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Firewalls, Hawking (Tolman) Radiation, and a Tentative Resolution of the Firewall-Mass Problem

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Abstract

It has been theorized that black holes are surrounded by firewalls, although there is not universal agreement concerning this. We first review basic concepts pertaining to Schwarzschild black holes and Hawking radiation. Then we discuss the anticipation of Hawking radiation—albeit from *non*-black holes—initially by R. C. Tolman alone and shortly thereafter with P. Ehrenfest. We compare evaporation into a vacuum at absolute zero (0 K) of black holes with that of non-black holes, and show that not only black holes but also *non*-black holes evaporate within a *finite* time. The times required for evaporation of black holes and non-black holes are compared. We show that a 1 -solar-mass black dwarf and Earth will completely Tolman-evaporate into a vacuum at absolute zero (0 K) in much less time than required for a 1 -solar-mass black hole and 1 -Earth-mass black hole, respectively, to completely Hawking-evaporate into a vacuum at absolute zero (0 K). We also compare the times required for a minimal (1-Planck-mass) black hole to Hawking-evaporate and a minimal (1-Planck-mass) near-but-non-black hole to Tolman-evaporate into a vacuum at absolute zero (0 K), mentioning but not resolving a discrepancy related thereto. Next, we show that (i) if firewalls exist, they can originate via Hawking radiation at the minimum possible ruler distance (the Planck length) beyond the Schwarzschild horizon, where it has not suffered any gravitational redshift, or, alternatively, suffered maximal gravitational blueshift and (ii) the firewall temperature is on the order of the Planck temperature, *in*dependently of the mass and hence also of the Schwarzschild radius of a Schwarzschild black hole. We then explain the exponential nature of the gravitational frequency shift as a function of the gravitational potential. Next, we consider the firewall-mass problem, and provide an at least *prima facie* tentative resolution thereto based on: (i) the mass of a firewall being canceled by the negative gravitational mass = (negative gravitational energy)$/c^2$ accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem (actually first discovered by Jørg Tofte Jebsen). We then consider one aspect of thermodynamics in gravitational fields, showing that equilibrium relativistic gravitational temperature gradients can*not* be exploited to violate the Second Law of Thermodynamics. Following a concluding synopsis, auxiliary topics are discussed in the Notes.

Keywords: Schwarzschild black holes; Schwarzschild non-black holes; ruler distance; planck units; Hawking radiation; Tolman radiation; gravitational frequency shift; firewall mass; negative gravitational mass-energy; Birkhoff's theorem; thermodynamic equilibrium; first law of thermodynamics; second law of thermodynamics

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1. Introduction

It has been theorized that black holes are surrounded by firewalls [1–7], although there is not universal agreement concerning this [1–7]. There is a vast literature exploring this topic, of which we have cited only a tiny sample. But the many works cited, and discussed, in our cited Refs. [1–7] could provide at least the beginning of a thorough literature search [1–7].

