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Article

Generalized EP Elements in Banach *-Algebras

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Abstract: We introduce a new generalized inverse (i.e., generalized EP element) which is a natural generalization of EP and *-DMP elements in a Banach *-algebra. We present polar-like characterizations of generalized EP elements. The necessary and sufficient conditions under which the sum of two generalized EP elements is a generalized EP element are investigated. Finally, the generalized core-EP orders for generalized EP elements are characterized.

Keywords: EP element; *-DMP element; generalized core-EP inverse; additive property; generalized core-EP order; Banach algebra

MSC: 15A09; 16U90; 46H05

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra with involution $*$. An element a in $\mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that $ax^2 = x, (ax)^* = ax, a = xa^2$. Such x is unique if it exists and is denoted by $a^{\textcircled{\#}}$. The core inverse was extensively considered in the context of Banach algebras, e.g., [1,6,17–19,26].

An element a in $\mathcal{A}$ is EP (i.e., an EP element) if there exists some $x \in \mathcal{A}$ such that $ax^2 = x, (ax)^* = xa, a = xa^2$. Evidently, $a \in \mathcal{A}$ is EP if and only if there exists $x \in \mathcal{A}$ such that $a^2x = a, ax = xa, (ax)^* = ax$ if and only if there exists $x \in \mathcal{A}$ such that $ax^2 = x, (xa)^* = xa, xa^2 = a$ if and only if $a \in \mathcal{A}^{\#}$ and $(aa^{\#})^* = aa^{\#}$ ([2,16,23–25,27]). Here, $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that $ax^2 = x, ax = xa, a = xa^2$. Such x is unique if exists, denoted by $a^{\#}$, and called the group inverse of a.

An element a in a Banach *-algebra $\mathcal{A}$ is *-DMP (i.e., *-DMP element) if there exist $m \in \mathbb{N}$ and $x \in \mathcal{A}$ such that $ax^2 = x, (ax)^* = xa = ax, a^m = xa^{m+1}$. As is well known, $a \in \mathcal{A}$ is *-DMP if and only if $a^m \in \mathcal{A}$ is EP for some $m \in \mathbb{N}$ (see [6,11,13]). In [21], Mosic and Djordjevic introduced and studied the gDMP inverse for a Hilbert space operator using its generalized Drazin inverse and its Moore-Penrose inverse.

The motivation of this paper is to introduce and study a new kind of generalized inverse as a natural generalization of EP and *-DMP elements mentioned above. Let $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim\limits_{n\to\infty} \| x^n \|^{\frac{1}{n}} = 0\}$. As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$.

Definition 1.1. *An element $a \in \mathcal{A}$ is generalized EP (i.e., generalized EP element) if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A} \text{ is EP}, \ y \in \mathcal{A}^{qnil}.$$

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that $ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}$. Such x is unique, if exists, and denote it by a^d. AS it is well known, a has g-Drazin inverse if and only if a has quasi-polar property, i.e., there exists an idempotent $p \in \mathcal{A}$ such that $a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$ (see [3]). In Section 2, we investigate polar-like characterizations of generalized EP elements. We prove that $a \in \mathcal{A}$ is generalized EP if and only if there exists a projection $p \in \mathcal{A}$ (i.e., $p^2 = p = p^*$)such that $a + p \in \mathcal{A}^{-1}, ap = pa$ and $ap \in \mathcal{A}^{qnil}$.

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In Section 3, we are concerned with additive properties of generalized EP elements. The necessary and sufficient conditions under which the sum of two generalized EP elements is a generalized EP element are investigated by using orthogonal and commuting perturbations.

An element $a \in \mathcal{A}$ has generalized core-EP inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

The preceding x is unique if it exists, and denoted by $a^{\text{\textcircled{d}}}$. Let $a, b \in \mathcal{A}$ have generalized core-EP inverses. Recall that $a \leq^{\text{\textcircled{d}}} b$ if $aa^{\text{\textcircled{d}}} = ba^{\text{\textcircled{d}}}$ and $a^{\text{\textcircled{d}}}a = a^{\text{\textcircled{d}}}b$. We refer the reader to [4] for properties of generalized core-EP inverses in a Banach *-algebra. Finally, in Section 4, the generalized core EP-orders for generalized EP elements in a Banach *-algebra are characterized. The properties of core-EP orders are thereby extended to wider cases.

Throughout the paper, all Banach *-algebras are complex with an identity. An element p in $\mathcal{A}$ is a projection provided that $p^2 = p = p^*$. We use $\mathcal{A}^{\text{\textcircled{d}}}$ and $\mathcal{A}^{\text{\textcircled{e}}}$ to denote the sets of all generalized core-EP invertible and generalized EP elements in $\mathcal{A}$. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$.

2. Polar-like Characterizations

In this section, we present a polar-like property for EP elements in a Banach *-algebra. The related characterize of EP elements are thereby derived. We begin with

Lemma 2.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\text{\textcircled{e}}}$.
(2) There exists $x \in comm(a)$ such that

$$ax^2 = x, (ax)^* = ax, a - xa^2 \in \mathcal{A}^{qnil}.$$

(3) There exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = xa, a - xa^2 \in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that $a = z + y, z^*y = yz = 0, z \in \mathcal{A}$ is EP, $y \in \mathcal{A}^{qnil}$. Set $x = z^{\text{\textcircled{\#}}}$. Then $zz^{\text{\textcircled{\#}}} = z^{\text{\textcircled{\#}}}z$ by [27, Lemma 3.4]. Hence $z^{\text{\textcircled{\#}}}y = z^{\text{\textcircled{\#}}}zz^{\text{\textcircled{\#}}}y = z^{\text{\textcircled{\#}}}(zz^{\text{\textcircled{\#}}})^*y = z^{\text{\textcircled{\#}}}(z^{\text{\textcircled{\#}}})^*(z^*y) = 0$. We check that $ax = (z+y)z^{\text{\textcircled{\#}}} = zz^{\text{\textcircled{\#}}}, xa = z^{\text{\textcircled{\#}}}(z+y) = z^{\text{\textcircled{\#}}}z; ax^2 = (z+y)(z^{\text{\textcircled{\#}}})^2 = z(z^{\text{\textcircled{\#}}})^2 = z^{\text{\textcircled{\#}}} = x, xa^2 = z^{\text{\textcircled{\#}}}(z+y)^2 = z^{\text{\textcircled{\#}}}z^2 = z$. Therefore $(ax)^* = (zz^{\text{\textcircled{\#}}})^* = zz^{\text{\textcircled{\#}}} = ax, ax = zz^{\text{\textcircled{\#}}} = z^{\text{\textcircled{\#}}}z = xa, xa^2 - a = z - a = -y \in \mathcal{A}^{qnil}$.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) By hypotheses, we have $z \in \mathcal{A}$ such that $az^2 = z, (az)^* = za, a - za^2 \in \mathcal{A}^{qnil}$. Then $(za)^* = \big((az)^*\big)^* = az$. In view of [25, Lemma 2.1], $az = za$. Set $x = aza$ and $y = a - aza$. We claim that x is EP. Evidently, we verify that $zx^2 = za(za^2z)a = za^2za = aza = x, xz^2 = azaz^2 = az^2 = z, xz = azaz = zaza = zx, (xz)^* = (azaz)^* = (az)^* = az = (aza)z = xz$. Therefore $x \in \mathcal{A}$ is EP.

