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Estimates for Approximation and Eigenvalues of the Resolvent of a Class of Singular Operators of Parabolic Type

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Abstract: In this paper, we study a differential operator of parabolic type with a variable and unbounded coefficient, defined on an infinite strip. Sufficient conditions for the existence and compactness of the resolvent are established, and an estimate for the maximum regularity of solutions of the equation $Lu = f \in L_2(\Omega)$ is obtained. Two-sided estimates for the distribution function of approximation numbers are obtained. As is known, estimates of approximation numbers show the rate of best approximation of the resolvent of an operator by finite-dimensional operators. The paper proves the assertion about the existence of positive eigenvalues among the eigenvalues of the given operator and finds two-sided estimates for them.

Keywords: parabolic type operator; an eigenvalues; a singular numbers; separability; an unbounded domain.

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1. Introduction

Let us consider an operator of parabolic type with an unbounded coefficient

$$(L + \mu I)u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + q(x)u + \mu u, \mu \geq 0 \tag{1}$$

defined on $C^{\infty}_{0,\pi}(\bar{\Omega})$ where $\bar{\Omega} = \{(t,x) | -\pi \leq t \leq \pi, -\infty < x < \infty\}$. $C^{\infty}_{0,\pi}(\bar{\Omega})$ is a set consisting of infinitely differentiable finite functions with respect to the variable x and satisfying with respect to the variable t the condition

$$u(-\pi, x) = u(\pi, x). \tag{2}$$

Let $q(x)$ satisfy the following conditions:

$i) q(x) \geq \delta > 0$ is the continuous function in $\mathbb{R} = (-\infty, \infty)$;

$ii) m = \sup\limits_{|x-t|\leq 1} \frac{q(x)}{q(t)} < \infty.$

Here $q(x)$ can be an unbounded function.

It is easy to see that the operator L admits closure in $L_2(\Omega)$. We denote the closure also by L.

Considerable literature is devoted to the study of differential operators with unbounded coefficients [1–18] and works cited there.

In contrast to these papers, this article deals with the existence, compactness of the resolvent, estimates of approximation numbers, as well as the existence of an estimate

$$\|\frac{\partial u}{\partial t}\|_{L_2(\Omega)} + \|\frac{\partial^2 u}{\partial x^2}\|_{L_2(\Omega)} + \|q(x)u\|_{L_2(\Omega)} \leq c(\|Lu\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}), \tag{3}$$

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for the singular parabolic operator (1) in the case of an unbounded domain with a strongly growing coefficient $q(x)$ at infinity.

2. Results

Let us formulate the main results.

Theorem 1. *Let condition i) be satisfied. Then the operator $L + \mu I$ as $\mu \geq 0$ is continuously invertible in the space $L_2(\Omega)$, and the following equalities*

$$u(t,x) = (L+\mu I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(x) \cdot e^{int}, \tag{4}$$

hold, where $f(t,x) \in L_2(\Omega)$, $f(t,x) = \sum_{n=-\infty}^{\infty} f_n(x) \cdot e^{int}$, $f_n(x) =< f(t,x), e^{int} >$, $i^2 = -1$, $< \cdot, \cdot >$ is a scalar product in $L_2(\Omega)$, $(l_n + \mu I)u(x) = -u''(x) + (in + q(x) + \mu)u, u \in D(l_n), n = 0, \pm 1, \pm 2, ...$

Definition 1. *We say that a parabolic operator L is separable if the estimate (3) holds for all $u(t,x) \in D(L)$.*

The term "separability" was rst used in the papers of W. Everitt and M. Giertz [3]. The papers [4–8] and the articles cited there are devoted to the issues of separability of differential operators of elliptic, hyperbolic and mixed types, given in an unbounded domain. In this work, apparently for the first time, the separability of the operator of parabolic type.

Theorem 2. *Let conditions i)-ii) are fulfilled. Then L is a separable.*

Example 1. *Let $q(x) = e^{100|x|}, -\infty < x < \infty$. Then it is easy to verify that all conditions of Theorems 1-2 are satisfied. Therefore $L + \mu I$ is continuously invertible and separable in $L_2(\Omega)$ as $\mu \geq 0$, i.e., the following estimate*

$$\|\frac{\partial u}{\partial t}\|_2 + \|\frac{\partial^2 u}{\partial x^2}\|_2 + \|e^{100|x|} u\|_2 \leq c(\|Lu\|_2 + \|u\|_2),$$

holds, where $c > 0$ is any constant, $\| \cdot \|_2$ is a norm of $L_2(\Omega)$.

