_2\hat{L}}\hat{J}_1]\hat{\rho}(t).
\end{aligned} \tag{13}
$$

By substituting these expressions in (10) and compare it with equation (7) we obtain the set of first order differential equations

$$
\frac{df_0}{dt} - 4k_1k_2 f_1 e^{2k_{12}f_3+f_2\hat{L}}\frac{df_4}{dt} = -2k_1, \tag{14}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2023 doi:10.20944/preprints202310.0404.v1

$$\begin{gathered}\frac{df_1}{dt}+\frac{df_2}{dt}f_1\hat{L}+2k_{12}\frac{df_3}{dt}f_1+4k_1k_2f_1^2e^{2k_{12}f_3+f_2\hat{L}}\frac{df_4}{dt}=1,\\ \frac{df_2}{dt}=1,\\ \frac{df_3}{dt}+\frac{4k_1k_2}{k_{12}}f_1e^{2k_{12}f_3+f_2\hat{L}}\frac{df_4}{dt}=1,\\ e^{2k_{12}f_3+f_2\hat{L}}\frac{df_4}{dt}=1.\end{gathered}$$

The above system of differential equations depends on the superoperator $\hat{L}$, but, because there are no other superoperators in it, a simple solution may be found for the different functions

$$f_0\left(t,\hat{L}\right)=\left[\frac{\left(\hat{L}+2k_{12}\right)-4k_1}{2}\right]t, \tag{15}$$

$$\begin{gathered}+\ln\left(\frac{\cos\left(\theta(\hat{L})\right)}{\cos\left(\frac{\beta(\hat{L})}{2}t+\theta(\hat{L})\right)}\right),\\ f_1\left(t,\hat{L}\right)=\frac{\beta(\hat{L})}{8k_1k_2}\tan\left(\frac{\beta(\hat{L})}{2}t+\theta(\hat{L})\right)+\frac{\left(\hat{L}+2k_{12}\right)}{8k_1k_2},\end{gathered}$$

$$\begin{gathered}f_2\left(t\right)=t,\\ f_3\left(t,\hat{L}\right)=\frac{-\hat{L}}{2k_{12}}t+\frac{1}{k_{12}}\ln\left(\frac{\cos\left(\frac{\beta(\hat{L})}{2}t+\theta(\hat{L})\right)}{\cos\left(\theta(\hat{L})\right)}\right)\\ f_4\left(t,\hat{L}\right)=\frac{2\cos^2\left(\theta(\hat{L})\right)}{\beta(\hat{L})}\tan\left(\frac{\beta(\hat{L})}{2}t+\theta(\hat{L})\right)+\frac{2\left(\hat{L}+2k_{12}\right)\cos^2\left(\theta(\hat{L})\right)}{\left(\beta(\hat{L})\right)^2},\end{gathered}$$

with

$$\beta(\hat{L})=\sqrt{16k_1k_2-\left(\hat{L}+2k_{12}\right)^2}$$

and

$$\theta(\hat{L})=\arctan\left(-\frac{\left(\hat{L}+2k_{12}\right)}{\sqrt{16k_1k_2-\left(\hat{L}+2k_{12}\right)^2}}\right).$$

3. Linear entropy for the quantized field

Once we have calculated the total density matrix we may calculate the reduced field operator.

$$\hat{\rho}_f\left(t\right)=Tr_A\{\hat{\rho}\left(t\right)\} \tag{16}$$

where $\hat{\rho}\left(0\right)=\hat{\rho}_f\left(0\right)\hat{\rho}_A\left(0\right)$ is the initial density matrix for the total, atom-field, system. We consider an arbitrary initial field and the atom in a superposition of its excited and ground state, $\hat{\rho}_A\left(0\right)=|\psi_A\left(0\right)\rangle\langle\psi_A\left(0\right)|$, with $|\psi_A\left(0\right)\rangle=\frac{1}{\sqrt{2}}\left(|e\rangle+|g\rangle\right)$ therefore $\hat{\rho}_A\left(0\right)=\frac{1}{2}\left(\hat{I}+|e\rangle\langle g|+|g\rangle\langle e|\right)$.

