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Prachi Sharma and Asher Yahalom *

Posted Date: 27 September 2023

doi: 10.20944/preprints202309.1830.v1

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Generalized Chi and Eta Cross Helicities in Non-Ideal Magnetohydrodynamics

Prachi Sharma [1] and Asher Yahalom [1,2,*]

[1] Ariel University, Faulty of Engineering, Department of Electrical & Electronic Engineering, Ariel 40700, Israel; asya@ariel.ac.il

[2] Ariel University, Center for Astrophysics, Geophysics, and Space Sciences (AGASS), Ariel 40700, Israel;

* Correspondence: asya@ariel.ac.il; Tel. +972-54-7740294:

Abstract: We study the generalized χ and η cross helicities for non-ideal non-barotropic magnetohydrodynamics (MHD). χ and η, the additional label translation symmetry group, have been used to generalize the cross helicity in ideal flows. Both new helicities are additional topological invariants of ideal MHD. To study there behaviour in non-ideal MHD, we derive the time derivative of both helicities using the non-ideal MHD equations taking viscosity, finite resistivity and heat conduction into account. Physical variables are divided into ideal and non-ideal quantities separately during the mathematical analysis for simplification. The analytical results indicate that χ and η cross helicities are not strict constant of motion in the non-ideal MHD and show the rate of dissipation which is compared to the dissipation of other topological constants of motion.

Keywords: MHD; topological constants of motion; non ideal flows

1. Introduction

Topological invariants have always been useful for several decades in various physical systems, and there are such invariants in MHD flows. For example, the importance of two helicities i.e., magnetic helicity and cross-helicity have long been discussed in relation to the controlled nuclear fusion problem and in numerous astrophysical scenarios. In the absence of dissipation, cross helicity is conserved. Earlier work [1–3] have studied the relations between the helicities and symmetries of ideal MHD. MHD connects Maxwell's equations with hydrodynamics of highly conductive flows to explain the macroscopic behaviour of conducting fluid such as plasma. Considering the fact that ideal MHD does not describe precisely the behaviour of real plasmas was the paramount motivation for the present work. Some important realistic processes are missing in the ideal description such as resistive heating, heat conduction and viscous effects. Viscosity plays an important role on dissipation scale while investigating the plasma turbulence in solar wind and elsewhere. Similarly magnetic diffusivity is one of the reasons for the happening of magnetic reconnection phenomena. Thermal conductivity is also a substantial process needed to understand real plasmas. It causes the perturbations of physical variables to spread out through the plasmas. These essential properties of all three dissipative processes are the stimulus for the authors to study this current analysis.

The mathematical expression for cross helicity is defined as [4,5]:

$$H_C = \int \vec{B} \cdot \vec{v} \, d^3x \tag{1}$$

in which the integral is taken over the entire flow domain. Here H_C is conserved for barotropic or incompressible MHD (but not for non-barotropic MHD) and is given a topological interpretation in terms of the knottiness of magnetic and flow field lines. This correlation is having great importance in case of Alfven waves. However, it shows relation with magnetosonic waves in the compressible case [6]. A generalization for non-barotropic MHD of this quantity was given by [7,8]. This resembles the generalization of barotropic fluid dynamics conserved quantities including helicity to non barotropic flows including topological constants of motion derived by [9]. Conservation law of cross helicity

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for non barotropic MHD has been discussed by [10] in a multi-symplectic formulation of MHD. A potential vorticity conservation equation for non-barotropic MHD was derived by [11] by using Noether's second theorem.

Recently the non-barotropic cross-helicity was generalized using additional label translation symmetry groups (χ and η translations) [12], this led to additional topological conservation laws, the χ and η cross-helicities. The functions χ and η are sometimes denoted 'Euler potentials', 'Clebsch variables' and also 'flux representation functions' [13].