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In Section 2, we review basic concepts pertaining to Schwarzschild black holes and Hawking radiation. In Section 3, we discuss the anticipation of Hawking radiation—albeit from *non*-black holes—initially by R. C. Tolman alone and shortly thereafter with P. Ehrenfest, and Tolman's [8,9] proof, bolstered with Ehrenfest [10], that *any* gravitator—black hole or *non*-black hole—*must radiate* and hence can*not* be in thermodynamic equilibrium with a surrounding vacuum at (or sufficiently close to) absolute zero (0 K), but *must completely evaporate into that vacuum within a finite time* (see also Garrod [11]). This has been corroborated by recent research [12]. The times required for evaporation of black holes and non-black holes are compared. We show that a 1-solar-mass black dwarf (completely-cooled white dwarf) and Earth will completely Tolman-evaporate into a vacuum at absolute zero (0 K) in much less time than required for a 1-solar-mass black hole and 1-Earth-mass black hole, respectively, to completely Hawking-evaporate into a vacuum at absolute zero (0 K). [A black hole formed by stellar gravitational collapse must have an initial mass of at least about $2\frac{1}{2}$ solar masses. But a 1-solar-mass black hole can be construed as a partially Hawking-evaporated stellar black hole, and a 1-Earth-mass black hole as a more-completely Hawking-evaporated one (assuming that the Universe expands forever and hence that the temperature of the cosmic background radiation eventually drops low enough to allow such evaporation). Moreover, the laws of physics allow the existence of primordial 1-solar-mass and 1-Earth-mass black holes, even though none have yet been discovered.] We also compare the times required for a minimal (1-Planck-mass) black hole to Hawking-evaporate and a minimal (1-Planck-mass) near-but-non-black hole to Tolman-evaporate into a vacuum at absolute zero (0 K), mentioning but not resolving a discrepancy related thereto. In Section 4, we show that (i) if firewalls exist, they can originate via Hawking radiation at the minimum possible ruler distance [13] (the Planck length [14–16]) beyond the Schwarzschild horizon, where it has not suffered any gravitational redshift [17,18], or, alternatively, suffered maximal gravitational blueshift [17,18] and (ii) the firewall temperature is on the order of the Planck temperature [19], *in*dependently of the mass and hence also of the Schwarzschild radius of a Schwarzschild black hole. We emphasize and focus on *ruler* distance [13] because, unlike other distance measures in General Relativity [13], ruler distance—*uniquely!* [13]—is actual *physical* distance: the distance between two points as measured by rulers laid upon the shortest possible spatial path separating them and hence the *physical* distance separating them [13]. (We employ other distance measures where they are more applicable.) In Section 5, we explain the exponential nature of the gravitational frequency shift as a function of the gravitational potential. In Section 6, we consider the firewall-mass problem [3], and provide an at least *prima facie* tentative resolution thereto based on: (i) the mass of a firewall being canceled by the *negative* gravitational mass [17,18] $=$ (*negative* gravitational energy)$/c^2$ [17,18] accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem [20–31] (actually first discovered by Jørg Tofte Jebsen [25–28]). We show that the mass of a firewall is *exactly* counterbalanced by the (negative) gravitational mass-energy accompanying its formation. Perhaps this may complement other lines of reasoning [4] disputing massiveness [3] of firewalls. (There is a caveat [28–31][1] with respect to Birkhoff's Theorem [20–31], but it [28–31][1] is *not* relevant with respect to our considerations.[1]) In Section 7, we consider one aspect of thermodynamics in gravitational fields, showing that equilibrium relativistic gravitational temperature gradients can*not* be exploited to violate the Second Law of Thermodynamics. A concluding synopsis is provided in Section 8. Auxiliary topics are discussed in the Notes.

2. Review of Basic Concepts Pertaining to Schwarzschild Black Holes and Hawking Radiation

The Schwarzschild metric of a Schwarzschild black hole of mass M and Schwarzschild radius $r_S = 2GM/c^2$ is [32]

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= \left(1 - \frac{r_S}{r}\right)c^2dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= \left(1 - \frac{r_S}{r}\right)c^2dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\left(\sin^2\theta\right)d\phi^2 \\ &= c^2d\tau^2 - dl_r^2 - dl_\theta^2 - dl_\phi^2. \end{aligned} \tag{1}$$

Schwarzschild-coordinate radial distance r is *not* radial *ruler* distance but is radial area distance and also radial distance from apparent size [13]. By contrast, in accordance with the *Euclidean* form of the angular part of the Schwarzschild metric [the last term in the first and second lines of Equation (1) and the sum of the last two terms in the third and fourth lines thereof], a spherical shell at r_{shell} has ruler-distance circumference $C_{\text{shell}} = 2\pi r_{\text{shell}}$ and ruler-distance surface area $A_{\text{shell}} = 4\pi r_{\text{shell}}^2$ [13]. At $r \geq r_S$, $l_r = (1 - \frac{r_S}{r})^{-1/2}dr$—*not* r itself—is radial *ruler* distance, $l_\theta = rd\theta$ is θ-directional ruler distance, $l_\phi = r(\sin\theta)d\phi$ is ϕ-directional ruler distance, t is Schwarzschild-coordinate time (proper time measured by a clock at rest at $r \to \infty$) [17,18], and τ is proper time measured by a clock at rest at *any* given $r \geq r_S$ [17,18]. [A clock at rest *at* $r = r_S$—if such a clock can exist—must be constructed entirely of photons (and/or other zero-rest-mass particles)!]