After some algebra we obtain

$$\begin{aligned}\hat{\rho}_f\left(t\right) \quad = \quad & \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}d_1\left(t,r,k\right)\hat{a}^{\dagger k}e^{-k_{12}f_3(t,0)\hat{n}}\\ & \times[e^{i\chi\hat{n}t}\left(\hat{a}^r\hat{\rho}_f\left(0\right)\hat{a}^{\dagger r}\right)e^{-i\chi\hat{n}t}\\ & +e^{-i\chi\hat{n}t}\left(\hat{a}^r\hat{\rho}_f\left(0\right)\hat{a}^{\dagger r}\right)e^{i\chi\hat{n}t}]\\ & \times e^{-k_{12}f_3(t,0)\hat{n}}\hat{a}^k,\end{aligned} \tag{17}$$

with

$$d_1\left(t,r,k\right)=e^{f_0(t,0)}\frac{\left(2k_2f_4\left(t,0\right)\right)^r}{2r!}\frac{\left(2k_1f_1\left(t,0\right)\right)^k}{k!}. \tag{18}$$

If we consider that $\hat{\rho}_f(0) = |\alpha\rangle\langle\alpha|$, with the coherent state given by [13]$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, we obtain

$$\begin{aligned}\hat{\rho}_f(t) &= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2d_1(t,r,k) \\ &\times \frac{|\alpha|^{2r}\alpha^n\alpha^m}{\sqrt{n!}\sqrt{m!}} e^{-|\alpha|^2 - k_{12}f_3(t,0)(n+m)} \\ &\times \hat{a}^{\dagger k}|n\rangle\langle m|\hat{a}^k\cos(\chi(n-m)t)\end{aligned} \tag{19}$$

we need to calculate $\hat{a}^{\dagger k}|n\rangle$ which is easy to see that is given by

$$\hat{a}^{\dagger k}|n\rangle = \sqrt{\frac{(n+k)!}{n!}}|n+k\rangle$$

and substitituing $d_1(t,r,k)$, us to write

$$\begin{aligned}\hat{\rho}_f(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-k_{12}f_3(t,0)(n+m)}(2k_1f_1(t,0))^k \\ &\times \frac{\sqrt{(n+k)!(m+k)!}\cos(\chi(n-m)t)}{k!n!m!} \\ &\times \alpha^{n+m}e^{f_0(t,0)+(2k_2f_4(t,0)-1)|\alpha|^2}|n+k\rangle\langle m+k|\end{aligned} \tag{20}$$

setting $j = n+k$ y $l = m+k$ we find the reduced field operator

$$\begin{aligned}\hat{\rho}_f(t) &= e^{f_0(t,0)+(2k_2f_4(t,0)-1)|\alpha|^2}\sum_{r=0}^{\infty}\sum_{g=r}^{\infty}\sum_{h=r}^{\infty} \\ &\times \frac{\sqrt{g!h!}\left(\alpha e^{-k_{12}f_3(t,0)}\right)^{(g+h-2r)}}{r!(g-r)!(h-r)!} \\ &\times (2k_1f_1(t,0))^r\cos(\chi(g-h)t)|g\rangle\langle h|\end{aligned} \tag{21}$$

Once we have calculated the reduced field operator we may calculate the field entropy $\hat{S}_f$

$$\hat{S}_f(t) = 1 - Tr\left\{\hat{\rho}_f^2(t)\right\} \tag{22}$$

Substitituing the found solution for the reduced field operator (21) in the above equation, after some algebra we obtain the field entropy

$$\begin{aligned}\hat{S}_f(t) &= 1 - \left(e^{f_0(t,0)+(2k_2f_4(t,0)-1)|\alpha|^2}\right)^2 \\ &\times \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{n!j!(2k_1f_1(t,0))^{k+r}}{(n-r)!(j-r)!k!r!} \\ &\times \frac{\left(\alpha e^{-k_{12}f_3(t,0)}\right)^{2(j+n-k-r)}\cos^2(\chi(j-n)t)}{(j-k)!(n-k)!}\end{aligned} \tag{23}$$

4. Numerical results

We plot the linear field entropy (23) with the help of numerical analysis. In the Figures 1–3 we set the parameter $k_1 = 0.01$ and $k_1 = 0.05$ that correspond to a decay constant $C = 0.08$ and an average number of photons $\bar{n} = 0.25$. The linear entropy suffers oscillations that stabilize around a value of

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2023 doi:10.20944/preprints202310.0404.v1

$S_f \approx 0.33$. It may be seen that for smaller values of the coherent intensity, namely, $\alpha = 1$ in Figure 1 such stabilization is faster. This is because of the fact that there are not many coherent photons, initially, inside the cavity and they are replaced by the thermal field as the cavity is at a finite temperature. Eventually, the cavity is filled by a thermal field

$$\rho_{Th} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} |n\rangle\langle n|, \tag{24}$$

that has a linear entropy of [see equation (3)]

$$S_{Th} = 1 - \sum_{n=0}^{\infty} \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \right)^2, \tag{25}$$

that, for the parameters chosen in Figures 1–3, $S_{Th} \approx 0.33$.