Theorem 3. *Let conditions i)-ii) be satisfied. Then the resolution of the operator L is compact if and only if*

$$\lim_{|x|\to\infty} q(x) = \infty. \tag{$*$}$$

Definition 2. *[19] Let y_n be the collection of all finite-dimensional operators of dimension $\leq n$ and let A be a linear completely continuous operator, then the numbers*

$$s_{n+1}(A) = \min_{k \in y_n} \|A - k\|_{2\to 2}, n = 0, 1, 2, ...$$

are called approximation numbers, where $\| \cdot \|_{2\to 2}$ is the norm of an operator from $L_2(\Omega)$ to $L_2(\Omega)$.

A nonzero s-numbers of $(L + \mu I)^{-1}$ will be numbered in descending order, taking into account their multiplicity.

We introduce the following counting function $N(\lambda) = \sum_{s_k > \lambda} 1$ is a number of s_k greater than $\lambda > 0$ of the operator $(L + \lambda I)^{-1}$. As is known, singular numbers (s-numbers) are recovered from their counting function [20].

Theorem 4. *Let the conditions i)-ii) and (*) are fulfilled. Then the estimate*

$$c^{-1}\sum_{n=-\infty}^{\infty}\lambda^{-\frac{1}{2}}mes(x\in\mathbb{R}:(|n|+q(x))\leq c^{-1}\lambda^{-1})\leq N(\lambda)\leq$$

$$\leq c\sum_{n=-\infty}^{\infty}\lambda^{-\frac{1}{2}}mes(x\in\mathbb{R}:(|n|+q(x))\leq c^{-1}\lambda^{-1}), \tag{5}$$

hold for $N(\lambda)$ of the operator $(L+\lambda I)^{-1}$, where $c>0$ is any constant, mes is the Lebesgue measure.

It follows from Representation (4) that if the s is a singular point of $(L+\mu I)^{-1}$, then s is the singular value of one of the operators $(l_n+\mu I)^{-1}$, $n=0,\pm1,\pm2,...$ and vice versa. We denote by $s_{k,n}(k=1,2,...)$ the singular values of $(l_n+\mu I)^{-1}$, $n=0,\pm1,\pm2,...$ as $\mu\geq0$.

Now, we separately consider the case $n=0$. In this case the operator $(l_0+\mu I)u=-u''(x)+(q(x)+\mu)u$ will be a self-adjoint and positive definite operator. Therefore, according to the results of [11], it follows that $s_{k,0}=\lambda_{k,0}$, where $\lambda_{k,0}$ are the eigenvalues of the operator $(l_0+\mu I)^{-1}$.

From here and from Theorem 4 we can obtain some important properties of the eigenvalues of the operator (1).

Corollary 1. *Let conditions i) and (*) be satisfied. Then*

a) there exists a sequence of positive eigenvalues of the resolvent of the operator (1);

b) the following two-sided estimate holds for $N(\lambda)$ of this sequence

$$c^{-1}\lambda^{-\frac{1}{2}}mes(x\in R:q(x)\leq c^{-1}\lambda^{-1}\leq N(\lambda)\leq c\lambda^{\frac{1}{2}}mes(x\in R:q(x)\leq c^{-1}\lambda^{-1}), \tag{6}$$

where $c>0$ is any constant.

Example 2. *We will show estimates of the positive eigenvalues of the following operator*

$$(L+\mu I)u=\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+(|x|+1)u+\mu u, u\in D(L),\mu\geq0.$$

Here the coefficient $q(x)=|x|+1$ is chosen so that $N(\lambda)$ can be easily calculated.

Now, using Corollary 1 and inequality (6), we obtain the following two-sided estimate

$$\frac{c^{-1}}{k^{\frac{2}{3}}}\leq\lambda_{k,0}\leq\frac{c}{k^{\frac{2}{3}}}, k=1,2,3,...,$$

where $c>0$ is any constant, $\lambda_{k,0}$ are positive eigenvalues.

This shows that estimates (5) and (6) can be effectively used to derive asymptotic formulas for the eigenvalues.

3. The Existence of A Resolvent

Lemma 1. *Let condition i) be fulfilled and $\mu\geq0$. Then the inequality*

$$\|(L+\mu I)u\|_2\geq(\delta+\mu)\|u\|_2, \tag{7}$$

holds for all $u\in D(L)$, where $\|\cdot\|_2$ is a norm in $L_2(\Omega)$.