The concept of metage as a label for fluid elements along a vortex line in ideal fluids was first introduced by Lynden-Bell & Katz [14]. A translation group of this label was found to be connected to the conservation of Moffat's [15] helicity by Yahalom [3] using a **Lagrangian** variational principle. The concept of metage was later generalized by Yahalom & Lynden-Bell [1] for barotropic MHD, but now as a label for fluid elements along magnetic field lines which are comoving with the flow in the case of ideal MHD. Yahalom & Lynden-Bell [1] has also shown that the translation group of the magnetic metage is connected to Woltjer [4,5] conservation of cross helicity for barotropic MHD. Recently the concept of metage was generalized also for non barotropic MHD in which magnetic field lines lie on entropy surfaces [16]. This was later generalized by dropping the entropy condition on magnetic field lines [17]. In those papers the metage translation symmetry group was used to generate a non-barotropic cross helicity generalization using a **Lagrangian** variational principle.

Cross helicity is expected to play an important role in several MHD plasma phenomena such as global magnetic-field generation, turbulence suppression, etc. It provides a measure of the degree of linkage of the vortex tubes of the velocity field with the flux tubes of the magnetic field. Cross helicity plays an important role in turbulent dynamo [18]. The cross helicity density conservation law for barotropic flows is important in MHD turbulence theory [19–21]. Plasma velocity and magnetic field measurements from the Voyager 2 mission are used to study solar wind turbulence in the slow solar wind [22] and characterize its cross helicity. [23] has discussed MHD turbulence in his review paper in detail. He has examined the Alfvénic MHD turbulence with zero and non-zero helicities. The energy fluxes of MHD turbulence provide a measure for transfers of energy among velocity and magnetic fields [24,25].

Magnetic helicity is also one of the important topological invariants in fluid dynamics. [26] has already been showed the well conservation of magnetic helicity in turbulent flows. However when flux tubes diffuse through one another on resistive time scales, magnetic helicity dissipates[27]. [28] has studied the role of magnetic helicity in the plasmas stabilization in details by doing series of experiments and numerical simulations. Further, its important role in determining the structures, dynamics and heating of the solar corona has been well explained by [29].

The importance of the χ and η cross helicities is expected to be not less important than their more older predecessors the magnetic and generalized cross helicities, for stability and dynamics of MHD [12]. Thus it is of paramount importance to study the effects of non-ideal processes on their development.

The plan of this paper is as follows: There are three sections in the current paper. Section 2 describes the basic quantities and equations in non-ideal MHD. Calculations for time derivative of χ helicity is performed in section 3 and section 4 deals with the mathematics of η helicity.

2. Standard Formulation of Non-ideal Non-barotropic MHD

The standard set of equations solved for non-ideal non-barotropic MHD is given below (Here we use the EMU system of units):

$$\frac{\partial \vec{v}_n}{\partial t} = -(\vec{v}_n \cdot \vec{\nabla})\vec{v}_n - \frac{\vec{\nabla} p_n}{\rho_n} + \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} - \vec{\nabla}\phi + \frac{1}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} \tag{2}$$

$$\frac{\partial \rho_n}{\partial t} + \vec{\nabla} \cdot (\rho_n \vec{v}_n) = 0 \tag{3}$$

$$\vec{\nabla}\cdot\vec{B}_n = 0 \tag{4}$$

$$\frac{\partial\vec{B}_n}{\partial t} = \vec{\nabla}\times(\vec{v}_n\times\vec{B}_n) + \frac{\eta_v}{4\pi}\nabla^2\vec{B}_n \tag{5}$$

$$\rho_n T_n \frac{ds_n}{dt} = \sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \eta_v J_n^2 + \vec{\nabla}\cdot(k\vec{\nabla}T_n) \tag{6}$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the partial temporal derivative. $\vec{\nabla}$ has its standard meaning in vector calculus. We use the subscript $_n$ to describe non-ideal processes. Thus $\vec{v}$ is the ideal velocity and $\vec{v}_n$ is the velocity for non-ideal fluid etc. . ρ_n is the density, p_n is the pressure respectively which depends through the equation of state on the density and entropy s_n (the non-barotropic case), T_n is the temperature and ϕ is a potential.
The stress tensor is defined as:

$$\sigma'_{ik} = \mu_v\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{nl}}{\partial x_l}\right) \tag{7}$$

in which μ_v is a coefficient of kinematic viscosity (not to be confused with the metage μ defined in [12]). Notice that we take coefficient of second viscosity (or volume viscosity) is zero for the sake of simplicity. The entropy equation (6) depends on the heat conduction coefficient k. According to classical kinetic theory, viscosity arises from collisions between particles. The justification for those equations and the conditions under which they apply can be found in standard books on MHD (see for example [30–33]).

The current density $\vec{J}_n$ and the magnetic field $\vec{b}_n$ of MHD are related by Ampere's law:

$$\vec{\nabla}\times\vec{B}_n = 4\pi\vec{J}_n \tag{8}$$

in which the displacement current is neglected. Equation (5) depends on the non-ideal magnetic diffusivity η_v (not to be confused with the label η introduced in [12]). In the limit of ideal flows all the non-ideal coefficients μ_v, η_v, k tend to zero and the ideal MHD equations are recovered we prime the difference between ideal and non-ideal quantities, for example: $\vec{v}' = \vec{v}_n - \vec{v}$. It is clear that in the ideal limit all primed quantities tend to zero, for example: $\lim_{(\mu_v,\eta_v,k)\to 0}\vec{v}' = 0$.

3. Direct derivation of the constancy of non-barotropic χ cross-helicity

We introduce the abstract 'magnetic fields' as follows [34]:

$$\vec{B}_\chi = \vec{\nabla}\mu\times\vec{\nabla}\eta \tag{9}$$

Non-barotropic χ cross-helicity is given by:

$$H_{CNB\chi} = \int \vec{v}_{nt}\cdot\vec{B}_\chi d^3x \tag{10}$$

Here the topological non-ideal velocity field is defined as $\vec{v}_{nt} = \vec{v}_n - \sigma_n\vec{\nabla}s_n$ [34], σ_n is auxiliary variable, which depends on the Lagrangian time integral of the temperature i.e.:

$$\frac{d\sigma_n}{dt} = T_n. \tag{11}$$

Please refer [16] for the detailed justification for the relation of non-barotropic cross helicity.

Taking the temporal derivative of the non-barotropic χ cross-helicity:

$$\frac{dH_{CNB\chi}}{dt} = \int d^3x\,\left(\vec{v}_{nt}\cdot\frac{\partial\vec{B}_\chi}{\partial t} + \vec{B}_\chi\cdot\frac{\partial\vec{v}_{nt}}{\partial t}\right) \tag{12}$$

Now, find out the value of first term of RHS, we take the time derivative of equation (9):

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla}(\frac{\partial \mu}{\partial t}) \times \vec{\nabla}\eta + \vec{\nabla}\mu \times \vec{\nabla}(\frac{\partial \eta}{\partial t}) \tag{13}$$

Notice that both the labels are comoving and conserved under ideal material derivative [12], thus:

$$\frac{\partial \mu}{\partial t} + (\vec{v} \cdot \vec{\nabla})\mu = 0. \tag{14}$$

Similarly:

$$\frac{\partial \eta}{\partial t} + (\vec{v} \cdot \vec{\nabla})\eta = 0. \tag{15}$$

So:

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla}[(-\vec{v} \cdot \vec{\nabla}\mu)] \times \vec{\nabla}\eta + \vec{\nabla}\mu \times \vec{\nabla}[(-\vec{v} \cdot \vec{\nabla}\eta)] \tag{16}$$

Using vector identity:

$$\vec{\nabla} \times (\psi\vec{a}) = \vec{\nabla}\psi \times \vec{a} + \psi\vec{\nabla} \times \vec{a} \tag{17}$$

equation (16) takes the form:

$$\frac{\partial \vec{B}_\chi}{\partial t} = [\vec{\nabla} \times \{\vec{\nabla}\mu(\vec{v} \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{v} \cdot \vec{\nabla}\mu)\}] \tag{18}$$