Although not necessary for our derivations, it may be helpful, as an aside, to briefly remark on the following three features of the Schwarzschild metric [Equation (1)]: (i) Setting $ds^2 = 0$ in Equation (1) shows that the *physical* radial velocity of light $V_{\text{phys,light}} = dl_r/d\tau = c$ at *all* $r \geq r_S$ [32]; but, by contrast, the Schwarzschild-*coordinate* radial velocity of light $V_{\text{coor,light}} = dr/dt = c[1 - (r_S/r)]$ decreases monotonically with decreasing r from c at $r \to \infty$ to zero at $r = r_S$ [32]. (ii) We focus on distance [13], especially on *ruler* distance [13], and most especially on *radial ruler* distance [13], *beyond* the Schwarzschild horizon r_S in Schwarzschild spacetime [32]. But it may be interesting to note that, in accordance with the *angular* part of the Schwarzschild metric [the last term in the first and second lines of Equation (1) and the sum of the last two terms in the third and fourth lines thereof] being of the *identical Euclidean* form at *all* $r \geq 0$ [32],[2] *even within* r_S,[2] where a spherical shell can*not* be at rest but *must* be collapsing, while falling through a given $r_{\text{shell}} < r_S$ it has ruler-distance circumference $C_{\text{shell}} = 2\pi r_{\text{shell}}$ and ruler-distance surface area $A_{\text{shell}} = 4\pi r_{\text{shell}}^2$ [32]:[2] at *all* $r \geq 0$, i.e., *even within* r_S,[2] $l_\theta = rd\theta$ is θ-directional ruler distance and $l_\phi = r(\sin\theta)d\phi$ is ϕ-directional ruler distance. *Even within* r_S,[2] where r becomes time*like*, r does *not* become time itself:[2] *un*like time itself r still retains these (Euclidean) *spatial* geometrical attributes [32].[2] Time itself has *no spatial* geometrical attributes (Euclidean or otherwise). (iii) Because the gravitational field of a Schwarzschild black hole is purely radial, it seems intuitive that this gravitational field's stretching [33,34][3] of space from the Euclidean [33,34][3] is *purely* in the *vertical*[3] radial r direction, and *not at all* in the *horizontal* angular θ and ϕ directions: thus the *identical Euclidean* form of the angular part of the Schwarzschild metric [the last term in the first and second lines of Equation (1) and the sum of the last two terms in the third and fourth lines thereof] at *all* $r \geq 0$ [32]. Indeed, more generally, intuition suggests that stretching [33,34][3] of space from the Euclidean by *any* gravitational field is *purely* in the *vertical* direction [33,34],[3] and *not at all* in any *horizontal* direction [33,34].[3] This intuition augments the immediately preceding Item (ii), and is perhaps most clear in relation to Sakharov's elastic-strain theory of gravity [35–40].[4,5] (Thus, might space be the ether [41,42]?[4,5])

We will be concerned only with *radial* motions of photons in the gravitational fields of Schwarzschild black holes, because only *radial* motions can result in gravitational frequency shifts. In this regard we will be concerned only with the *temporal-radial* part of the Schwarzschild metric

[Equation (1)], hence ignoring the angular part thereof [the last term in the first and second lines of Equation (1) and the sum of the last two terms in the third and fourth lines thereof]. Therefore, for brevity in notation, henceforth we omit the subscript $_r$ on l: henceforth l is to be construed as *radial* ruler distance (except that l_r appears in Note 2, because Note 2 has already been referred to).

Hawking radiation, the (at least essentially) blackbody radiation from a Schwarzschild black hole, is most typically construed to have the temperature [16,43–49]

$$T_{\mathrm{H},r\to\infty} = \frac{\hbar c^3}{8\pi G k_B M} = \frac{\hbar c^3}{8\pi G k_B \frac{r_S c^2}{2G}} = \frac{\hbar c}{4\pi k_B r_S}, \tag{2}$$