Proof. Since the operator L has a real coefficient, it suffices to prove estimate (7) for real-valued functions. We compose the scalar product $<(L+\mu I)u,u>, u\in C_{0,\pi}^{\infty}(\Omega)$. Integrating by parts, we obtain

$$\|(L+\mu I)u\|_2\geq(\delta+\mu)\|u\|_2.$$

Due to the continuity of the norm, the last estimate is true for all $u\in D(L)$. □

It is easy to verify that the operator (1) can be reduced using the Fourier method to the study of the following operator

$$(l_n + \mu I)u(x) = -u''(x) + (in + q(x) + \mu)u, u \in D(L), n = 0, \pm 1, \pm 2, ...$$

We present a series of assertions that reduce questions about the existence of a resolvent and separability of the operator $l_n + \mu I$ with an unbounded coefficient $q(x)$ to the case of an operator with bounded coefficients.

Take a set of non-negative functions $\{\varphi_j\}$ from $C_0^\infty(\mathbb{R})$ such that $\sum_j \varphi_j^2 = 1, supp\varphi_j \subseteq \Delta_j, \bigcup_j \Delta_j = \mathbb{R}$, where $\Delta_j = (j-1, j+1), j \in \mathbb{Z}$ [4,17].

Extend $q(x)$ from Δ_j to the whole $\mathbb{R}$ so that its extension $q_j(x)$ is a bounded and periodic function of the same period.

Denote by $l_{n,j} + \mu I$ the closure of the operator

$$(l_{n,j} + \mu I)u(x) = -u''x + (in + q_j(x) + \mu)u$$

is defined on $C_0^\infty(\mathbb{R})$.

Lemma 2. *Let the condition i) be fulfilled. Then the following estimates*

$$\|(l_{n,j} + \mu I)u\|_2 \geq (\delta + \mu)\|u\|_2, n = 0, \pm 1, \pm 2, ..., j \in \mathbb{Z}; \tag{8}$$

$$\|(l_{n,j} + \mu I)u\|_2 \geq |n| \cdot \|u\|_2, n = \pm 1, \pm 2, ..., j \in \mathbb{Z}, \tag{9}$$

hold for all $u \in D(l_{n,j})$, *where* $\| \cdot \|_2$ *is a norm in* $L_2(\mathbb{R})$.

Proof. Let $u(x) \in C_0^\infty(\mathbb{R})$. Then the equality

$$< (l_{n,j} + \mu I)u, u >= \int_{-\infty}^{\infty} (-u''(x) + (in + q_j(x) + \mu))u)\bar{u}dx. \tag{10}$$

holds.

From here and using the Cauchy-Bunyakovsky inequality, we obtain inequalities (8) and (9) for all $u \in C_0^\infty(\mathbb{R})$. Since the norm is continuous, it follows that inequalities (8) and (9) are valid for all $u \in D(l_{n,j})$. □

Lemma 3. *Let the condition i) be fulfilled. Then the operator* $l_{n,j} + \mu I$ *continuously invertible in space* $L_2(\mathbb{R})$ *as* $\mu \geq 0$.

Proof. It follows from inequality (9) that $\|(l_{n,j} + \mu I)^{-1}u\|_{2\to 2} \to 0$ as $|n| \to \infty$. Therefore, it suffices to prove the lemma for any finite $n (n = 0, \pm 1, \pm 2, ...)$. Next, repeating the calculations and arguments used in the proof of Lemma 2.2 in [18], we obtain the proof of Lemma 3. □

Lemma 4. *Let the condition i) be fulfilled. Then the estimates*

a) $\|(l_{n,j} + \mu I)^{-1}\|_{2\to 2} \leq \frac{1}{\delta+\mu}$;

b) $\|\frac{d}{dx}(l_{n,j} + \mu I)^{-1}\|_{2\to 2} \leq \frac{1}{(\delta+\mu)^{\frac{1}{2}}}$, *hold for* $(l_{n,j} + \mu I)^{-1}$, *where* $c > 0$ *is any constant.*

Proof. The proof of estimate a) of Lemma 4 follows from Lemma 2. By Inequality (8) we get

$$\frac{1}{\delta + \mu}\|(l_{n,j} + \mu I)u\|_2 \geq \|u\|_2. \tag{11}$$

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From Equality (10) we have that

$$|<(l_{n,j}+\mu I)u,u>|\geq\int\limits_{-\infty}^{\infty}(|u'|^2+q_j(x)+\mu)|u|^2dx. \quad (12)$$

