Short Note

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Disproof of Several Recently Published Results in Ring Theory

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Short Note

Disproof of Several Recently Published Results in Ring Theory

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Abstract: In this note we consolidate and give a brief description of several recently published results in ring theory having disproof. These results have been published during 2016 to 2023 in the so called non-predatory reputed mathematical journals indexed in the well known database like Scopus. We have considered results on rings in which each element is a sum of two idempotents appeared in Canad. Math. Bull. (2016), weakly tripotent rings appeared in Bull. Korean Math. Soc. (2018) and Rendiconti Sem. Mat. Univ. Pol. Torino (2021), invo-regular unital rings appeared in Ann. Univ. Mariae Curie-Sklodowska Sect. A Mathematica (2018), locally invo-regular rings appeared in Azerbijan Journal of Mathematics (2021), involution t-clean rings appeared in Eur. J. Pure Appl. Math (2022) and strongly 2-nil clean rings with units of order two appeared in Eur. J. Pure Appl. Math (2023).

Keywords: ring theory; algebra; mathematics; disproof; counterexamples; involution t-clean ring; weakly tripotent ring; invo-regular ring; locally invo-regular ring; strongly 2-nil clean ring; strongly nil clean ring

MSC: 16U40; 16E50

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1. Introduction.

Recently we have come across more than a dozen of results in ring theory having disproof. These results have been published during 2016 to 2023 in the so called non-predatory reputed mathematical journals indexed in the well known database like Scopus.

Here we provide a brief description of some of such results. We hope that this work will be useful for researchers in ring theory and in mathematics in general. Moreover this work will eventually suggest to the reviewers and editors of mathematical journals to be more cautious while considering a mathematical paper for publication.

It may be emphasized that if a result is published then generally its validity is taken to be granted by the readers. However if it is wrong and it remains unnoticed, then it can damage the existing literature drastically. As there are great chances of being forwarded from one journal to the other and it may inculcate its validity in the mind of readers leading further for more wrong results in mathematics. Hence it is very important to find and publish the counterexamples for existing incorrect mathematical results.

For the sake of convenience and completeness we include some relevant definitions as follows. We recall that a Boolean ring is a ring in which the square of each element is equal to the element itself and such an element is known as idempotent element. A weakly tripotent ring A is a ring in which either $x^3 = x$ or $(1-x)^3 = 1-x$ holds for every element $x \in A$ [5–7] and a ring A is called invo-regular if for each $x \in A$ there exists $y \in A$ satisfying $y^2 = 1$ such that $x = xyx$ [9]. Similarly A ring A is called locally invo-regular if $x = xyx$ or $1-x = (1-x)y(1-x)$ holds for each $x \in A$ and some $y \in A$ satisfying $y^2 = 1$ [13] and a ring A is called strongly involution t-clean ring if every element of A can be written as $u+t$ for some $u \in R$ with $u^2 = 1$ and some $t \in A$ with $t^3 = 1$ such that $ut = tu$. Further a ring A is called strongly 2-nil clean rings with units of order two if each unit $u \in A$ satisfies $u^2 = 1$ and for each $a \in A$ we have

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$a = e_1 + e_2 + q$. Here e_1, e_2 are some idempotent elements in A and q is any nilpotent element of A that commute with each other [15].

We consolidate and describe some of these results published during 2016 to 2023 in the next section.

2. Some Results Having Disproof

Here every ring A is an associative ring with identity element.

Result 1 ([1]). *Every element of a ring A is a sum of two idempotents iff $A \cong B \times C$, here $ch(B) = 2$ and every element of B is a sum of two idempotents, and C is zero or a subdirect product of the field of order three.*

Disproof. For the disproof of this result we refer to [2]. It is worth mentioning that in [1] this result was proved by assuming that 3 exists and is non-zero. It also appears that without stating in the initial setup it was assumed that idempotents commute. It may be noted that if A is a noncommutative ring in which each element is a sum of two idempotents, then there must exist an element $u \in A$ such that $u = u_1 + u_2$ with $u_1^2 = u_1$, $u_2^2 = u_2$ and $u_1 u_2 \neq u_2 u_1$. It is well known that if each element of a ring is a sum of two commuting idempotents, then the ring is always commutative and it is isomorphic to a subdirect product of copies the field of order two and the field of order three [3,4].