By hypothesis, $a(1 - za) \in \mathcal{A}^{qnil}$. By virtue of Cline's formula (see [3, Theorem 15.1.14]), $y = a - za^2 \in \mathcal{A}^{qnil}$. Moreover, we see that $x^*y = (aza)^*(1 - az)a = a^*(az)^*(1 - az)a = a^*(az)(1 - az)a = 0, yx = (a - aza)aza = a(a - za^2)za = 0$. This completes the proof. □

We are ready to prove:

Theorem 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

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(1) $a \in \mathcal{A}^{\circledcirc}$.
(2) *There exists a projection* $p \in comm(a)$ *such that* $a + p \in \mathcal{A}^{-1}$ *and* $ap \in \mathcal{A}^{qnil}$.

Proof. (1) $\Rightarrow$ (2) In view of Lemma 2.1, there exists $x \in comm(a)$ such that $ax^2 = x, (ax)^* = ax, a - xa^2 \in \mathcal{A}^{qnil}$. Let $p = 1 - ax$. Then $p = p^2 \in comm(a)$ and $a + p = a + 1 - ax$. We check that $(a+1-ax)(x+1-ax) = ax + a(1-ax) + (1-ax)x + (1-ax) = 1$. Likewise, $(x+1-ax)(a+1-ax) = 1$. Then $a + p \in \mathcal{A}^{-1}$. Additionally, $ap = a - a^2x \in \mathcal{A}^{qnil}$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $p \in comm(a)$ such that $a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$. Set $x = (1-p)(a+p)^{-1}$. Then $x \in comm(a)$ and $ax^2 = a(1-p)(a+p)^{-2} = (1-p)(a+p)(a+p)^{-2} = (1-p)(a+p)^{-1} = x$. Moreover, we verify that $a - a^2x = a - a^2(1-p)(a+p)^{-1} = a - (a+p)^2(1-p)(a-p)^{-1} = a - (a+p)(1-p) = ap \in \mathcal{A}^{qnil}$, as desired. $\square$

Corollary 2.3. *Every generalized EP element in a Banach *-algebra is the sum of three invertible elements.*

Proof. Let $a \in \mathcal{A}^{\circledcirc}$. In view of Theorem 2.2, we have $p^2 = p = p^* \in \mathcal{A}$ such that $u := a + p \in \mathcal{A}^{-1}$. Then $a = u - p$. Obviously, $-p = \frac{1-2p}{2} - \frac{1}{2}$. It is easy to verify that

$$(\frac{1-2p}{2})^2 = \frac{1}{4},$$

and so

$$(\frac{1-2p}{2})^{-1} = 2(1-2p).$$

Therefore $a = u + \frac{1-2p}{2} - \frac{1}{2}$, as desired. $\square$

Theorem 2.4. *Every Hermitan periodic element (i.e.,* $a^* = a$ *and* $a^k = a^k$ *for some distinct* $k, l \in \mathbb{N}$*) in a Banach *-algebra is generalized EP.*

Proof. Assume that $a^* = a$ and $a^k = a^k$ for some positive integers $k, l (k > l)$. Then $a^l = a^k = a^{(k-l)+l} = \cdots = a^{l(k-l)+l}$, and so $a^l = (a^l)^{k-l+1}$. Choose $m = l(k-l)$. Then $a - a^{m+1} \in \mathcal{A}^{nil}$. According to the Dirichlet Theorem, there exists a prime k such that $k = sm + 1$ for some $s \in \mathbb{N}$. One easily checks that

$$\begin{aligned} a - aa^m &\in \mathcal{A}^{nil}, \\ aa^m - aa^{2m} = (a - a^{m+1})a^m &\in \mathcal{A}^{nil}, \\ aa^{2m} - aa^{3m} = (a - a^{m+1})a^{2m} &\in \mathcal{A}^{nil}, \\ &\vdots \\ aa^{(s-1)m} - aa^{sm} = (a - a^{m+1})a^{(s-1)m} &\in \mathcal{A}^{nil}. \end{aligned}$$

Therefore $a - a^k = a - a^{sm+1} = \sum\limits_{i=0}^{s-1}(aa^{im} - aa^{(i+1)m} \in \mathcal{A}^{nil}$. Set $n = k - 1$. Then $a - a^{n+1} \in \mathcal{A}^{nil}$.

In view of [9, Theorem 3.5], there exists some $e^{n+1} = e \in \mathbb{Z}[a, \frac{1}{n}]$ such that

$$w := a - e \in \mathbb{Z}[\frac{1}{n}](a^{n+1} - a) \subseteq \mathcal{A}^{nil}.$$

Thus, $ea = ae$. Let $q = e^{n-1}$. Then $q^2 = e^{2n-1} = e^{n+1}e^{n-2} = e^{n-1} = q \in comm(a)$. Since $a = a^*$, we see that $q = q^*$. Since $a - e \in \mathcal{A}^{nil}$ and $ea = ae$, we see that $a^{n-1} - q \in \mathcal{A}^{nil}$. Set $p = 1 - q$. Then $p^2 = p^* = p \in comm(a)$. One easily checks that

$$a^{n-1} + p^{n-1} = a^{n-1} + p = (a^{n-1} - q) + 1 \in \mathcal{A}^{-1}.$$

Since $n-1=k$ is prime, we see that $a+p\in \mathcal{A}^{-1}$. Moreover, we have that

$$\begin{array}{rcl} a^{n-1}p & = & a^{n-1}(1-q)=[a^{n-1}-q](1-q) \\ & \in & \mathcal{A}^{nil}. \end{array}$$

Hence $(ap)^{n-1}\in \mathcal{A}^{nil}$, and so $ap\in \mathcal{A}^{nil}$. Therefore $a\in \mathcal{A}^{\circledcirc}$ by Theorem 2.2. □

We are ready to prove:

Theorem 2.5. *Let $a\in \mathcal{A}$. Then the following are equivalent:*

(1) $a\in \mathcal{A}^{\circledcirc}$.
(2) There exists $x\in comm(a)$ such that

$$(ax)^*=ax, a-a^2x\in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) This is obvious by Lemma 2.1.

(2) $\Rightarrow$ (1) By hypothesis, there exists some $x\in comm(a)$ such that $(ax)^*=ax, a-a^2x\in \mathcal{A}^{qnil}$. Set $s=xax$. Then $z\in comm(a)$. We check that

$$\begin{array}{rcl} a-a^2s & = & a-axaxa \\ & = & (1+ax)(a-a^2x) \\ & \in & \mathcal{A}^{qnil}, \\ s-s^2a & = & xax-xaxaxax \\ & = & x(a-a^2x)x+xax(a-a^2x)x \\ & \in & \mathcal{A}^{qnil}. \end{array}$$

$$as-(as)^2=(a-a^2s)s\in \mathcal{A}^{qnil}.$$

Let

$$q=as-(as)^2, z=-\frac{1}{2}\sum_{k=1}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)\left(4q(4q-1)^{-1}\right)^k, e=as-(2as-1)z.$$

As in the proof of [3, Lemma 15.1.2], we have an idempotent $e\in comm^2(az)$ such that $az-e\in \mathcal{A}^{qnil}$. We easily check that

$$(a+1-as)((s+1-as)=1+(a-a^2s)(1-s)+(s-s^2a).$$

Hence,

$$\begin{array}{rcl} a+1-e & = & (a+1-as)+(as-e)\in \mathcal{A}^{-1}, \\ a(1-e) & = & (a-a^2s)+a(as-e)\in \mathcal{A}^{qnil}. \end{array}$$

Since $a\in comm(as)$, we have $ea=ae$. As $(ax)^*=ax$, we see that $(as)^*=as$. This implies that $q^*=q$, and then $z^*=z$. Therefore $e^*=e$. In light of Theorem 2.2, $a\in \mathcal{A}^{\circledcirc}$. □

Theorem 2.6. *Let $a\in \mathcal{A}$. Then the following are equivalent:*

(1) $a\in \mathcal{A}^{\circledcirc}$.
(2) *There exists a projection $p\in comm(a)$ such that*

$$1-p\in a\mathcal{A}a \text{ and } pap\in \mathcal{A}^{qnil}.$$

(3) *There exists a projection $p\in comm(a)$ such that*

$$pap\in (p\mathcal{A}p)^{qnil} \text{ and } (1-p)a(1-p)\in \left((1-p)\mathcal{A}(1-p)\right)^{-1}.$$

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Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 2.2, there exists a projection $p \in comm(a)$ such that $u := a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$. By using Cline's formula, we have $pap \in \mathcal{A}^{qnil}$. Moreover, $(1-p)a = (1-p)u$, and then $u^{-1}(1-p)a = 1-p$. Similarly, $a(1-p)u^{-1} = 1-p$. Accordingly, $1-p = a(1-p)u^{-2}(1-p)a \in a\mathcal{A}a$, as required.