Hence and using Cauchy inequality, we have

$$\|(l_{n,j}+\mu I)u\|_2\cdot\|u\|_2\geq\|u'\|_2^2.$$

Hence, using Inequality (11), we obtain

$$\frac{c}{\delta+\mu}\|(l_{n,j}+\mu I)u\|_2^2\geq\|u'\|_2^2.$$

Hence and by virtue of the definition of the norm, we obtain the proof of estimate b) of Lemma 4. □

Let

$$K_\mu f=\sum_{\{j\}}\varphi_j(l_{n,j}+\mu I)^{-1}\varphi_j f, f\in C_0^\infty(\mathbb{R}),$$

where $\{\varphi_j\}$ is a set of non-negative functions from Lemma 2.

Now, using the properties of the operators $(l_{n,j}+\mu I)^{-1}, j\in\mathbb{Z}$, we prove the existence of the resolvent of the operator $l_n+\mu I$. To do this, we act by the operator $l_n+\mu I$ on $K_\mu f$

$$(l_n+\mu I)K_\mu f=f-B_\mu f,$$

where $B_\mu f=\sum\limits_{\{j\}}\varphi_j''(l_{n,j}+\mu I)^{-1}f+2\sum\limits_{\{j\}}\varphi_j'\frac{d}{dx}(l_{n,j}+\mu I)^{-1}\varphi_j f.$

Lemma 5. *Let condition i) be fulfilled and $\mu_0>0$. Then the equality*

$$(l_n+\mu I)K_\mu f=f-B_\mu f$$

where $B_\mu f=\sum\limits_{\{j\}}\varphi_j''(l_{n,j}+\mu I)^{-1}f+2\sum\limits_{\{j\}}\varphi_j'\frac{d}{dx}(l_{n,j}+\mu I)^{-1}\varphi_j f.$

Proof. Acting on $K_\mu f$ by the operator $l_n+\mu I$, we obtain

$$(l_n+\mu I)K_\mu f=(l_n+\mu I)\sum_{\{j\}}\varphi_j(l_{n,j}+\mu I)^{-1}\varphi_j f=\sum_{\{j\}}\varphi_j(l_n+\mu I)(l_{n,j}+\mu I)^{-1}\varphi_j f-B_\mu f. \quad (13)$$

Since on the support φ_j the coefficients of the operators $l_n+\mu I$ and $l_{n,j}+\mu I$ coincide, hence the equality

$$(l_n+\mu I)(l_{n,j}+\mu I)^{-1}\varphi_j f=\varphi_j f.$$

Hence, using equality (13) we obtain

$$(l_n+\mu I)K_\mu f=\sum_{\{j\}}\varphi_j^2 f-B_\mu f.$$

Since $\sum\limits_{\{j\}}\varphi_j^2\equiv 1$, from the last equality we get

$$(l_n+\mu I)K_\mu f=f-B_\mu f \quad (14)$$

□

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Lemma 6. *Let condition i) be fulfilled. Then there exists a number $\mu_0 > 0$ for the operator B_μ such that $\|B_\mu\|_{2\to 2} < 1$ for all $\mu \geq \mu_0$, where $\|\cdot\|_{2\to 2}$ is the norm of the operator B_μ acting from $L_2(\Omega)$ to $L_2(\Omega)$.*

Proof. Using Lemma 4 and repeating the calculations and reasoning used in the proof of Lemma 3.2 in [8] and Lemma 9 in [17], we obtain the proof of Lemma 6. □

Lemma 7. *Let the function $q(x)$ satisfy condition i), then the inequality*

$$\|(l_n + \mu I)u\|_2 \geq (\delta + \mu)\|u\|_2$$

holds for any $u \in D(l_n)$.

Proof. Lemma 7 is proved in exactly the same way as Lemma 2. □

Lemma 8. *Let the condition i) be fulfilled. Then the operator $l_n + \mu I$ is continuously invertible in $L_2(\mathbb{R})$ as $\mu \geq \mu_0 > 0$ and the inverse operator satisfies the equality*

$$(l_n + \mu I)^{-1} = K_\mu (I - B_\mu)^{-1}.$$

Proof. The proof of Lemma 8 follows from representation (14) and from Lemmas 5, 6 and 7. □

Lemma 9. *[21] Let the operator $L + \mu_0 I$ bounded invertible for $\mu_0 > 0$ in $L_2(\Omega)$ and the estimate $\|(L + \mu I)u\|_2 \geq c\|u\|_2$ holds for all $u \in D(L)$ as $\mu \in [0, \mu_0]$, where $c > 0$. Then the operator $L : L_2(\Omega) \to L_2(\Omega)$ also bounded invertibility.*