where k_B is Boltzmann's constant, M is the mass of the black hole, and $r_S = 2GM/c^2$ is its Schwarzschild radius [16,43–49]. Note that $T_{\mathrm{H},r\to\infty}$ given by Equation (2) is the temperature of Hawking radiation at a great distance from a Schwarzschild black hole, i.e., at $r \to \infty$, hence after Hawking radiation having suffered the maximum possible gravitational redshift [16–18,43–49]. For all non-primordial black holes, which are all of stellar mass or larger, $T_{\mathrm{H},r\to\infty}$ is extremely low compared to the current temperature $T_{\mathrm{CBR}} = 2.725\,\mathrm{K}$ [50] of the cosmic background radiation [50]. For sufficiently small primordial black holes [51–61] ($M < \frac{\hbar c^3}{8\pi G k_B T_{\mathrm{CBR}}} = 4.50 \times 10^{22}\,\mathrm{kg} \iff r_S < \frac{\hbar c}{4\pi k_B T_{\mathrm{CBR}}} = 6.69 \times 10^{-5}\,\mathrm{m}$), $T_{\mathrm{H},r\to\infty} > T_{\mathrm{CBR}} = 2.725\,\mathrm{K}$ obtains. But as of this writing, to the best knowledge of the author, no such sufficiently small primordial black holes—indeed, no primordial black holes at all—have yet been discovered [51–61]. Moreover, while there are rationales according to which primordial black holes might contribute, perhaps significantly, to cold dark matter, there also are both theoretical and observational upper limits on their abundance [51–61] and therefore also on their actual contribution to cold dark matter [51–61]. Hence, while it is possible that they could contribute, perhaps significantly, to cold dark matter, as of this writing, to the best knowledge of the author, it is uncertain whether or not they actually exist [51–61].

Thus far we have considered $T_{\mathrm{H},r\to\infty}$. But $T_{\mathrm{H},r}$ at smaller values of r ($r_S \le r < \infty$) has been discussed as well [49]. Closer to r_S (at $r_S \le r < \infty$) [49] than at $r \to \infty$, Hawking radiation has suffered less [17,18,49] gravitational redshift [17,18] and hence has a higher [49] temperature [16,43–49]

$$T_{\mathrm{H},r} = T_{\mathrm{H},r\to\infty}\left(1 - \frac{r_S}{r}\right)^{-1/2} = \frac{\hbar c^3}{8\pi G k_B M}\left(1 - \frac{r_S}{r}\right)^{-1/2} = \frac{\hbar c}{4\pi k_B r_S}\left(1 - \frac{r_S}{r}\right)^{-1/2}. \tag{3}$$

3. Tolman's Anticipation of Hawking Radiation: *All* Gravitators—Black Holes *and Non*-Black Holes—*Must Radiate*

It is *extremely important* to note that Equation (3) is a special case of the more general result derived by Tolman [8,9], bolstered with Ehrenfest [10]. The last two terms of Tolman's Equation (128.6) and the entirety of Tolman's Equation (129.10) in Ref. [9] read:

$$T_{\mathrm{T}}(g_{tt})^{1/2} = C_{\mathrm{T}} \iff T_{\mathrm{T}} = C_{\mathrm{T}}(g_{tt})^{-1/2}, \tag{4}$$

where (i) C_{T} is Tolman's constant in Equations (128.6) and (129.10) of Ref. [9], and (ii) T_{T} is the Tolman gravitational thermodynamic-equilibrium temperature as measured by a local observer, written more explicitly as $T_{\mathrm{T}}(r, \theta, \phi)$, at a specified location [specified radial area distance and also radial distance from apparent size [13] r and specified direction (θ, ϕ)] from the center of mass of a gravitator (spherically symmetrical or otherwise) where the time-time component of the metric has the value $g_{tt}(r, \theta, \phi)$. [As per the first term of Equation (128.6) in Ref. [9] (this notation is also employed in Refs [8] and [10]), Tolman (with Ehrenfest in Ref. [10]) sets $e^{\nu} = g_{tt}$, but for uniformity in notation we employ only the g_{tt} symbolization. Tolman [8,9] (with Ehrenfest in Ref. [10]) employs the symbols T_0 instead of T_{T} (with the subscript $_0$ designating measurement by a local observer: see p. 489 of Ref. [9]), and C instead of C_{T}. (In Tolman's paper with Ehrenfest [10] const. is employed instead of C.) We employ the subscript $_{\mathrm{T}}$ to refer to Tolman's quantities. We take T_{T} to be the Tolman gravitational

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thermodynamic-equilibrium temperature as measured by a local observer (the subscript $_0$ omitted for brevity).]