(2) $\Rightarrow$ (3) As $pap \in \mathcal{A}^{qnil}$, we see that $pap \in (e\mathcal{A}e)^{qnil}$. Write $1-e = ara$ for some $r \in \mathcal{A}$. Hence, $(1-p)a(1-p)(1-p)ra(1-p) = 1-p = (1-p)ar(1-p)a(1-p)$. Therefore $(1-p)a(1-p) \in \big((1-p)\mathcal{A}(1-p)\big)^{-1}$, as desired.

(3) $\Rightarrow$ (1) By using Cline's formula, we have $pa, ap \in \mathcal{A}^{qnil}$. Since $(1-p)a(1-p) \in \big((1-p)\mathcal{A}(1-p)\big)^{-1}$, we have some $b \in \mathcal{A}$ such that $(1-p)a(1-p)b(1-p) = (1-p)b(1-p)a(1-p) = 1-p$. This implies that

$$(a+p)\big((1-p)b(1-p)+p\big) = a(1-p)b(1-p) + ap + p = 1 + ap \in \mathcal{A}^{-1}.$$

Analogously, $\big((1-p)b(1-p)+p\big)(a+p) = 1 + pa \in \mathcal{A}^{qnil}$. Accordingly, $a + p \in \mathcal{A}^{qnil}$ and $ap \in \mathcal{A}^{qnil}$, thus yielding the result. □

Corollary 2.7. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{e}}$.
(2) *There exists a projection $p \in comm(a)$ such that*

$$1 - p \in a\mathcal{A} \bigcap \mathcal{A}a \text{ and } pap \in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) This is clear by Theorem 2.6 as $a\mathcal{A}a \subseteq a\mathcal{A} \bigcap \mathcal{A}a$.

(2) $\Rightarrow$ (1) By hypothesis, there exists a projection $p \in comm(a)$ such that

$$1 - p \in a\mathcal{A} \bigcap \mathcal{A}a \text{ and } pap \in \mathcal{A}^{qnil}.$$

Write $1 - p = ax = ya$ for some $x, y \in \mathcal{A}$. Then $1 - p = ax = ax(1-p) = axyz \in a\mathcal{A}a$. This completes the proof by Theorem 2.6. □

An element $a \in \mathcal{A}$ has generalized core-EP inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n \to \infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

the preceding x is unique if it exists, and denoted by $a^{\textcircled{d}}$. We use $\mathcal{A}^{\textcircled{d}}$ to stand for the set of all generalized core-EP invertible element a in $\mathcal{A}$. We refer the reader to [4] for more properties of the generalized core-EP inverse in Banach *-algebra. We say that a has dual generalized core-EP inverse if $a^* \in \mathcal{A}^{\textcircled{d}}$ denote $a_{\textcircled{d}} = \big((a^*)^{\textcircled{d}}\big)^*$. We now derive

Theorem 2.8. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{e}}$.
(2) $a \in \mathcal{A}^d$ *and* a^π *is a projection.*
(3) $a \in \mathcal{A}^{\textcircled{d}}$ *and* $a^{\textcircled{d}} = a^d$.
(4) $a \in \mathcal{A}^{\textcircled{d}} \bigcap \mathcal{A}_{\textcircled{d}}$ *and* $a^{\textcircled{d}} = a_{\textcircled{d}}$.

Proof. (1) $\Rightarrow$ (2) This is obvious by Lemma 2.1.

(2) $\Rightarrow$ (1) Since $a \in \mathcal{A}^d$, there exists $x \in comm(a)$ such that $ax^2 = x, a - a^2x \in \mathcal{A}^{qnil}$. By hypothesis, $a^\pi = 1 - ax$ is a projection. Hence, $(ax)^* = ax$. In view of Lemma 2.1, $a \in \mathcal{A}^{\textcircled{e}}$.

(1) $\Rightarrow$ (3) Since every EP element has core inverse, it follows by [4, Theorem 1.2] that $a \in \mathcal{A}^{\textcircled{d}}$; hence, $a \in \mathcal{A}^d$. By the uniqueness of the g-Drazin inverse of a, we have $a^{\textcircled{d}} = a^d$.

$(3) \Rightarrow (1)$ Since $a \in \mathcal{A}^{\textcircled{d}}$, there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

By hypothesis, $x = a^d$, nd then $x \in comm(a)$. This implies that $\lim_{n\to\infty} ||(a - xa^2)^n||^{\frac{1}{n}} = 0$, and so $a - xa^2 \in \mathcal{A}^{qnil}$. Therefore $a \in \mathcal{A}^{\textcircled{e}}$ by Theorem 2.5.

$(1) \Rightarrow (4)$ By the discussion above, $a \in \mathcal{A}^{\textcircled{d}}$ and $a^{\textcircled{d}} = a^d$. Dually, we have $a_{\textcircled{d}} = \big((a^*)^d\big)^* = a^d = a^{\textcircled{d}}$, as desired.

$(4) \Rightarrow (1)$ Since $a \in \mathcal{A}^{\textcircled{d}}$, it follows by [4, Theorem 1.2] that there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\textcircled{\#}},\ y \in \mathcal{A}^{qnil}.$$

By hypothesis, $x^{\textcircled{\#}} = a^{\textcircled{d}} = a_{\textcircled{d}}\big((a^*)^{\textcircled{d}}\big)^*$. Hence, $[a^*(x^{\textcircled{\#}})^*]^* = a^*(x^{\textcircled{\#}})^*$. This implies that $(x^{\textcircled{\#}}a)^* = x^{\textcircled{\#}}a$. We easily check that

$$\begin{array}{rcl} x^{\textcircled{\#}}a & = & x^{\textcircled{\#}}(x+y) \\ & = & x^{\textcircled{\#}}x + x^{\textcircled{\#}}xx^{\textcircled{\#}}y \\ & = & x^{\textcircled{\#}}x + x^{\textcircled{\#}}(x^{\textcircled{\#}})^*(x^*y) \\ & = & x^{\textcircled{\#}}x. \end{array}$$

Then $(x^{\textcircled{\#}}x)^* = x^{\textcircled{\#}}x$. In view of [27, Theorem 2.2], $x \in \mathcal{A}^{\textcircled{\#}}$. Accordingly, $a \in \mathcal{A}^{\textcircled{e}}$, as asserted. □

Corollary 2.9. *Let $A \in \mathcal{B}(X)$. Then the following are equivalent:*

(1) $A \in \mathcal{B}(X)^{\textcircled{e}}$.
(2) $A \in \mathcal{B}(X)^d$ *and* $AA^{\textcircled{d}}A \in \mathcal{B}(X)$ *is EP.*
(3) $A \in \mathcal{B}(X)^d$ *and* $A^{\textcircled{d}}A^*AA^{\pi} = 0$.