The same occurs with the linear entropies plotted in Figures 4–6. In these figures, we set $k_1 = 0.005$ and $k_2 = 0.01$ which deliver a decay constant $C = 0.01$, smaller than the previous examples, which means that the emptying of the coherent filed and the filling of the thermal field are slower. On the other hand, the average number of photons for this choice of parameters is about $\bar{n} = 1$ that produces a linear entropy for the thermal field of about $S_{Th} \approx 0.66$, precisely the stabilization line shown in Figures 4–6.

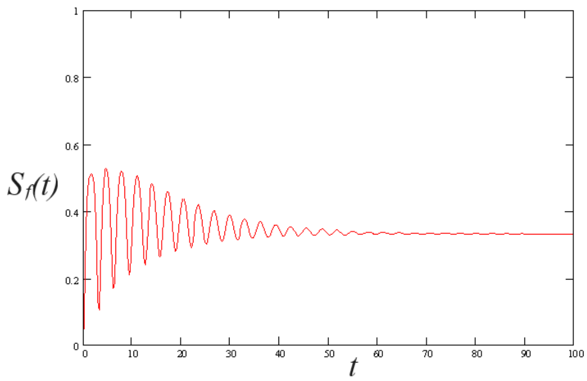

Figure 1. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the parameters $\chi = 1$, $k_1 = 0.01$, $k_2 = 0.05$, and $\alpha = 1$.

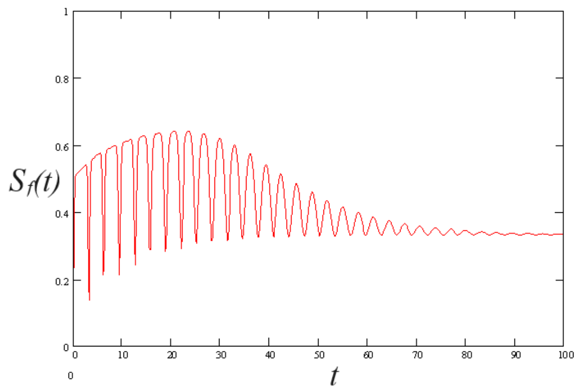

Figure 2. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the same parameters as in Figure 1 except the amplitude of the initial coherent field: $\alpha = 3$.

 doi:10.20944/preprints202310.0404.v1

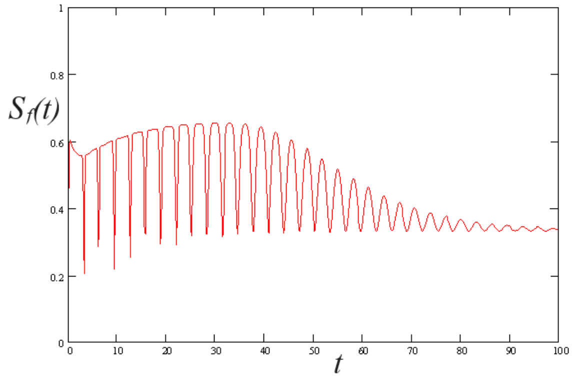

Figure 3. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the same parameters as in Figure 1 except the amplitude of the initial coherent field: $\alpha = 5$.

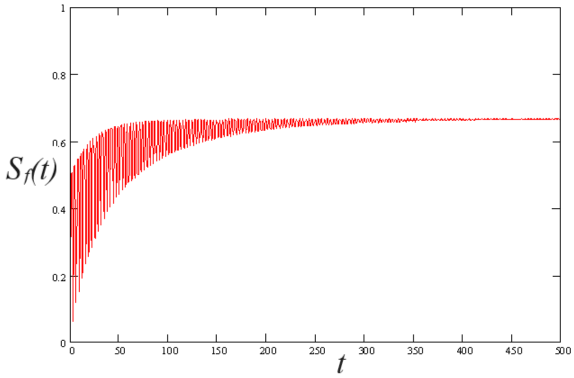

Figure 4. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the parameters $\chi = 1$, $k_1 = 0.005$, $k_2 = 0.01$, and $\alpha = 1$.