Proof. Proof of Theorem 1. It follows from Lemma 8 that

$$u_k(t,x) = \sum_{n=-k}^{k} (l_n + \mu I)^{-1} f_n(x) \cdot e^{int} \tag{15}$$

is a solution of

$$(L + \mu I)u_k(t,x) = f_k(t,x),$$
$$u_k(-\pi, x) = u_k(\pi, x),$$

where $f_k(t,x) \xrightarrow{L_2} f(t,x)$, $f_k(t,x) = \sum_{n=-k}^{k} f_n(x) \cdot e^{int}$, $i^2 = -1$. From Lemma 1 we get that

$$\|u_k(t,x) - u_m(t,x)\|_2 \leq \frac{1}{\delta + \mu}\|f_k(t,x) - f_m(t,x)\|_2 \to 0$$

as $k, m \to \infty$.

Hence it follows that the sequence u_k is fundamental, therefore, due to the completeness of the space $L_2(\Omega)$, we have

$$u_k(t,x) \xrightarrow{L_2} u(t,x) \tag{16}$$

as $k \to \infty$.

Using equalities (15) and (16), we obtain

$$u(t,x) = (L + \mu I)^{-1} f(t,x) = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(x) \cdot e^{int} \tag{17}$$

is a strong solution to the following problem

$$(L + \mu I)u = f, u(-\pi, x) = u(\pi, x) \tag{18}$$

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for any $f(t,x) \in) L_2(\Omega)$.

Definition 3. *A function $u \in L_2(\Omega)$ is called a strong solution to Problem (18) if there exists a sequence $\{u_k(t,x)\} \subset C^\infty_{0,\pi}(\Omega)$ such that*

$$\|u_k - u\|_2 \to 0, \|(L+\mu I)u_k - f\|_2 \to 0$$

as $k \to \infty$.

Hence, it is easy to verify that formula (17) is an inverse operator to the closed operator $L + \mu I$.

According to Lemmas 1, 9 and equality (17), we have that Theorem 1 is valid for all $\mu_0 \geq 0$. Theorem 1 is completely proved. □

4. Separability

To prove the separability of the operator $L + \mu I(\mu \geq 0)$, we first prove several lemmas.

Lemma 10. *Let conditions i)-ii) be fulfilled. Then the estimates*

$$\|(l_{n,j} + \mu I)^{-1}\|_{2\to 2} \leq \frac{1}{|n|}, n = \pm 1, \pm 2, ..., j \in \mathbb{Z}; \tag{19}$$

$$\|(l_{n,j} + \mu I)^{-1}\|_{2\to 2} \leq \frac{1}{q(x_j) + \mu}, n = 0, \pm 1, \pm 2, ..., j \in \mathbb{Z}, \tag{20}$$

hold, where $q(x_j) = \min_{x \in \bar{\Delta}_j} q_j(x)$

Proof. The proof of estimate (19) follows from Lemma 2. We prove inequality (20). It is easy to see that from inequality (12), we obtain

$$| < (l_{n,j} + \mu I)u, u > | \geq \int_{-\infty}^{\infty} (q_j(x) + \mu)|u|^2 dx$$

Hence, using the Cauchy inequality, we have

$$\|(l_{n,j} + \mu I)u\|_2 \geq ((q_j(x_j) + \mu)\|u\|_2, \tag{21}$$

where $q_j(x_j) = \min_{x \in \bar{\Delta}_j} q_j(x)$

From the construction of the segments, it follows that on the segment $q_j(x) = q(x)$. Therefore $q_j(x_j) = \min_{x \in \bar{\Delta}_j} q(x) = q(x_j)$.

Now, using the last equality, we obtain from Inequality (21) that

$$\|(l_{n,j} + \mu I)u\|_2 \geq (q(x_j) + \mu)\|u\|_2,$$

This inequality proves the estimate (20). □

We now give some auxiliary estimates for the resolvent of the operator $l_n + \mu I, n = 0, \pm 1, \pm 2, ...$

Lemma 11. *Let the conditions i)-ii) be fulfilled and $\mu > 0$ be such that $\|B\mu\|_{2\to 2} < 1$. Then the inequality*

$$\|q(x)(l_n + \mu I)^{-1}\|_{2\to 2} \leq c(\mu) \cdot \sup_{\{j\}} \|q(x)\varphi_j((l_{n,j} + \mu I)^{-1}\|^2_{2\to 2} \tag{22}$$

holds.