Now, with the help of identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}), \tag{19}$$

we obtain:

$$\frac{\partial \vec{B}_\chi}{\partial t} = [\vec{\nabla} \times \{\vec{v} \times (\vec{\nabla}\mu \times \vec{\nabla}\eta)\}]. \tag{20}$$

Substituting $\vec{B}_\chi$ defined in equation (9), we obtain:

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}_\chi). \tag{21}$$

Next, we dot both sides with $\vec{v}_{nt}$:

$$\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} = \vec{v}_{nt} \cdot \vec{\nabla} \times (\vec{v} \times \vec{B}_\chi), \tag{22}$$

and using a well known vector identity we obtain:

$$\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\chi) \times \vec{v}_{nt}\} + (\vec{v} \times \vec{B}_\chi) \cdot \vec{\omega}_{nt}, \tag{23}$$

where we define the topological vorticity of the non-ideal flow field as:

$$\vec{\omega}_{nt} \equiv \vec{\nabla} \times \vec{v}_{nt}. \tag{24}$$

Next we calculate second term on the RHS of equation (12):

$$\partial_t \vec{v}_{nt} \cdot \vec{B}_\chi = \vec{B}_\chi \cdot \partial_t(\vec{v}_n - \sigma_n \vec{\nabla} s_n) = \vec{B}_\chi \cdot (\partial_t \vec{v}_n - \partial_t \sigma_n \vec{\nabla} s_n - \sigma_n \vec{\nabla}\partial_t s_n). \tag{25}$$

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Now we simplify right hand side of equation (25) in three steps. The first term will be calculated with the help of equation (2):

$$\frac{\partial \vec{v}_n}{\partial t} = (\vec{v}_n \times \vec{\omega}_n) + \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} - \vec{\nabla}(\frac{v_n^2}{2}) - \vec{\nabla} w_n + T_n \vec{\nabla} s_n - \vec{\nabla}\phi + \frac{1}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k}, \tag{26}$$

in which the non-ideal vorticity is:

$$\vec{\omega}_n \equiv \vec{\nabla} \times \vec{v}_n \tag{27}$$

and we have used the thermodynamical identity:

$$dw_n = d\varepsilon_n + d(\frac{p_n}{\rho_n}) = T_n ds_n + \frac{1}{\rho_n} dp_n \Rightarrow \vec{\nabla} w_n = T_n \vec{\nabla} s_n + \frac{1}{\rho_n}\vec{\nabla} p_n. \tag{28}$$

Thus:

$$\vec{B}_\chi \cdot \frac{\partial \vec{v}_n}{\partial t} = \vec{B}_\chi \cdot \Big[(\vec{v}_n \times \vec{\omega}_n) - \vec{\nabla}(\frac{v_n^2}{2} + w_n) + T_n \vec{\nabla} s_n - \vec{\nabla}\phi\Big] + \frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n}. \tag{29}$$

In the second term we use equation (11) to obtain:

$$-\frac{\partial \sigma_n}{\partial t}\vec{\nabla} s_n = (\vec{v}_n \cdot \vec{\nabla}\sigma_n - T_n)\vec{\nabla} s_n. \tag{30}$$

In the third term we use equation for the rate of change of entropy in non-ideal MHD also known as the heat equation (6):

$$-\sigma_n \vec{\nabla}\frac{\partial s_n}{\partial t} = \sigma_n \vec{\nabla}[\vec{v}_n \cdot \vec{\nabla} s_n - \frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} - \frac{\eta_v}{\rho_n T_n} J_n^2 - \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k \vec{\nabla} T_n)]. \tag{31}$$