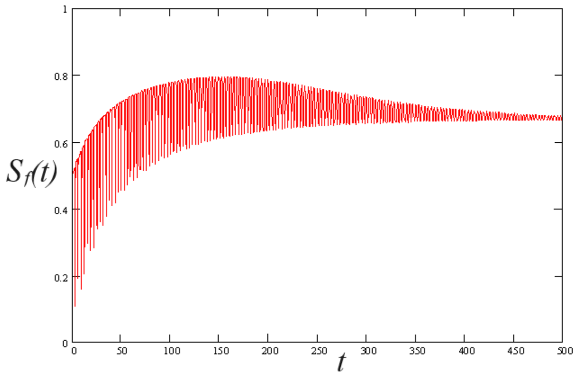

Figure 5. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the same parameters as in Figure 4 except the amplitude of the initial coherent field: $\alpha = 3$.

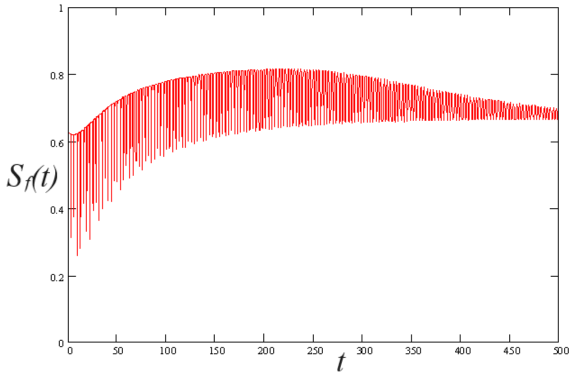

Figure 6. Plot of the linear entropy, $S_f(t)$, as a function of time. We set the same parameters as in Figure 4 except the amplitude of the initial coherent field: $\alpha = 5$.

5. Conclusions

We have studied the atom-field interaction at finite temperature when the field is initially in a coherent state and in the dispersive regime. By using superoperator techniques, we manage to solve the master equation for this system and have calculated the linear entropy. We have shown that the linear entropy for the dispersive interaction subject to decay at $T > 0$ approaches the limit given by the thermal distribution of the bath.

Appendix A

Here we list the commutators between the different superoperators needed to solve equation (7),

$$\begin{aligned}
\left[\hat{S},\hat{J}_1\right]\hat{\rho} &= -2k_{12}\hat{J}_1\hat{\rho},\\
\left[\hat{R},\hat{J}_2\right]\hat{\rho} &= \hat{L}\hat{J}_2\hat{\rho},\\
\left[\hat{R},\hat{J}_1\right]\hat{\rho} &= -\hat{L}\hat{J}_1\hat{\rho},\\
\left[\hat{S},\hat{J}_2\right]\hat{\rho} &= 2k_{12}\hat{J}_2\hat{\rho},\\
\left[\hat{J}_1,\hat{J}_2\right]\hat{\rho} &= \frac{4k_1k_2}{k_{12}}\hat{S}\hat{\rho} - 4k_1k_2\hat{\rho},\\
\left[\hat{R},\hat{L}\right]\hat{\rho} &= 0,\\
\left[\hat{R},\hat{S}\right]\hat{\rho} &= 0,\\
\left[\hat{S},\hat{L}\right]\hat{\rho} &= 0,\\
\left[\hat{J}_1,\hat{L}\right]\hat{\rho} &= 0,\\
\left[\hat{J}_2,\hat{L}\right]\hat{\rho} &= 0.
\end{aligned}$$

It is worth to mention that the superoperators involved in the master equation, namely, $\hat{J}_1$, $\hat{J}_2$ and $\hat{S}$ commute in such a way that only give the same superoperators, allowing an algebraic solution.

References

1. C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, 2005).
2. S. Abdel-Kahlek, K. Berrada, and H. Eleuch, Effect of the time-dependent coupling on a superconducting qubit-field system under decoherence: Entanglement and wehrl entropy, Ann. of Phys **361**, 247–258. (2015).
3. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955).
4. H. Araki and E. H. Lieb, Entropy inequalities, Commun. Math. Phys. **18**, 160–170. (1970).
5. S. J. D. Phoenix and P. L. Knight, Fluctuations and entropy in models of quantum optical resonance, Annals of Physics **186**, 381–407. (1988).
6. A. Zúñiga-Segundo, R. Juárez-Amaro, O. Aguilar-Loreto, and H. Moya-Cessa, Field's entropy in the atom-field interaction: Statistical mixture of coherent states, Annals of Physics **379**, 150–158. (2017).