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Then, using Lemma 8 and taking into account the properties of the functions $\varphi_j (j \in \mathbb{Z})$, we have

$$\|q(x)(l_n + \mu I)f\|_2^2 \le \sum_{\{j\}} \int_{\Delta_j} \left| \sum_{k=j-1}^{j+1} q(x)\varphi_k(l_{n,k} + \mu I)^{-1}\varphi_k(I - B\mu)^{-1} f \right|^2 dx.$$

From here and using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, and also by the method of [17] (Lemma 3.7), we obtain the estimate (22). □

Lemma 12. *Let the conditions i)-ii) be fulfilled. The the estimate*

$$\|q(x)(l_n + \mu I)^{-1}\|_{2\to 2} \le c < \infty,$$

holds, where $c > 0$ is any constant.

Proof. From Inequalities (20) and (22) we have

$$\|q(x)(l_n + \mu I)^{-1}\|_{2\to 2} \le c(\mu) \sup_{\{j\}} \|q(x)\varphi_j((l_{n,j} + \mu I)^{-1}\|_{2\to 2} \le$$

$$\le c(\mu) \frac{\max_{x \in \bar{\Delta}_j} |q(x)\varphi_j|}{q(x_j) + \mu} \le c(\mu) \sup_{|x-t| \le 1} \frac{q(x)}{q(t)} \le c(\mu) \cdot m \le c < \infty.$$

□

Lemma 13. *Let the conditions i)-ii) be fulfilled. The the following estimates*

$$\|inu\|_2 \le \|(l_n + \mu I)u\|_2, u \in D(l_n); \quad (23)$$

$$\|q(x)u\|_2 \le c(\mu) \cdot \|(l_n + \mu I)u\|_2, u \in D(l_n); \quad (24)$$

$$\| -u''\|_2 \le c(\mu) \cdot \|(l_n + \mu I)u\|_2, u \in D(l_n) \quad (25)$$

hold.

Proof. The proof of inequalities (23) and (24) follows from Lemmas 10 and 11.

Using inequalities (23), (24) and Lemma 7, we obtain

$$\| -u''\|_2 = \|(l_n + \mu I)u - inu + q(x)u - \mu u\|_2 \le$$

$$\le \|(l_n + \mu I)u\|_2 + \|inu\|_2 + \|q(x)\|_2 + \|\mu u\|_2 \le c(\mu)\|(l_n + \mu I)u\|_2, \quad (26)$$

$c(\mu) > 0$.

From Inequality (26) we obtain Estimate (25). □

We also note that inequality (25) and according to the definition of the norm of an operator, we obtain

$$\|\frac{d^2}{dx^2}(l_n + \mu I^{-1})\|_{2\to 2} \le c(\mu) < \infty, n = 0, \pm 1, \pm 2, ..., \quad (27)$$

where $\| \cdot \|_{2\to 2}$ is the norm of the operator B_μ operating from $L_2(\Omega)$ to $L_2(\Omega)$.

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Proof. Proof of Theorem 2. From Theorem 1 and representation (17) it follows that

$$\frac{\partial u}{\partial t} = \sum_{n=-\infty}^{\infty} in(l_n + \mu I)^{-1} f_n(x) e^{int}.$$

Hence, due to the orthonormality of the system $\{e^{int}\}_{n=-\infty}^{n=\infty}$ we obtain

$$\|\frac{\partial u}{\partial t}\|_2^2 \le \sup_{\{n\}} \|in(l_n + \mu I)^{-1}\|_{2\to 2}^2 \cdot 2\pi \sum_{n=-\infty}^{\infty} \|f_n(x)\|_2^2.$$

From the last inequality, using estimate (19), we have

$$\|\frac{\partial u}{\partial t}\|_2^2 \le \|f(t,x)\|_2^2. \tag{28}$$

where $\|f(t,x)\|_2^2 = 2\pi \sum_{n=-\infty}^{\infty} \|f_n(x)\|_2^2$.