Proof. $(1) \Rightarrow (2)$ In view of Theorem 2.8, $A \in \mathcal{B}(X)^d$. By hypothesis, there exist $X, Y \in \mathcal{B}(X)$ such that

$$A = X + Y, X^*Y = YX = 0, X \in \mathcal{B}(X) \text{ is EP},\ Y \in \mathcal{B}(X)^{qnil}.$$

In view of [24, Corollary 5], $X = AA^{\textcircled{d}}A$. Hence, $AA^{\textcircled{d}}A \in \mathcal{B}(X)$ is EP.

$(2) \Rightarrow (1)$ In view of [24, Theorem 4], there exist unique $X, Y \in \mathcal{B}(X)$ such that

$$A = X + Y, X^*Y = 0, YX = 0, X \in \mathcal{B}(X)^{\#},\ Y \in \mathcal{B}(X)^{qnil}.$$

Explicitly, $X = AA^{\textcircled{d}}A$. By hypothesis, X is EP. Therefore $A \in \mathcal{B}(X)^{\textcircled{e}}$.

$(1) \Rightarrow (3)$ Clearly, $A \in \mathcal{B}(X)^d$. By hypothesis, there exist $X, Y \in \mathcal{B}(X)$ such that

$$A = X + Y, X^*Y = YX = 0, X \in \mathcal{B}(X) \text{ is EP},\ Y \in \mathcal{B}(X)^{qnil}.$$

By virtue of [24, Corollary 1], $X = A^2A^{\textcircled{d}}$. Therefore

$$X^*Y = \big(A^2A^{\textcircled{d}}\big)^*\big(A - A^2A^{\textcircled{d}}\big) = 0.$$

Hence,

$$AA^{\textcircled{d}}A^*A\big(I - AA^{\textcircled{d}}\big) = 0.$$

$$A^{\textcircled{d}}A^*AA^{\pi} = 0.$$

$(3) \Rightarrow (1)$ In view of [24, Theorem 1], there exist unique $X, Y \in \mathcal{B}(X)$ such that

$$A = X + Y, YX = 0, X \in \mathcal{B}(X) \text{ is EP},\ Y \in \mathcal{B}(X)^{qnil}.$$

Explicitly, $X = A^2A^{\textcircled{d}}$. By hypothesis, $X^*Y = 0$. Therefore $A \in \mathcal{B}(X)^{\textcircled{e}}$. □

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3. Additive Properties

In this section, we are concerned with additive properties of generalized EP elements. Let $a, p^2 = p \in \mathcal{A}$. Then a has the Pierce decomposition relative to p, and we denote it by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p$. We now derive

Lemma 3.1. *Let p be a projection, $a \in (p\mathcal{A}p)^d$ and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p$. Then $x \in \mathcal{A})$ is generalized EP if and only if $a, d \in \mathcal{A}$ are generalized EP and*

$$\sum_{n=0}^{\infty} a^n a^{\pi} b (d^d)^{n+2} = 0,$$
$$\sum_{n=0}^{\infty} (a^d)^{n+2} b d^n d^{\pi} = 0.$$

Proof. $\Longrightarrow$ Since $x \in \mathcal{A}^{ⓔ}$, it follows by [4, Theorem 1.2] that $x^d \in \mathcal{A}^{\#}$. In this case, $(x^d)^{\#} = (x^{ⓔ})^d$ and $x^{ⓔ} = (x^d)^2 (x^d)^{\#}$. In view of Theorem 2.8, $x^{ⓔ} = x^d$. By virtue of [8, Theorem 2.1], we can write $x^{ⓔ} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and so $(x^{ⓔ})^d = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. This implies that $(x^d)^{\#} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Obviously, we have $x^d = \begin{pmatrix} a^d & z \\ 0 & d^d \end{pmatrix}_p$, where

$$z = \sum_{n=0}^{\infty} (a^d)^{n+2} b d^n d^{\pi} + \sum_{n=0}^{\infty} a^n a^{\pi} b (d^d)^{n+2} - a^d b d^d.$$

In light of [26, Theorem 2.5], $a^d, d^d \in \mathcal{A}^{\#}$ and $(a^d)^{\pi} z = 0$. This implies that $a^{\pi} z = 0$. Since $x^d \in \mathcal{A}_{\#}$, then $(x^d)^* \in \mathcal{A})^{\#}$. It follows by [26, Theorem 2.5] that $((d^*)^d)^{\pi} z^* = 0$, and so $z(d^d)^{\pi} = 0$, i.e., $zd^{\pi} = 0$. We easily check that

$$\begin{aligned} a^{\pi} z &= \sum_{n=0}^{\infty} a^n a^{\pi} b (d^d)^{n+2}, \\ z d^{\pi} &= \sum_{n=0}^{\infty} (a^d)^{n+2} b d^n d^{\pi}. \end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty} a^n a^{\pi} b (d^d)^{n+2} = 0,$$
$$\sum_{n=0}^{\infty} (a^d)^{n+2} b d^n d^{\pi} = 0.$$

In view of Theorem 2.8, $a, d \in \mathcal{A}^{ⓔ}$, as desired.

$\Longleftarrow$ Since $a, d \in \mathcal{A}$ are generalized EP, it follows by Theorem 2.8 that $a^d, d^d \in \mathcal{A}^{\#} \bigcap \mathcal{A}_{\#}$. In view of [8, Theorem 2.1], $x \in \mathcal{A}^d$ and $x^d = \begin{pmatrix} a^d & z \\ 0 & d^d \end{pmatrix}$, where

$$\begin{aligned} z &= \sum_{n=0}^{\infty} (a^d)^{n+2} b d^n d^{\pi} + \sum_{n=0}^{\infty} a^n a^{\pi} b (d^d)^{n+2} - a^d b d^d \\ &= -a^d b d^d. \end{aligned}$$

We easily check that $a^{\pi} z = 0$ and $z d^{\pi} = 0$. In light of [26, Theorem 2.5], $x^d \in \mathcal{A}^{\#}$. In this case,

$$(x^d)^{\#} = \begin{pmatrix} (a^d)^{\#} & -(a^d)^{\#} z (d^d)^{\#} \\ 0 & (d^d)^{\#} \end{pmatrix}.$$

Thus, we have

$$
\begin{aligned}
x^{\textcircled{d}} &= (x^d)^2(x^d)^{\#} \\
&= \begin{pmatrix} a^d & -a^d b d^d \\ 0 & d^d \end{pmatrix}^2 \begin{pmatrix} (a^d)^{\#} & (a^d)^{\#} a^d b d^d (d^d)^{\#} \\ 0 & (d^d)^{\#} \end{pmatrix} \\
&= \begin{pmatrix} (a^d)^2 & -(a^d)^2 b d^d - a^d b (d^d)^2 \\ 0 & (d^d)^2 \end{pmatrix} \begin{pmatrix} (a^d)^{\#} & (a^d)^{\#} a^d b d^d (d^d)^{\#} \\ 0 & (d^d)^{\#} \end{pmatrix} \\
&= \begin{pmatrix} a^{\textcircled{d}} & -a^d b d^d \\ 0 & d^{\textcircled{d}} \end{pmatrix}.
\end{aligned}
$$

This implies that $x^{\textcircled{d}} = x^d$. According to Theorem 2.8, $x \in \mathcal{A}$ is generalized EP. □

Lemma 3.2. *Let $a, b \in \mathcal{A}$ be generalized EP. If $ab = ba = a^*b = 0$, then $a + b$ is generalized EP.*

Proof. In view of Theorem 2.8 , $a, b \in \mathcal{A}^{\textcircled{d}}$, $a^{\textcircled{d}} = a^d$ and $b^{\textcircled{d}} = b^d$. Since $a^*b = 0$, we have $b^*a = 0$. By virtue of [5, Theorem 3.4], we have $(a+b)^{\textcircled{d}} = a^{\textcircled{d}} + b^{\textcircled{d}}$. Clearly, $(a+b)^d = a^d + b^d$, and then $(a+b)^{\textcircled{d}} = (a+b)^d$. By using Theorem 2.8 again, $a + b$ is generalized EP. □

We come now to the demonstration for which this section has been developed.