We choose Tolman's constant $C_{\rm T}$ for a given gravitator to be equal to $C_{{\rm T},r\to\infty} = \lim_{r\to\infty} T_{\rm T}(r,\theta,\phi)[g_{tt}(r,\theta,\phi)]^{1/2} = T_{{\rm T},r\to\infty}$, the Tolman gravitational thermodynamic-equilibrium temperature measured by a local observer at radial area distance and also radial distance from apparent size [13] $r\to\infty$ in *any* specified direction (θ,ϕ) from the center of mass of *any* gravitator (spherically symmetrical or otherwise), because spacetime approaches the Minkowskian and hence $g_{tt,r\to\infty} = \lim_{r\to\infty} g_{tt}(r,\theta,\phi) = 1$ at radial area distance and also radial distance from apparent size [13] $r\to\infty$ in *any* specified direction (θ,ϕ) from the center of mass of *any* gravitator (spherically symmetrical or otherwise):

$$C_{\rm T} = T_{{\rm T},r\to\infty}(g_{tt,r\to\infty})^{1/2} = T_{{\rm T},r\to\infty}\times 1 = T_{{\rm T},r\to\infty}. \tag{5}$$

Hence Equation (4) can be rewritten as:

$$\begin{aligned} &T_{\rm T}(r,\theta,\phi) = T_{{\rm T},r\to\infty}[g_{tt}(r,\theta,\phi)]^{-1/2} \text{ (in general)} \\ &\Longrightarrow T_{\rm T}(r) = T_{{\rm T},r\to\infty}[g_{tt}(r)]^{-1/2} = T_{{\rm T},r\to\infty}\left(1-\frac{r_{\rm S}}{r}\right)^{-1/2} \text{(spherical symmetry).} \end{aligned} \tag{6}$$

Tolman presents this general result not only via the last two terms of Equation (128.6) and the entirety of Equation (129.10) in Ref. [9], but also in Ref. [8] via the second equation in the Abstract and Equations (27), (28), (42), (53), and (54). It is also presented in Ref. [10] with Ehrenfest via Equations (2), (3), and (30), and discussed in some detail in Section 7 thereof. [In the weak-field limit ($M \ll r_{\rm S}c^2/2G \iff r \gg r_{\rm S} = 2GM/c^2$ given spherical symmetry) this result is presented via Equations (8) and (29) and the last (unnumbered) equation in Section 8 of Ref. [8], and also via Equations (128.4), (128.5), and (128.10) in Ref. [9].] Tolman also evaluates the gradient $d\{\ln[T_{\rm T}(r)/T_{{\rm T},r\to\infty}]\}/dr$ at and near Earth's surface in the last (unnumbered) equation in Section 2 of Ref. [8] and in Equation (128.7) of Ref. [9]. (In Earth's weak gravitational field, this gradient can of course to within at most negligible error be construed as being either with respect to radial ruler distance [13] or with respect to radial area distance and/or radial distance from apparent size [13].) We focus on the last two terms of Equation (128.6) in Ref. [9] and of Equation (3) in Ref. [10], on Equation (129.10) in Ref. [9], and on Equation (30) and Section 7 in Ref. [10]. (See also Garrod [11].)

For Schwarzschild black holes, we should expect that $T_{\rm T} = T_{\rm H}$. Indeed, letting $T_{\rm T}\to T_{\rm H}$, the second line of Equation (6) is identical to Equation (3), taking $T_{{\rm T},r\to\infty} = T_{{\rm H},r\to\infty}$ in accordance with Equation (2). For $T_{{\rm T},r\to\infty}$ of Schwarzschild (spherically-symmetrical, non-rotating) non-black holes, we will give a plausibility argument (albeit not a proof) for a conjecture concerning the value of $T_{{\rm T},r\to\infty}$.

We define a Schwarzschild *non*-black hole (of mass $M_{\rm NBH}$) as a spherically-symmetrical, non-rotating gravitator whose radius $r_{\rm NBH}$ (as per radial area distance/radial distance from apparent size [13]) exceeds its Schwarzschild radius $r_{\rm S} = 2GM_{\rm NBH}/c^2$. (The subscript $_{\rm NBH}$ may be omitted for brevity when that will not result in confusion.)