Taking into account that $(L + \mu I)u = f(t,x)$, we obtain from (28)

$$\|\frac{\partial u}{\partial t}\|_2^2 \le \|(L + \mu I)u\|_2^2. \tag{29}$$

Repeating the above calculations and reasoning, taking into account Lemma 12, we have

$$\|q(x)u(t,x)\|_2^2 \le c^2(\mu)\|(L + \mu I)u\|_2^2. \tag{30}$$

Similarly, from representation (17), taking into account inequality (27), we obtain

$$\|u_{xx}\|_2^2 \le \sup_{\{n\}} \|\frac{d^2}{dx^2}(l_n + \mu I)^{-1}\|_{2\to 2}^2 \cdot \|f(t,x)\|_2^2 \le c^2(\mu)\|(L + \mu I)u\|_2^2 \tag{31}$$

where $(L + \mu I)u = f(t,x)$.

Now it is easy to see that from (29)-(31), it follows that

$$\|\frac{\partial u}{\partial t}\|_2 + \|\frac{\partial^2 u}{\partial x^2}\|_2 + \|q(x)u\|_2 \le c(\mu)\|(L + \mu I)u\|_2 \le c(\mu)(\|Lu\|_2 + \|u\|_2)$$

□

5. The Compactness of the Resolvent

We need the following lemmas to prove Theorem 3.

Lemma 14. *Let condition i) be fulfilled. Then the estimate*

$$\|(l_n + \mu I)^{-1}\|_{2\to 2} \le \frac{1}{|n|}, n = \pm 1, \pm 2, ...$$

holds.

Proof. Lemma 14 can be proved in the same way as Lemma 10. □

Lemma 15. *Let conditions i)-ii) be fulfilled. Then the resolvent of the operator l_n is compact if and only if*

$$\lim_{|x|\to\infty} q(x) = \infty$$

Proof. Lemma 15 is proved in exactly the same way as Theorem 3 in [17]. □

Proof of Theorem 3. Since for each n $(n = 0, \pm 1, \pm 2, ...)$ by Lemma 15 the operator $(l_n + \mu I)^{-1}$ is completely continuous, from Theorem 1 and from the representation (17) Using well-known tricks with the ε-net, one can show that the operator $(L + \mu I)^{-1}$ is completely continuous if and only if

$$\lim_{n\to\infty} \|(l_n + \mu I)^{-1}\|_{2\to 2} = 0. \tag{32}$$

Now, it is easy to see that Equality (32) follows from Lemma 14 Theorem 3 is proved.

6. Estimates of Approximation Numbers (S-Numbers)

To study the singular values of the operator $(L + \mu I)^{-1}$, we need the following lemmas. In what follows, since by assumption the function $q(x)$ is bounded from below, we can assume without loss of generality that the condition $q(x) \geq 1$ is satisfied for all $x \in \mathbb{R}$.

We introduce the following sets

$$M = \{u \in L_2(\mathbb{R}) : \|l_n u\|_2^2 + \|u\|_2^2 \leq 1\},$$

where $\| \cdot \|_2^2$ is a norm in $L_2(\mathbb{R})$;

$$\tilde{M}_{c_0} = \{u \in L_2(\mathbb{R}) : \| - u''\|_2^2 + \|inu\|_2^2 + \|q(x)u\|_2^2 \leq c_0\};$$

$$\tilde{M}_{c_0^{-1}} = \{u \in L_2(\mathbb{R}) : \| - u''\|_2^2 + \|inu\|_2^2 + \|q(x)u\|_2^2 \leq c_0^{-1}\},$$

where $c_0 > 0$ is a constant number independent of $u(x)$, n.

Lemma 16. *Let conditions i)-ii) be fulfilled. Then the inclusions*

$$\tilde{M}_{c_0^{-1}} \subseteq M \subseteq \tilde{M}_{c_0},$$

where $c_0 > 0$ is a constant number independent of $u(x) and n$.

Proof. Let $u \in \tilde{M}_{c_0^{-1}}$. Then

$$\|l_n u\|_2^2 + \|u\|_2^2 \leq \| - u''\|_2^2 + \|inu\|_2^2 + \|q(x)u\|_2^2 + \|u\|_2^2 \leq$$

$$\leq c_0(\| - u''\|_2^2 + \| - inu\|_2^2 + \|q(x)u\|_2^2)$$

where $c_0 = 2$.

Since $u \in \tilde{M}_{c_0^{-1}}$, then it follows from the last inequality that

$$\|l_n u\|_2^2 + \|u\|_2^2 \leq c_0 \cdot c_0^{-1} \leq 1.$$

Hence $\tilde{M}_{c_0^{-1}} \subseteq M$. The left inclusion is thus proved.