7. B. M. Rodríguez-Lara and H. M. Moya-Cessa, Por qué y cómo encontramos funciones de matrices: entropia en mecánica cuántica, Revista Mexicana de Física E **51**, 87–98 (2005).
8. A. Motazedifard, S. A. Madani, and N. S. Vayaghan, Measurement of entropy and quantum coherence properties of two type-i entangled photonic qubits, Opt. Quant. Electr. **53**, 378–401. (2021).
9. E. T. Jaynes and F. W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, Proc. IEEE. **51**, 89–109. (1963).
10. M. Satyanarayana, P. Rice, V. R., and H. Carmichael, Ringing revivals in the interaction of a two-level atom with squeezed light, J. Opt. Soc. Am. B **6**, 228–237. (1989).
11. H. Moya-Cessa and A. Vidiella-Barranco, Interaction of squeezed states of light with two-level atoms, J. of Mod. Optics **39**, 2481–2499. (1992).
12. A. Vidiella-Barranco, H. Moya-Cessa, and V. Buzek, Interaction of superpositions of coherent states of light with two-level atoms, J. of Mod. Optics **39**, 1441–1459. (1992).
13. R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. **131**, 2766 (1963).
14. S. M. Barnett, A. Beige, A. Ekert, B. M. Garraway, C. H. Keitel, V. Kendon, M. Lein, G. J. Milburn, H. Moya-Cessa, M. Murao, J. K. Pachos, G. M. Palma, E. Paspalakis, S. J. D. Phoenix, B. Piraux, M. B. Plenio, B. C. Sanders, J. Twanmley, A. Vidiella-Barranco, and M. S. Kim, Journeys from quantum optics to quantum technology, Progress in Quantum Electronics **54**, 19–45. (2017).
15. J. Gea-Banacloche, Atom- and field-state evolution in the jaynes-cummings model for large initial fields, Phys. Rev. A **44**, 5913–5931. (1991).
16. R. Juárez-Amaro, L. M. Rosales, J. A. Anaya-Contreras, Zúñiga-Segundo, and H. M. Moya-Cessa, Relation between the entropy and the linear entropy in the ion–laser interaction, Journal of Modern Optics **67**, 805–810. (2020).
17. S. J. D. Phoenix, Wave-packet evolution in the damped oscillator, Phys. Rev. A **41**, 5132–5138. (1990).
18. L. M. Arévalo-Aguilar and H. M. Moya-Cessa, Quantum bits and superposition of displaced fock states of the cavity field, Revista Mexicana de Física **48**, 423 (2002).
19. H. Wu, X. Fan, and L. Chen, How gravitational fluctuations degrade the high dimensional spatial entanglement, Phys. Rev. D **106**, 045023. (2022).
20. H. Wu, X. Fan, and L. Chen, Twisting and entangling gravitons in high-dimensional orbital agular momentum states ia photon-graviton conversion, Phys. Rev. D **107**, 125027 (2023).
21. A. Perez-Leija, D. Guzmán-Silva, L.-M. R. de J., M. Gräfe, M. Heinrich, H. M. Moya-Cessa, K. Busch, and A. Szameit, Endurance of quantum coherence due to particle indistinguishability in noisy quantum networks, npj Quantum Information **4**, 45 (2018).
22. N. Yazdanpanah, M. K. Tavassoly, R. Juárez-Amaro, and H. M. Moya-Cessa, Reconstruction of quasiprobability distribution functions of the cavity field considering field and atomic decays, Optics Communications **400**, 69–73. (2017).
23. H. Moya-Cessa, Entropy operator and associated wigner function, International Journal of Quantum Information **5**, 149 (2023).
24. J. Moya-Cessa, L. Berriel-Valdos, and H. Moya-Cessa, Optical production of the husimi function of two gaussian functions, Applied Mathematics Infromation Sciences **2**, 309–316 (2008).
25. A. B. Klimov and L. L. Sanchez-Soto, Method of small rotations and effective hamiltonians in nonlinear quantum optics, Physical Review A **61**, 063802 (2000).
26. S. M. Chumakov, A. B. Klimov, and C. Saavedra, Dispersive atomic evolution in a dissipative-driven cavity, Physical Review A **61**, 033814 (2000).
27. A. B. Klimov, L. L. Sanchez-Soto, and J. Delgado, Mimicking a kerrlike medium in the dispersive regime of second-harmonic generation, Optics Communications **191**, 419 (2001).
28. L. M. Arévalo-Aguilar and H. Moya-Cessa, Cavidad con pérdidas: una descripción usando superoperadores, Revista Mexicana de Física **42**, 675 (1996)