The apparent gravitational acceleration $\mathcal{G}_{\rm BH}(r)$ towards a Schwarzschild black hole of mass $M_{\rm BH}$, as measured by dangling a unit mass m at $r_{\rm S}$ from a higher altitude $r > r_{\rm S}$ (with a massless string), and the limit thereof as $r\to\infty$, are [62]:

$$\begin{aligned} \mathcal{G}_{\rm BH}(r) &= \frac{c^4}{4GM_{\rm BH}}\left(1-\frac{r_{\rm S}^2}{r^2}\right)^{-1/2} = \frac{GM_{\rm BH}}{\left(\frac{2GM_{\rm BH}}{c^2}\right)^2}\left(1-\frac{r_{\rm S}^2}{r^2}\right)^{-1/2} = \frac{GM_{\rm BH}}{r_{\rm S}^2}\left(1-\frac{r_{\rm S}^2}{r^2}\right)^{-1/2} \\ \Longrightarrow \lim_{r\to\infty}\mathcal{G}_{\rm BH}(r) &\equiv \mathcal{G}_{\rm BH} = \frac{c^4}{4GM_{\rm BH}} = \frac{GM_{\rm BH}}{r_{\rm S}^2}. \end{aligned} \tag{7}$$

The limiting value of $\mathcal{G}_{\rm BH}(r)$ as $r\to\infty$—as per the second line of Equation (7)—is sometimes called the *surface gravity* of a Schwarzschild black hole [62]. *Fortuitously*, *as if* Newtonian theory was adequate for Schwarzschild black holes [62]! *Fortuitously* despite Newtonian theory taking all distance measures to be equivalent—not distinguishing, for example, between ruler distance [13] on the one hand, and area distance and/or distance from apparent size [13] on the other. *Fortuitously* because enormous values of $\mathcal{G}_{\rm BH}(r)$ if r is only slightly greater than $r_{\rm S}$ suffer gravitational redshift down to Newtonian values in the limit $r \to \infty$. (We employ g to denote components—we focus on the time-time component—of the spacetime metric, G to denote the universal gravitational constant, and $\mathcal{G}$ to denote acceleration due to gravity.) For a Schwarzschild (spherically-symmetrical non-rotating) non-black hole of mass $M_{\rm NBH}$ in the weak-field limit ($M_{\rm NBH} \ll r_{\rm S}c^2/2G \iff r_{\rm NBH} \gg r_{\rm S} = 2GM_{\rm NBH}/c^2$)

$$\mathcal{G}_{\rm NBH} = \frac{GM_{\rm NBH}}{r_{\rm NBH}^2}. \tag{8}$$

Based on the fortuitous Newtonian-equivalence of the forms of the second line of Equation (7) on the one hand and Equation (8) on the other, and applying Equation (2), we *prima facie* suggest the following conjecture for $T_{{\rm T},r\to\infty}$ of a Schwarzschild non-black hole of the *same mass* M as a Schwarzschild black hole ($M_{\rm NBH} = M_{\rm BH} = M$) but with radius (radial area distance and also radial distance from apparent size [13]) from the center to the surface of the Schwarzschild non-black hole) $r_{\rm NBH} > r_{\rm S} = 2GM/c^2$:

$$\begin{aligned} T_{{\rm T},r\to\infty} &= T_{{\rm H},r\to\infty}\frac{\mathcal{G}_{\rm NBH}}{\mathcal{G}_{\rm BH}} = \frac{\hbar c^3}{8\pi G k_B M} \times \frac{\mathcal{G}_{\rm NBH}}{\mathcal{G}_{\rm BH}} \text{ for all } r_{\rm NBH} > r_{\rm S} \\ &= T_{{\rm H},r\to\infty}\frac{\frac{GM}{r_{\rm NBH}^2}}{\frac{GM}{r_{\rm S}^2}} = T_{{\rm H},r\to\infty}\left(\frac{r_{\rm S}}{r_{\rm NBH}}\right)^2 = \frac{\hbar c^3}{8\pi G k_B M}\left(\frac{r_{\rm S}}{r_{\rm NBH}}\right)^2 \left\{ \begin{array}{c} \text{weak-field limit:} \\ r_{\rm NBH} \gg r_{\rm S} \end{array} \right. . \end{aligned} \tag{9}$$

We recognize that while the conjecture given by Equation (9) may, at least *prima facie*, seem *plausible*, we have not *proven* it. Nonetheless at least *prima facie* it seems a reasonable conjecture, especially given that, since

$$\lim_{r_{\rm NBH}\to r_{\rm S}} T_{{\rm T},r\to\infty} = T_{{\rm H},r\to\infty} = \frac{\hbar c^3}{8\pi G k_B M}, \tag{10}$$

at least it is consistent with Equation (2). However, irrespective of the validity (or lack thereof) of this conjecture, two paragraphs hence we will prove the *extremely important point* that $T_{{\rm T},r\to\infty}$ *must* be finitely greater than absolute zero (0 K) for *all* non-black holes [as $T_{{\rm H},r\to\infty}$ is finitely greater than absolute zero (0 K) for *all* black holes].