Now, we prove the right inclusion. Let $u \in M$. This means $u \in D(l_n + \mu I)$. Therefore, by virtue of Lemma 13, we have

$$\| - u''\|_2^2 + \|inu\|_2^2 + \|q(x)u\|_2^2 \leq c_0(\|l_n u\|_2^2 + \|u\|_2^2) \tag{33}$$

where $c_0 > 0$ is a constant number independent of $u(x)$ and n $(n = 0, \pm 1, \pm 2, ...)$.

Since $u \in M$, the inequality $\|l_n u\|_2 + \|u\|_2 \leq 1$ is valid. Taking into account the last inequality from (33), we find

$$\| -u''\|_2^2 + \|inu\|_2^2 + \|q(x)u\|_2^2 \leq c_0(\|l_n u\|_2^2 + \|u\|_2^2) \leq c_0$$

From there we get that $u \in \tilde{M}_{c_0}$, i.e., $M \subseteq \tilde{M}_{c_0}$. □

Definition 4. *[19] The Kolmogorov k-width of a set M in $L_2(\mathbb{R})$ is called the quantity*

$$d_k = \inf_{\{y_k\}} \sup_{u \in M} \inf_{v \in y_k} \|u - v\|_2,$$

where y_k is the set of all subspaces in $L_2(\mathbb{R})$, whose dimension does not exceed k.

Remark 1. *The Kolmogorov widths and the approximation numbers coincide in the Hilbert space $L_2(\mathbb{R})$, i.e., $s_{k+1}((l_n)^{-1}) \equiv d_k(M)$ [19].*

The following lemmas hold.

Lemma 17. *Let conditions i)-ii) be fulfilled. Then the estimate*

$$c^{-1}\tilde{d}_k \leq s_{k+1} \leq c\tilde{d}_k, k = 1, 2, ...,$$

holds, where $c > 0$ is any constant, s_k is a s-numbers of $(l_n + \mu I)^{-1}$, $\mu \geq 0$, d_k, $\tilde{d}_k$ are Kolmogorov widths of the corresponding sets $M, \tilde{M}$.

Lemma 18. *Let conditions i)-ii) be fulfilled. Then the estimate*

$$\tilde{N}(c\lambda) \leq N(\lambda) \leq \tilde{N}(c^{-1}\lambda)$$

holds, where $N(\lambda) = \sum_{s_{k+1} > \lambda} 1$ is a number s_{k+1} of $(l_n + \mu I)^{-1}$ greater than $\lambda > 0$, $\tilde{N(\lambda)} = \sum_{\tilde{d}_k > \lambda} 1$ is a number $\tilde{d}_k$ greater than $\lambda > 0$.

Proof. Lemmas 17 and 18 can be proved in exactly the same way as Lemmas 4.3 and 4.4 in [17]. □

Lemma 19. *Let conditions i)-ii) be fulfilled. Then the estimate*

$$c^{-1}\lambda^{-\frac{1}{2}} mes(x \in \mathbb{R} : (|n| + q(x)) \leq c^{-1}\lambda^{-1}) \leq N(\lambda) \leq c\lambda^{-\frac{1}{2}} mes(x \in \mathbb{R} : (|n| + q(x)) \leq c^{-1}\lambda^{-1})$$

holds for $N(\lambda)$ of $(l_n + \lambda I)^{-1}$, where $c > 0$ is a constant number independent of $n, q(x)$ and $\lambda > 0$.

Proof. Denote by $L_2^2(\mathbb{R}, (|n| + q(x)))$ the space obtained by completing $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|_{L_2^2(\mathbb{R}, (|n|+q(x)))} = \left(\int_{\mathbb{R}} (|u''|^2 + (|n| + q(x))^2|u|^2) dx \right)^{\frac{1}{2}}$$

It follows from Lemma 16. that $M \subset L_2^2(\mathbb{R}, (|n| + q(x)))$. Hence, repeating the calculations and reasoning used in the proof of Theorem 1.4 in [17], we obtain the proof of Lemma 19. □

Proof. Proof of Theorem 4. From Theorem 1 it follows that

$$u(x, y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}$$

It follows that if s is a singular point of the operator $(L+\mu I)^{-1}$, then s is a singular value of one of the operators $(l_n+\mu I)^{-1}(n=0,\ pm1,\pm 2,...)$, and reversely, if s is a singular value of one of the operators $(l_n+\mu I)^{-1}(n=0,\pm 1,\pm 2,...)$, then s is a singular point of the operator $(L+\mu I)^{-1}$. The proof of Theorem 4 follows from the above considerations and Lemma 19. □

Corollary 1 follows from Theorems 1 and 4.

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