Theorem 3.3. *Let $a, b, a^\pi b \in \mathcal{A}$ be generalized EP. If $a^\pi ab = a^\pi ba = a^\pi a^* b = 0$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}$ is generalized EP.
(2) $(a+b)aa^d \in \mathcal{A}$ is generalized EP

$$
\sum_{n=0}^{\infty} (a+b)^n aa^d (a+b)^\pi aa^d b (a^\pi b^d)^{n+2} = 0,
$$
$$
\sum_{n=0}^{\infty} ((a+b)^d aa^d)^{n+2} ba^\pi (a^n + b^n) a^\pi b^\pi = 0.
$$

Proof. Let $p = aa^d$. By hypothesis, $p^\pi bp = (1 - aa^d)baa^d = (a^\pi ba)a^d = 0$. So we get

$$
a = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p.
$$

Hence

$$
a + b = \begin{pmatrix} a_1 + b_1 & b_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p.
$$

Here, $a_1 = a^2 a^d$ and $b_1 = aa^d baa^d = baa^d$. Then

$$
a_1 + b_1 = (a+b)aa^d.
$$

Moreover, we see that

$$
\begin{aligned}
(a_1 + b_1)^i &= (a+b)^i aa^d, \\
(a_1 + b_1)^d &= (a+b)^d aa^d, \\
(a_1 + b_1)^\pi &= 1 - (a+b)(a+b)^d aa^d.
\end{aligned}
$$

Also we have $a_4 = aa^\pi$ and $b_4 = a^\pi ba^\pi = a^\pi b$, and so

$$
a_4 + b_4 = a^\pi a + a^\pi b.
$$

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Then we check that

$$\begin{array}{rcl}(a_4+b_4)^i &=& a^\pi(a^i+b^i),\\ (a_4+b_4)^d &=& a^\pi b^d,\\ (a_4+b_4)^\pi &=& 1-a^\pi bb^d.\end{array}$$

Clearly, $a^\pi a$ is generalized EP. By hypothesis, $a^\pi b$ is generalized EP. Further, we see that

$$\begin{array}{rcl}(a^\pi a)(a^\pi b) &=& 0,\\ (a^\pi b)(a^\pi a) &=& 0,\\ (a^\pi a)^*(a^\pi b) &=& 0.\end{array}$$

In view of Lemma 3.2, a_4+b_4 is generalized EP.

In light of Lemma 3.1, $a+b$ is generalized EP if and only if a_1+b_1 is generalized EP and

$$\sum_{n=0}^{\infty}(a_1+b_1)^n(a_1+b_1)^\pi b_2((a_4+b_4)^d)^{n+2}=0,$$
$$\sum_{n=0}^{\infty}((a_1+b_1)^d)^{n+2}b_2(a_4+b_4)^n(a_4+b_4)^\pi=0.$$

Therefore $a+b$ is generalized EP if and only if $(a+b)aa^d$ is generalized EP and

$$\sum_{n=0}^{\infty}(a+b)^n aa^d(a+b)^\pi aa^d b(a^\pi b^d)^{n+2}=0,$$
$$\sum_{n=0}^{\infty}((a+b)^d aa^d)^{n+2}ba^\pi(a^n+b^n)a^\pi b^\pi=0.$$

This completes the proof. □

Corollary 3.4. *Let $a,b,a^\pi b\in\mathcal{A}$ be EP. If $a^\pi ba=0$, then the following are equivalent:*

(1) $a+b\in\mathcal{A}$ is generalized EP.
(2) $(a+b)aa^\#\in\mathcal{A}$ is generalized EP and

$$\sum_{n=0}^{\infty}(a+b)^n aa^\#(a+b)^\pi aa^\# b(a^\pi b^\#)^{n+2}=0,$$
$$\sum_{n=0}^{\infty}((a+b)^d aa^\#)^{n+2}ba^\pi(a^n+b^n)a^\pi b^\pi=0.$$

Proof. Since $a\in\mathcal{A}$ is EP, we see that $a^\pi a=a^\pi a^*=0$. Therefore we complete the proof by Theorem 3.3. □

Lemma 3.5. *Let $a,b\in\mathcal{A}$ be generalized EP. If $ab=ba$ and $a^*b=ba^*$, then the following are equivalent:*

(1) $a+b\in\mathcal{A}$ is generalized EP.
(2) $1+ba^d\in\mathcal{A}$ is generalized EP.

Proof. (1) ⇒ (2) In view of [5, Theorem 3.4], $1+a^d b\in\mathcal{A}^d$ and

$$(1+a^d b)^d=a^\pi+a^2a^d(a+b)^d.$$

Then

$$\begin{array}{rcl}(1+a^d b)(1+a^d b)^d &=& a^\pi+(1+a^d b)a^2a^d(a+b)^d\\ &=& a^\pi+aa^d(a+b)(a+b)^d\\ &=& 1-aa^d(a+b)^\pi.\end{array}$$

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In view of Theorem 2.8, aa^d are $(a+b)^{\pi}$ are projections. Then $(1+a^db)(1+a^db)^d$ is a projection, and so is $(1+a^db)^{\pi}$. Therefore $1+a^db \in \mathcal{A}$ is generalized EP by Theorem 2.8. Since $ab = ba$, it follows by [3, Theorem 15.2.12] that $a^db = ba^d$, and so $1+ba^d$ is generalized EP.

(2) $\Rightarrow$ (1) Since $1+a^db = 1+ba^d \in \mathcal{A}$ is generalized EP, it follows by Theorem 2.8 that $1+a^db \in \mathcal{A}^d$ and $(1+a^db)^{\pi}$ is a projection. In view of [28, Theorem 3.3], $a+b \in \mathcal{A}^d$ and

$$(a+b)^d = (1+a^db)^da^d + b^d(1+aa^{\pi}b^d)^{-1}a^{\pi}.$$

Since $(1-a^{\pi}bb^d)(1+aa^{\pi}b^d) = 1-a^{\pi}bb^d$, we have

$$(1-a^{\pi}bb^d)(1+aa^{\pi}b^d)^{-1} = 1-a^{\pi}bb^d.$$

Then we check that

$$\begin{array}{rl} & (a+b)(1+b)^d \\ = & a^d(a+b)(1+a^db)^d + (a+b)a^{\pi}b^d(1+aa^{\pi}b^d)^{-1} \\ = & aa^d(1+a^db)(1+a^db)^d + (aa^{\pi}b^d + a^{\pi}bb^d)(1+aa^{\pi}b^d)^{-1} \\ = & aa^d(1+a^db)(1+a^db)^d + 1 - (1-a^{\pi}bb^d)(1+aa^{\pi}b^d)^{-1} \\ = & aa^d(1+a^db)(1+a^db)^d + 1 - [1-a^{\pi}bb^d] \\ = & aa^d(1+a^db)(1+a^db)^d + a^{\pi}bb^d. \end{array}$$

Therefore

$$\begin{array}{rcl} (a+b)^{\pi} & = & 1 - aa^d(1+a^db)(1+a^db)^d - a^{\pi}bb^d \\ & = & aa^d - aa^d(1+a^db)(1+a^db)^d + a^{\pi}b^{\pi} \\ & = & aa^d(1+a^db)^{\pi} + a^{\pi}b^{\pi}. \end{array}$$