Result 2 ([1]). *Let every element of a ring A is a sum of two idempotents. Then $C(A) = E \times F$, Here E is Boolean and F is zero or a subdirect product of the field of order three.*

Disproof. The disproof of this result directly follows from the disproof of Result 1. It may be noted that in [1], Result 2 has been proved by assuming that the characteristic of E is two. This suggests that as per [1] E is a non-zero Boolean ring.

Result 3 ([5–7]). *A commutative ring A is a weakly tripotent ring iff $A = A_1 \times A_2$ such that A_2 is a tripotent ring of characteristic three or $A_2 = 0$ and $A_1 = 0$ or A_1 can be embedded as a subring of a direct product $A_0 \times \left(\prod_{i \in I} A_i\right)$ such that A_0 is a weakly tripotent ring without nontrivial idempotents, and all A_i are Boolean rings.*

Disproof. For the disproof of this result we refer to [8]. It has noted in [8] that if $A = A_1 \times A_2$ is a ommutative weakly tripotent ring and $A_1 = 0$, then $0 \neq A_2$ need not be a weakly tripotent ring of characteristic three. Similarly it has been noted in [8] that if $A_2 = 0$, then $0 \neq A_1$ need not be embedded as a subring of a direct product $A_0 \times \left(\prod_{i \in I} A_i\right)$ such that A_0 is a weakly tripotent ring without nontrivial idempotents, and all A_i are Boolean rings.

Result 4 ([9]). *A is an invo-regular ring iff $A \cong B \times C$, here B is an invo-regular ring with $ch(B) = 2$ and C is an invo-regular ring with $ch(C) = 3$.*

Disproof. The supposed validity of result 1 given above might have led to this result on invo-regular rings. For further details we refer to [10].

Result 5 ([9]). *If A is an invo-regular ring and $A \cong B \times C$, then B is a Boolean ring of characteristic two (i. e. a non-zero Boolean ring).*

Disproof. The supposed validity of result 2 given above might have led to this result on invo-regular rings. For further details we refer to [10]. It may be noted that B is a Boolean ring of characteristic two implies that B is a non-zero Boolean ring.

Result 6 ([11]). *Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $a^2 = 2a = 0$ holds for each $a \in J(A)$.*

Disproof. For the disproof of this result we refer to [12]. It has been seen in [12] that $a^3 = 4a = 0$ for each $a \in J(A)$ does not necessarily imply that $a^2 = 2a = 0$ for each $a \in J(A)$. However $a^2 = 2a = 0$ implies that $a^3 = 4a = 0$ for each $a \in J(A)$.

Result 7 ([13]). *Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $a^2 = 2a = 0$ holds for each $a \in J(A)$.*

Disproof. The disproof of this result directly follows from the disproof of Result 6. For further details we refer to [11]. One may note that this result has been forwarded from [11–13].

Result 8 ([14]). *Let A is a ring such that $J(A)$ is strongly involution t-clean, then $J(A)$ is nil with index of nilpotency at most 3 and the characteristic of $J(A)$ is four.*

Disproof. We refer [11]. It has been noted in [11] that if $J(A)$ is strongly involution t-clean then the characteristic of $J(A)$ can be different from four.

Result 9 ([15]). *Let A is a 2-nil clean ring with units of order two in which 3 is a unit. Then A is a strongly nil clean ring of characteristic eight.*

Disproof. Let $A = \{0, 1, a, 1+a\}$. Then A is commutative ring under addition and multiplication. Here we have $a^2 = 1$ and $(1+a)^2 = 0$. One can easily verify that A is a 2-nil clean ring with units of order two in which 3 is a unit. We recall that a ring A is called strongly nil clean ring if each element of A is a sum of an idempotent and a nilpotent that commute [15]. We note that the characteristic of A is two and not eight. Hence our disproof is complete.

Result 10 ([15]). *Let A is a 2-nil clean ring with units of order two. Then the characteristic of the Jacobson radical $J(A)$ is four.*

Disproof. Let us consider the ring A given above (we refer to the Disproof of Result 9). We have $J(A) = \{0, 1+a\}$. Clearly the characteristic of $J(A)$ is two but not four. This completes the disproof.

Result 11 ([15]). *Let A is a 2-nil clean ring and $q^2 + 2q = 0$ for each nilpotent $q \in A$. Then each unit of A has order four (and the characteristic of A is 48) ([15], Page 1676).*

Disproof. Let us consider the ring A given above (we refer to the Disproof of Result 9). Clearly $q^2 + 2q = 0$ holds for each nilpotent $q \in A$. However, the order of any unit is not four (also the characteristic of A is not 48). Thus the disproof is complete.

Conflict of Interest: There is no conflict of interest.

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