Combining equation (29), equation (30) and equation (31) we obtain:

$$\begin{aligned} \vec{B}_\chi \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{B}_\chi \cdot [(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\ &- \sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}] \\ &+ \frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} \end{aligned} \tag{32}$$

in which the current density is given by

$$\vec{J}_n = \frac{\vec{\nabla} \times \vec{B}_n}{4\pi} \Rightarrow \vec{\nabla} \cdot \vec{J}_n = 0. \tag{33}$$

Now,

$$\vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = \frac{1}{\rho_n}\vec{J}_n \cdot \vec{B}_n \times \vec{B}_\chi \tag{34}$$

$$\vec{B}_n \times \vec{B}_\chi = \vec{B}_n \times (\vec{\nabla}\mu \times \vec{\nabla}\eta) = \vec{\nabla}\mu(\vec{B}_n \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B}_n \cdot \vec{\nabla}\mu) \tag{35}$$

$$\vec{B}_n \times \vec{B}_\chi = \vec{\nabla}\mu(\vec{B} \cdot \vec{\nabla}\eta) + \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B} \cdot \vec{\nabla}\mu) - \vec{\nabla}\eta(\vec{B}' \cdot \vec{\nabla}\mu), \tag{36}$$

however, using equations (15) and (17) of [12]: $\vec{B} \cdot \vec{\nabla}\eta = 0$ and $\vec{B} \cdot \vec{\nabla}\mu = \rho$. And thus:

$$\vec{B}_n \times \vec{B}_\chi = \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B}' \cdot \vec{\nabla}\mu) - \rho\vec{\nabla}\eta \tag{37}$$

 doi:10.20944/preprints202309.1830.v1

So:

$$\vec{B}_n \times \vec{B}_\chi = \vec{B}' \times (\vec{\nabla}\mu \times \vec{\nabla}\eta) - \rho\vec{\nabla}\eta = \vec{B}' \times \vec{B}_\chi - \rho\vec{\nabla}\eta \tag{38}$$

It thus follows that:

$$\vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = -\frac{\rho}{\rho_n}(\vec{J}_n \cdot \vec{\nabla}\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n} = -\frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n}. \tag{39}$$

Now equation (32) can be written as:

$$\begin{aligned} \vec{B}_\chi \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{B}_\chi \cdot \Big[(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} - \sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} \\ &+ \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla}T_n)\}\Big] + \frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} \\ &- \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n}. \end{aligned} \tag{40}$$

Combining equation (23) and equation (40) and taking into account that:

$$\vec{B}_\chi \cdot (\vec{v}_n \times \vec{\omega}_t) = -(\vec{v}_n \times \vec{B}_\chi) \cdot \vec{\omega}_t \tag{41}$$

We obtain:

$$\begin{aligned} \vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} + \vec{B}_\chi \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\chi) \times \vec{v}_{nt}\} + \vec{B}_\chi \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\ &+ \frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\chi \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 \\ &+ \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla}T_n)\}] \\ &- \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\chi \times \vec{v}'). \end{aligned} \tag{42}$$

Now substituting equation (42) into equation (12) we obtain:

$$\begin{aligned} \frac{dH_{CNB\chi}}{dt} &= \int \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\chi) \times \vec{v}_{nt}\} + \vec{B}_\chi \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} d^3x \\ &+ \int [\frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\chi \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla}T_n)\}]] d^3x \\ &- \int [\frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) - \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n} - \vec{\omega}_{nt} \cdot (\vec{B}_\chi \times \vec{v}')] d^3x. \end{aligned} \tag{43}$$

Using Gauss divergence theorem, we can write part of this integral as a surface integral:

$$\begin{aligned} \frac{dH_{CNB\chi}}{dt} &= \oint [(\vec{v} \times \vec{B}_\chi) \times \vec{v}_{nt} + \vec{B}_\chi\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} - \vec{J}_n\eta] \cdot d\vec{S} \\ &+ \int [\frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\chi \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla}T_n)\}]] d^3x \\ &+ \int [\frac{\rho'}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\chi \times \vec{v}')] d^3x. \end{aligned} \tag{44}$$