Hence, $(a+b)^{\pi}$ is a projection. Accordingly, $a+b \in \mathcal{A}$ is generalized EP by Theorem 2.8. □

We are ready to prove:

Theorem 3.6. *Let $a, b, a^{\pi}b \in \mathcal{A}$ be generalized EP. If $a^{\pi}ab = a^{\pi}ba$ and $a^{\pi}a^*b = a^{\pi}ba^*$, then the following are equivalent:*

(1) $a+b \in \mathcal{A}$ is generalized EP.
(2) $1+ba^d \in \mathcal{A}$ is generalized EP and

$$\sum_{n=0}^{\infty}(a+b)^naa^d(a+b)^{\pi}aa^db(a^{\pi}b^{\pi})^{n+2} = 0,$$
$$\sum_{n=0}^{\infty}((a+b)^daa^d)^{n+2}ba^{\pi}(a^n+b^n)a^{\pi}b^{\pi} = 0.$$

Proof. Since $a^{\pi}ab = a^{\pi}ba$, we have $a(a^{\pi}b) = (a^{\pi}b)a$. In view of [3, Theorem 15.2.12], $a^d(a^{\pi}b) = (a^{\pi}b)a^d$. Hence, $a^{\pi}ba^d = 0$. Let $p = aa^d$. Then $p^{\pi}bp = (a^{\pi}ba^d)a = 0$, and then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p.$$

Thus,

$$a+b = \begin{pmatrix} a_1+b_1 & b_2 \\ 0 & a_4+b_4 \end{pmatrix}_p.$$

Here, $a_1 = a^2a^d$ and $b_1 = aa^dbaa^d = (1-a^{\pi})baa^d = baa^d$. Then

$$a_1 + b_1 = (a+b)aa^d = (1+ba^d)a^2a^d.$$

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We verify that

$$\begin{array}{rcl}(a_1+b_1)^i & = & (a+b)^i aa^d,\\ (a_1+b_1)^d & = & (a+b)^d aa^d,\\ (a_1+b_1)^\pi & = & 1-(a+b)(a+b)^d aa^d.\end{array}$$

Further, we have $a_4 = aa^\pi$ and $b_4 = a^\pi ba^\pi = a^\pi b$; hence,

$$a_4 + b_4 = a^\pi a + a^\pi b.$$

Then we check that

$$\begin{array}{rcl}(a_4+b_4)^i & = & a^\pi(a^i+b^i),\\ (a_4+b_4)^d & = & a^\pi b^d,\\ (a_4+b_4)^\pi & = & 1-a^\pi bb^d.\end{array}$$

Obviously, $a^\pi a$ and $a^\pi b$ are generalized EP. Furthermore, we have

$$\begin{array}{rcl}(a^\pi a)(a^\pi b) & = & a^\pi ab = a^\pi ba = (a^\pi b)(a^\pi a),\\ (a^\pi a)^*(a^\pi b) & = & a^\pi a^* b = a^\pi ba^* = a^\pi ba^\pi a^* = (a^\pi b)(a^\pi a)^*,\\ 1+(a^\pi a)^d(a^\pi b) & = & 1 \text{ has generalized core-EP inverse.}\end{array}$$

By virtue Lemma 3.5, $a_4 + b_4$ is generalized EP.

By virtue of Lemma 3.1, $a + b$ is generalized EP if and only if $a_1 + b_1$ is generalized EP and

$$\sum_{n=0}^{\infty}(a_1+b_1)^n(a_1+b_1)^\pi b_2((a_4+b_4)^d)^{n+2} = 0,$$
$$\sum_{n=0}^{\infty}((a_1+b_1)^d)^{n+2} b_2(a_4+b_4)^n(a_4+b_4)^\pi = 0.$$

Claim 1. Assume that $1 + ba^d$ is generalized EP. Then we see that

$$\begin{array}{rcl}(ba^d)(aa^d) & = & ba^d = (1-a^\pi)ba^d = (aa^d)(ba^d),\\ (ba^d)(aa^d)^* & = & (ba^d)(aa^d) = (aa^d)(ba^d) = (aa^d)^*(ba^d),\end{array}$$

and then

$$\begin{array}{rcl}(1+ba^d)(aa^d) & = & (aa^d)(1+ba^d),\\ (1+ba^d)(aa^d)^* & = & (aa^d)^*(1+ba^d).\end{array}$$

By virtue of Lemma 3.5, $a_1 + b_1 = (1 + ba^d)(aa^d)$ is generalized EP.

Claim 2. Assume that $a_1 + b_1 = (1 + ba^d)aa^d$ is generalized EP. Obviously, we have $a^\pi(1 + ba^d)aa^d = (a^\pi)^*(1 + ba^d)aa^d = (1 + ba^d)aa^d a^\pi = 0$. It follows by Lemma 3.2 that $1 + ba^d = a^\pi + (1 + ba^d)aa^d$ is generalized EP.

Thus, we conclude that $a_1 + b_1$ is generalized EP if and only if so is $1 + ba^d$. Therefore $a + b$ is generalized EP if and only if $1 + ba^d$ is generalized EP and

$$\sum_{n=0}^{\infty}(a+b)^n aa^d(a+b)^\pi aa^d b(a^\pi b^\pi)^{n+2} = 0,$$
$$\sum_{n=0}^{\infty}((a+b)^d aa^d)^{n+2} ba^\pi(a^n+b^n)a^\pi b^\pi = 0.$$

□

Corollary 3.7. *Let $a, b \in \mathcal{A}$ be EP. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}$ is generalized EP.
(2) $1 + ba^\# \in \mathcal{A}$ is generalized EP.

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Proof. This is obvious by Theorem 3.6. □

4. Generalized Core-EP Orders

This section is devoted to the generalized core-EP orders involved in generalized EP elements. We now extend [13, Theorem 4.4] as follows.

Theorem 4.1. *$a \in \mathcal{A}^{\textcircled{e}}, b \in \mathcal{A}^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a \leq^{\textcircled{d}} b$.
(2) $a^{\textcircled{d}} \leq^{\textcircled{d}} b^{\textcircled{d}}$ and $a^{\textcircled{d}}a = a^{\textcircled{d}}b$.
(3) $a^{\textcircled{d}}b^{\textcircled{d}} = b^{\textcircled{d}}a^{\textcircled{d}}$ and $a^{\textcircled{d}}a = a^{\textcircled{d}}b$.

Proof. (1) $\Rightarrow$ (3) By hypothesis, we have $a^{\textcircled{d}}a = a^{\textcircled{d}}b, aa^{\textcircled{d}} = ba^{\textcircled{d}}$. Since $a \in \mathcal{A}^{\textcircled{e}}$, it follows by Theorem 2.8 that $a^{\textcircled{d}} = a^d$, and then

$$aa^{\textcircled{d}}b = aa^{\textcircled{d}}a = ba^{\textcircled{d}}a = ba^da = baa^d = baa^{\textcircled{d}}.$$

Moreover, $(aa^{\textcircled{d}})^*b = b(aa^{\textcircled{d}})^*$, and so $aa^{\textcircled{d}}b^* = b^*aa^{\textcircled{d}}$. In light of [5, Lemma 3.2],

$$aa^{\textcircled{d}}b^{\textcircled{d}} = b^{\textcircled{d}}aa^{\textcircled{d}}.$$

Therefore

$$\begin{array}{ll} & ||a^{\textcircled{d}} - aa^{\textcircled{d}}b^{\textcircled{d}}|| \\ = & ||a^k(a^{\textcircled{d}})^{k+1} - b^{\textcircled{d}}b^{k+1}(a^{\textcircled{d}})^{k+1}|| \\ \leq & ||a^k(a^{\textcircled{d}})^{k+1} - b^k(a^{\textcircled{d}})^{k+1}|| + ||b^k - b^{\textcircled{d}}b^{k+1}||||(a^{\textcircled{d}})^{k+1}|| \\ = & ||b^k - b^{\textcircled{d}}b^{k+1}||||(a^{\textcircled{d}})^{k+1}||. \end{array}$$

Since

$$\lim_{k\to\infty} ||b^k - b^{\textcircled{d}}b^{k+1}||^{\frac{1}{k}} = 0,$$

we deduce that

$$\lim_{k\to\infty} ||a^{\textcircled{d}} - aa^{\textcircled{d}}b^{\textcircled{d}}||^{\frac{1}{k}} = 0.$$

Then $aa^{\textcircled{d}}b^{\textcircled{d}} = a^{\textcircled{d}}$; hence,

$$a^{\textcircled{d}}[aa^{\textcircled{d}}b^{\textcircled{d}}] = (a^{\textcircled{d}})^2.$$

Accordingly,

$$a^{\textcircled{d}}b^{\textcircled{d}} = (a^{\textcircled{d}})^2 = a^{\textcircled{d}}a^d = b^{\textcircled{d}}a^d = b^{\textcircled{d}}a^{\textcircled{d}}.$$

(3) $\Rightarrow$ (1) Since $a, b \in \mathcal{A}^{\textcircled{e}}$, then $a^{\textcircled{d}} = a^d$ and $b^{\textcircled{d}} = b^d$. We verify that

$$\begin{array}{rcl} ba^{\textcircled{d}} & = & b(a^{\textcircled{d}})^2a = b(a^{\textcircled{d}})^2b = b(a^{\textcircled{d}})^{k+1}b^k, \\ bb^{\textcircled{d}}aa^{\textcircled{d}} & = & bb^{\textcircled{d}}(a^{\textcircled{d}})^{k+1}b^{k+1} = b(a^{\textcircled{d}})^{k+1}b^{\textcircled{d}}b^{k+1}, \\ aa^{\textcircled{d}} & = & a^{\textcircled{d}}a = a^{\textcircled{d}}b = (a^{\textcircled{d}})^kb^k, \\ bb^{\textcircled{d}}aa^{\textcircled{d}} & = & b^{\textcircled{d}}(a^{\textcircled{d}})^kb^{k+1} = (a^{\textcircled{d}})^kb^{\textcircled{d}}b^{k+1}. \end{array}$$

Then

$$||ba^{\textcircled{d}} - bb^{\textcircled{d}}aa^{\textcircled{d}}||^{\frac{1}{k}} \leq ||b||^{\frac{1}{k}}||a^{\textcircled{d}}||^{1+\frac{1}{k}}||b^k - b^{\textcircled{d}}b^{k+1}||^{\frac{1}{k}}.$$

Since $\lim_{k\to\infty} ||b^k - b^{\textcircled{d}}b^{k+1}||^{\frac{1}{k}} = 0$, we have

$$\lim_{k\to\infty} ||ba^{\textcircled{d}} - bb^{\textcircled{d}}aa^{\textcircled{d}}||^{\frac{1}{k}} = 0.$$

This implies that $ba^{\textcircled{d}} = bb^{\textcircled{d}}aa^{\textcircled{d}}$. Likewise, $aa^{\textcircled{d}} = bb^{\textcircled{d}}aa^{\textcircled{d}}$. Therefore $ba^{\textcircled{d}} = bb^{\textcircled{d}}aa^{\textcircled{d}} = aa^{\textcircled{d}}$, as required.

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(2) $\Rightarrow$ (3) By hypothesis, we have $(a^{\textcircled{d}})^{\textcircled{d}}a^{\textcircled{d}} = (a^{\textcircled{d}})^{\textcircled{d}}b^{\textcircled{d}}$. In view of [5, Theorem 3.5], $(a^{\textcircled{d}})^{\textcircled{d}} = a^2a^{\textcircled{d}}$. Then

$$a^2a^{\textcircled{d}}a^{\textcircled{d}} = a^2a^{\textcircled{d}}b^{\textcircled{d}}.$$

Hence, $aa^{\textcircled{d}} = a^2a^{\textcircled{d}}b^{\textcircled{d}}$.

On the other hand, $a^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}} = b^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}}$. Then $a^{\textcircled{d}}a^2a^{\textcircled{d}} = b^{\textcircled{d}}a^2a^{\textcircled{d}}$. This implies that $aa^{\textcircled{d}} = b^{\textcircled{d}}a^2a^{\textcircled{d}}$. Therefore

$$\begin{array}{rcl} a^{\textcircled{d}}b^{\textcircled{d}} & = & (a^{\textcircled{d}})^2(a^2a^{\textcircled{d}}b^{\textcircled{d}}) \\ & = & (a^{\textcircled{d}})^2(aa^{\textcircled{d}}) = (a^{\textcircled{d}})^2 = aa^{\textcircled{d}}(a^{\textcircled{d}})^2 \\ & = & (b^{\textcircled{d}}a^2a^{\textcircled{d}})(a^{\textcircled{d}})^2 = b^{\textcircled{d}}a^{\textcircled{d}}, \end{array}$$

as desired.

(3) $\Rightarrow$ (2) In view of [5, Theorem 3.5], $(a^{\textcircled{d}})^{\textcircled{d}} = a^2a^{\textcircled{d}}$. Then we check that

$$\begin{array}{rcl} (a^{\textcircled{d}})^{\textcircled{d}}a^{\textcircled{d}} & = & a^2(a^{\textcircled{d}})^2 = a^2(aa^{\textcircled{d}})(a^{\textcircled{d}})^2 = a^2b(a^{\textcircled{d}})^3 = a^2b(aa^{\textcircled{d}})(a^{\textcircled{d}})^3 \\ & = & a^2b^2(a^{\textcircled{d}})^4 = \cdots = a^2b^k(a^{\textcircled{d}})^{k+2}, \\ (a^{\textcircled{d}})^{\textcircled{d}}b^{\textcircled{d}} & = & a^2a^{\textcircled{d}}b^{\textcircled{d}} = a^2b^{\textcircled{d}}a^{\textcircled{d}} = a^2b^{\textcircled{d}}a(a^{\textcircled{d}})^2 = a^2b^{\textcircled{d}}b^{k+1}(a^{\textcircled{d}})^{k+2} \end{array}$$

Hence,

$$\begin{array}{rl} & ||(a^{\textcircled{d}})^{\textcircled{d}}a^{\textcircled{d}} - a^{\textcircled{d}})^{\textcircled{d}}b^{\textcircled{d}}||^{\frac{1}{k}} \\ \leq & ||a^2||^{\frac{1}{k}}||b^k - b^{\textcircled{d}}b^{k+1}||^{\frac{1}{k}}||a^{\textcircled{d}}||^{1+\frac{2}{k}}. \end{array}$$

Since $\lim_{k\to\infty}||b^k - b^{\textcircled{d}}b^{k+1}||^{\frac{1}{k}} = 0$, we deduce that

$$\lim_{k\to\infty}||(a^{\textcircled{d}})^{\textcircled{d}}a^{\textcircled{d}} - a^{\textcircled{d}})^{\textcircled{d}}b^{\textcircled{d}}||^{\frac{1}{k}} = 0.$$

This implies that

$$(a^{\textcircled{d}})^{\textcircled{d}}a^{\textcircled{d}} = (a^{\textcircled{d}})^{\textcircled{d}}b^{\textcircled{d}}.$$

On the other hand, we have

$$\begin{array}{rcl} a^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}} & = & a^{\textcircled{d}}a^2a^{\textcircled{d}} = (a^{\textcircled{d}}a)aa^{\textcircled{d}} \\ & = & b^{\textcircled{d}}a^2a^{\textcircled{d}} = b^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}}. \end{array}$$

Thus, $a^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}} = b^{\textcircled{d}}(a^{\textcircled{d}})^{\textcircled{d}}$. This completes the proof. □

The core-EP order for core-EP inverse of complex matrices was studied in [25, Theorem 4.2]. As an immediate consequence of Theorem 4.1, we give an alternative characterization of core-EP order for core-EP inverses as follows.