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Here, the surface integral encapsulates the volume for which the χ non barotropic cross helicity is calculated as well as a cut in case that η is multiple valued. If the surface integral vanishes:

$$\begin{aligned}\frac{dH_{CNB\chi}}{dt} &= \int [\frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\chi \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}]]d^3x \\ &+ \int [\frac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \frac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt}\cdot(\vec{B}_\chi\times\vec{v}')]d^3x. \end{aligned} \quad (45)$$

To conclude we notice that is not possible to partition the time derivative of the non barotropic χ cross helicity in accordance with the different non ideal processes: viscosity, finite conductivity and heat conductivity. Each of this processes may contribute to the primed quantities directly or indirectly. In the limit that all non-ideal coefficients tend to zero it is easy to see that the non barotropic χ cross helicity is indeed conserved:

$$\frac{dH_{CNB\chi}}{dt} = 0. \quad (46)$$

As is expected for an ideal flow.

4. Direct derivation of the constancy of non-barotropic η cross-helicity

We introduce the abstract 'magnetic field' as follows [34]:

$$\vec{B}_\eta = \vec{\nabla}\chi \times \vec{\nabla}\mu \quad (47)$$

Non-barotropic η cross-helicity is given by:

$$H_{CNB\eta} = \int \vec{v_{nt}} \cdot \vec{B}_\eta d^3x. \quad (48)$$

Taking the temporal derivative of the non-barotropic η cross-helicity

$$\frac{dH_{CNB\eta}}{dt} = \int d^3x\,(\vec{v}_{nt}\cdot\frac{\partial \vec{B}_\eta}{\partial t} + \vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t}) \quad (49)$$

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla}(\frac{\partial \chi}{\partial t}) \times \vec{\nabla}\mu + \vec{\nabla}\chi \times \vec{\nabla}(\frac{\partial \mu}{\partial t}) \quad (50)$$

Again, both the labels are comoving and conserved under an ideal material derivative [12]:

$$\frac{\partial \mu}{\partial t} + (\vec{v}\cdot\vec{\nabla})\mu = 0, \qquad \frac{\partial \chi}{\partial t} + (\vec{v}\cdot\vec{\nabla})\chi = 0. \quad (51)$$

So:

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla}[(-\vec{v}\cdot\vec{\nabla}\chi)] \times \vec{\nabla}\mu + \vec{\nabla}\chi \times \vec{\nabla}[(-\vec{v}\cdot\vec{\nabla}\mu)] \quad (52)$$

Using vector identity:

$$\vec{\nabla} \times (\psi\vec{\nabla}\mu) = \vec{\nabla}\psi \times \vec{\nabla}\mu, \quad (53)$$

equation (52) takes the form.

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla} \times \{\vec{\nabla}\chi(\vec{v}\cdot\vec{\nabla}\mu) - \vec{\nabla}\mu(\vec{v}\cdot\vec{\nabla}\chi)\}. \quad (54)$$

Now, with the help of identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}), \quad (55)$$

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we obtain:

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla} \times \{\vec{v} \times (\vec{\nabla}\chi \times \vec{\nabla}\mu)\} = \vec{\nabla} \times (\vec{v} \times \vec{B}_\eta). \tag{56}$$

Thus:

$$\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\eta}{\partial t} = \vec{v}_{nt} \cdot [\vec{\nabla} \times (\vec{v} \times \vec{B}_\eta)] = \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt}\} + (\vec{v} \times \vec{B}_\eta) \cdot \vec{\omega}_{nt} \tag{57}$$

Next we calculate second term:

$$\partial_t \vec{v}_{nt} \cdot \vec{B}_\eta = \vec{B}_\eta \cdot \partial_t(\vec{v}_n - \sigma_n \vec{\nabla} s_n) = \vec{B}_\eta \cdot (\partial_t \vec{v}_n - \partial_t \sigma_n \vec{\nabla} s_n - \sigma_n \vec{\nabla} \partial_t s_n) \tag{58}$$

Now we simplify right hand side of equation (58) in three steps. First term will be calculated with the help of momentum equation (26) for the non ideal case:

$$\vec{B}_\eta \cdot \frac{\partial \vec{v}_n}{\partial\, t} = \vec{B}_\eta \cdot \left[(\vec{v}_n \times \vec{\omega}_n) - \vec{\nabla}(\frac{v_n^2}{2} + w_n) + T_n \vec{\nabla} s_n - \vec{\nabla}\phi\right] + \frac{B_\eta}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\eta \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} \tag{59}$$

For the second and third terms we use equation (30) and equation (31) to obtain:

$$\begin{aligned}
\vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial\, t} &= \vec{B}_\eta \cdot [(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\
&- \sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k \vec{\nabla} T_n)\}] \\
&+ \frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\eta \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n}
\end{aligned} \tag{60}$$

Now:

$$\vec{B}_\eta \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = \frac{1}{\rho_n}\vec{J}_n \cdot \vec{B}_n \times \vec{B}_\eta \tag{61}$$

And we have:

$$\begin{aligned}
\vec{B}_n \times \vec{B}_\eta &= \vec{B}_n \times (\vec{\nabla}\chi \times \vec{\nabla}\mu) = \vec{\nabla}\chi(\vec{B}_n \cdot \vec{\nabla}\mu) - \vec{\nabla}\mu(\vec{B}_n \cdot \vec{\nabla}\chi) \\
&= \vec{\nabla}\chi(\vec{B} \cdot \vec{\nabla}\mu) + \vec{\nabla}\chi(\vec{B}' \cdot \vec{\nabla}\mu) - \vec{\nabla}\mu(\vec{B} \cdot \vec{\nabla}\chi) - \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\chi),
\end{aligned} \tag{62}$$

however, using equations (15) and (17) of [12] which dictate:

$$\vec{B} \cdot \vec{\nabla}\chi = 0, \qquad \vec{B} \cdot \vec{\nabla}\mu = \rho, \tag{63}$$

it follows that:

$$\begin{aligned}
\vec{B}_n \times \vec{B}_\eta &= \vec{\nabla}\chi(\vec{B}' \cdot \vec{\nabla}\mu) - \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\chi) + \rho\vec{\nabla}\chi = \vec{B}' \times (\vec{\nabla}\chi \times \vec{\nabla}\mu) + \rho\vec{\nabla}\chi \\
&= \vec{B}' \times \vec{B}_\eta + \rho\vec{\nabla}\chi
\end{aligned} \tag{64}$$

Inserting equation (64) into equation (61) leads to the following identity:

$$\vec{B}_\eta \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = \frac{\rho}{\rho_n}(\vec{J}_n \cdot \vec{\nabla}\chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} = \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} \tag{65}$$

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So equation (60) becomes:

$$\begin{aligned}
\vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{B}_\eta \cdot [(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\
&- \sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}] \\
&+ \frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \frac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n}. \qquad (66)
\end{aligned}$$

Combining equation (57) and equation (66) we obtain:

$$\begin{aligned}
\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\eta}{\partial t} + \vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt}\} + \vec{B}_\eta \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\
&+ \frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 \\
&+ \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}] \\
&+ \frac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}'). \qquad (67)
\end{aligned}$$

Now substituting equation (67) into equation (49), we obtain:

$$\begin{aligned}
\frac{dH_{CNB\eta}}{dt} &= \int \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt}\} + \vec{B}_\eta \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} d^3x \\
&+ \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}]] d^3x \\
&+ \int [\frac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x. \qquad (68)
\end{aligned}$$