Corollary 4.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(1) $A \leq^{\textcircled{D}} B$.
(2) $A^{\textcircled{D}} \leq^{\textcircled{D}} B^{\textcircled{D}}$ and $A^{\textcircled{D}}A = A^{\textcircled{D}}B$.
(3) $A^{\textcircled{D}}B^{\textcircled{D}} = B^{\textcircled{D}}A^{\textcircled{D}}$ and $A^{\textcircled{D}}A = A^{\textcircled{D}}B$.

Theorem 4.3. *Let $a \in \mathcal{A}^{\textcircled{e}}, b \in \mathcal{A}^{\textcircled{d}}$. If $a \leq^{\textcircled{d}} b$, then the following are equivalent:*

(1) $b \in \mathcal{A}^{\textcircled{e}}$.
(2) $b(1 - aa^{\textcircled{d}}) \in \mathcal{A}^{\textcircled{e}}$.

Proof. Since $a \leq^{\textcircled{d}} b$, we have that $a^{\textcircled{d}}a = a^{\textcircled{d}}b$ and $aa^{\textcircled{d}} = ba^{\textcircled{d}}$. Then

$$aa^{\textcircled{d}}b = aa^{\textcircled{d}}a = ba^{\textcircled{d}}a = baa^{\textcircled{d}}.$$

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(1) $\Rightarrow$ (2) Since $b(1-aa^{\circledD}) = (1-aa^{\circledD})b$, it follows by [28, Theorem 3.1] that

$$\begin{aligned}[b(1-aa^{\circledD})]^d &= b^d(1-aa^d) = b^d - b^d aa^d = b^d - (baa^d)^d \\ &= b^d - (aa^d b)^d = b^d - (aa^d a)^d = b^d - a^d.\end{aligned}$$

We verify that

$$[b(1-aa^{\circledD})][b^{\circledD} - a^{\circledD}] = bb^d - aa^d;$$
$$\left[(b(1-aa^{\circledD}))(b^{\circledD} - a^{\circledD})\right]^* = [b(1-aa^{\circledD})][b^{\circledD} - a^{\circledD}],$$

Moreover, we check that

$$\begin{aligned} & \left(b(1-aa^{\circledD})\right)^n - \left(b^{\circledD} - a^{\circledD}\right)\left(b(1-aa^{\circledD})\right)^{n+1} \\ = & \; b^n(1-aa^{\circledD}) - \left(b^{\circledD} - a^{\circledD}\right)b^{n+1}(1-aa^{\circledD}) \\ = & \; b^n(1-aa^{\circledD}) - b^{\circledD}b^{n+1}(1-aa^{\circledD}) \\ = & \; (b^n - b^{\circledD}b^{n+1})(1-aa^{\circledD}).\end{aligned}$$

Since $\lim_{n\to\infty} ||b^n - b^{\circledD}b^{n+1}||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||\left(b(1-aa^{\circledD})\right)^n - \left(b^{\circledD} - a^{\circledD}\right)\left(b(1-aa^{\circledD})\right)^{n+1}||^{\frac{1}{n}} = 0.$$

Therefore

$$[b(1-aa^{\circledD})]^{\circledD} = b^{\circledD} - a^{\circledD} = [b(1-aa^{\circledD})]^d.$$

By virtue of Theorem 2.8, $b(1-aa^{\circledD})$ is generalized EP.

(2) $\Rightarrow$ (1) Obviously, $b = x + y$, where $x = baa^{\circledD}$ and $y = b(1-aa^{\circledD})$.

Claim 1. $x \in \mathcal{A}$ are generalized EP. We directly verify that $x^{\circledD} = a^{\circledD} = a^d = x^d$. In view of Theorem 2.8, $x \in \mathcal{A}^{\circledcirc}$.

Claim 2. $xy = yx = x^*y = 0$. Since $aa^{\circledD}a = baa^{\circledD}$, we have that $a^*aa^{\circledD} = aa^{\circledD}b^*$. Then we verify that

$$\begin{aligned} xy &= baa^{\circledD}b(1-aa^{\circledD}) = baa^{\circledD}a(1-aa^{\circledD}) = 0, \\ yx &= b(1-aa^{\circledD})ba^{\circledD}a = b(1-aa^{\circledD})aa^{\circledD}a = 0, \\ x^*y &= aa^{\circledD}b^*b(1-aa^{\circledD}) = a^*aa^{\circledD}b(1-aa^{\circledD}) \\ &= a^*aa^{\circledD}a(1-aa^{\circledD}) = 0.\end{aligned}$$

In light of Lemma 3.2, $b = x + y \in \mathcal{A}$ is generalized EP, as asserted. $\square$

As an immediate consequence, we now improve [13, Theorem 4.5] as follows.

Corollary 4.4. *Let $a \in \mathcal{A}^{\circledcirc}, b \in \mathcal{A}^{\circledD}$. If $a \leq^{\circledD} b$, then the following are equivalent:*

(1) $b \in \mathcal{A}$ is *-DMP.
(2) $b(1-aa^{\circledD}) \in \mathcal{A}$ is *-DMP.

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 2.8, $b(1-aa^{\circledD}) \in \mathcal{A}^{\circledcirc}$ and $[b(1-aa^{\circledD})]^{\circledD} = [b(1-aa^{\circledD})]^d$. Since $b \in \mathcal{A}$ is *-DMP, it follows by Theorem 2.8 that $b \in \mathcal{A}^D$. Since $a \leq^{\circledD} b$, we have

$$aa^d b = aa^{\circledD}b = aa^{\circledD}a = ba^{\circledD}a = baa^d.$$

In view of [28, Theorem 3.1], $aa^d b \in \mathcal{A}^D$. Obviously, $aa^d ba^{\pi} = baa^d a^{\pi} = 0$. Set $p = aa^d$. Then

$$b = \begin{pmatrix} aa^d b & 0 \\ a^{\pi}baa^d & ba^{\pi} \end{pmatrix}_p.$$

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By virtue of [8, Theorem 2.1], $ba^{\pi} \in \mathcal{A}^D$. Hence,

$$[b(1-aa^{\textcircled{d}})]^{\textcircled{d}} = [b(1-aa^{\textcircled{d}})]^D.$$

Therefore

$$\left[(b(1-aa^{\textcircled{d}}))(b(1-aa^{\textcircled{d}}))^D\right]^* = (b(1-aa^{\textcircled{d}}))(b(1-aa^{\textcircled{d}}))^D.$$

Accordingly, $b(1-aa^{\textcircled{d}}) \in \mathcal{A}$ is *-DMP.

(2) $\Rightarrow$ (1) In view of Theorem 4.3, $b \in \mathcal{A}^{\textcircled{\#}}$. Then $b \in \mathcal{A}^{\textcircled{d}}$ and $b^{\textcircled{d}} = b^d$. Since $b \in \mathcal{A}^{\textcircled{D}}$, it follows by [10, Theorem 2.3] that $b \in \mathcal{A}^D$. Therefore $b^{\textcircled{\#}} = b^D$. Then $(bb^D)^* = bb^D$. This implies that $b \in \mathcal{A}$ is *-DMP by [13, Lemma 2.2]. $\square$

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