Using Gauss divergence theorem, we obtain:

$$\begin{aligned}
\frac{dH_{CNB\eta}}{dt} &= \oint [(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt} + \vec{B}_\eta\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} + \vec{J}_n \chi] \cdot d\vec{S} \\
&+ \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}]] d^3x \\
&+ \int [-\frac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x. \qquad (69)
\end{aligned}$$

Here, the surface integral encapsulates the volume for which the χ non barotropic cross helicity is calculated (χ is usually single valued so no cut is required). If the surface integral vanishes:

$$\begin{aligned}
\frac{dH_{CNB\eta}}{dt} &= \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n} J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}]] d^3x \\
&+ \int [-\frac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x. \qquad (70)
\end{aligned}$$

To conclude we notice that is not possible to partition the time derivative of the non barotropic η cross helicity in accordance with the different non ideal processes: viscosity, finite conductivity and heat conductivity. Each of this processes may contribute to the primed quantities directly or indirectly. In the limit that all non-ideal coefficients tend to zero it is easy to see that the non barotropic η cross helicity is indeed conserved:

$$\frac{dH_{CNB\eta}}{dt} = 0. \quad (71)$$

As is expected for an ideal flow.

5. Conclusion

The current paper highlights on the constancy of two new topological invariants generalized χ and η cross helicities when fluid flow is not ideal, but the non ideal processes are less significant. We have showed that the helicities are not conserved and their time derivatives depends on dissipative processes present in the system. Non-zero cross helicity has an important consequences related to transport and [35] has been showed the close relation between momentum transfer and cross helicity. [36] has discussed its role in MHD turbulence by high resolution direct numerical simulations. Cross helicity is proportional to the correlation between velocity and magnetic field fluctuation and measures the relative importance of Alfvén waves in global fluctuation and [37] have been predicted the generation of interstellar turbulence caused by nonlinear interactions among Shear Alfvén waves. Many studies in the literature have shown that the cross helicity is correlated to the self production of turbulence.

The cascade process in MHD turbulence has been studied by [38] in details and concluded "cross helicity blocks the spectral energy transfer in MHD turbulence and results in energy accumulation in the system. This accumulation proceeds until the vortex intensification compensates the decreasing efficiency of nonlinear interactions". [39] examined the effect of cross helicity on the decay of isotropic MHD turbulence and concluded that an initial non zero cross helicity makes imbalanced MHD turbulence. The subtle anisotropic effect of cross helicity can be caused for the same. The technological (fusion) and astrophysical importance of cross helicity suggest that the newly discovered χ and η cross helicities may also play a pivotal rule. The reason for this is two fold. First, as we have shown above all processes that changes the quantities are slow dissipative processes thus any fast process conserve the χ and η cross helicities. Second, given that those quantities are approximately constant they serve as lower bounds to other quantities such as "energy":

$$|H_{CNB\chi}| = \left| \int \vec{B}_\chi \cdot \vec{v}_t d^3x \right| \leq \frac{1}{2} \int \left(\vec{B}_\chi{}^2 + \vec{v}_t{}^2 \right) d^3x, \quad (72)$$

$$|H_{CNB\chi}| = \left| \int \vec{B}_\chi \cdot \vec{v}_t d^3x \right| \leq \sqrt{\int \vec{v}_t{}^2 d^3x} \sqrt{\int \vec{B}_\chi{}^2 d^3x}, \quad (73)$$

$$|H_{CNB\eta}| = \left| \int \vec{B}_\eta \cdot \vec{v}_t d^3x \right| \leq \frac{1}{2} \int \left(\vec{B}_\eta{}^2 + \vec{v}_t{}^2 \right) d^3x, \quad (74)$$

$$|H_{CNB\eta}| = \left| \int \vec{B}_\eta \cdot \vec{v}_t d^3x \right| \leq \sqrt{\int \vec{v}_t{}^2 d^3x} \sqrt{\int \vec{B}_\eta{}^2 d^3x}, \quad (75)$$

And thus may prevent at least part of the most fast and dangerous instabilities which may provide some hope that controlled fusion is indeed feasible.

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