In Ref. [8] and in Sections 128 and 129 of Ref. [9], Tolman implies that our Equations (4)–(6) are valid in *any static* spacetime [63]. Together with Ehrenfest [10] this is also implied in Ref. [10]. But given rotation at *constant* angular velocity, a *time-independent* centrifugal potential can be incorporated into the *time-independent* gravitational potential that obtains in *static* spacetime, i.e., into that which obtains neglecting the rotation [63]. This *time-independent* gravitational-centrifugal potential would then of course be a function of θ as well as of r, but at any *given* θ it can still be expressed as a function of r alone. Thus we can construe Equations (4)–(6) to be valid in *any static or stationary* spacetime [63]. Hence these results proven by Tolman [8,9], bolstered with Ehrenfest [10], and summarized via our Equations (4)–(6) and the associated discussions, imply that at thermodynamic equilibrium temperature increases downwards[6] in *any static or stationary* gravitational field (the centrifugal contribution in a stationary field construed as incorporated therein given rotation at *constant* angular velocity) [63]. But for simplicity and definiteness we focus on the *static* spacetimes at $r \geq r_{\rm S}$ of *Schwarzschild*, i.e., spherically-symmetrical *non*-rotating (black hole and non-black-hole) gravitators.

Even more importantly, Tolman [8,9], bolstered with Ehrenfest [10], *furthermore* implies *more than that*, as summarized via our Equations (4)–(6): it is *furthermore* implied [8–10] that $T_{{\rm T},r\to\infty}$ *must be finitely higher than absolute zero* (0 K) for *all non*-black holes, as $T_{{\rm H},r\to\infty}$ is finitely higher than absolute zero (0 K) for *all* black holes. As per Equations (4)–(6) [most explicitly as per Equation (6)], *in*correctly assuming that $T_{{\rm T},r\to\infty} = 0\,{\rm K}$ *in*correctly implies that $T_{\rm T}(r,\theta,\phi) = 0\,{\rm K}$ obtains *everywhere*—at least, everywhere that $g_{tt}(r,\theta,\phi) > 0$ or equivalently everywhere that $[g_{tt}(r,\theta,\phi)]^{-1/2} < \infty$. In the Schwarzschild (spherically-symmetrical non-rotating) special case, wherein $g_{tt}(r) = 1 - \frac{r_{\rm S}}{r} \iff$

$[g_{tt}(r)]^{-1/2} = \left(1 - \frac{r_S}{r}\right)^{-1/2}$, for a black hole this *in*correct implication would pertain *everywhere* in the region $r \geq r_S$ except at *exactly* $r = r_S$; and for a non-black hole, whose radius r_{NBH} exceeds r_S, this *in*correct implication would pertain *everywhere without exception*. Thus the *correct* implication is that *any* gravitator—black hole or *non*-black hole—*must radiate*: a black hole surrounded by a vacuum colder than $T_{H,r\to\infty}$ and a non-black hole surrounded by a vacuum colder than $T_{T,r\to\infty}$ can*not* be in thermodynamic equilibrium with that vacuum, but *must radiate into that vacuum and completely evaporate into that vacuum within a finite time*! Thus at least the *qualitative* fact that Hawking (Tolman!) radiation emanates from *all* gravitators—not only from black holes but also from *non*-black holes—(even if not also *quantitative* values of $T_{T,r\to\infty}$ and $T_{H,r\to\infty}$ [16,43–49]) was discovered by Tolman [8,9], bolstered with Ehrenfest [10], at least as early as 1930! *Any* gravitator—black hole or *non*-black hole—surrounded by a 0 K vacuum can*not* be at thermodynamic equilibrium unless it is enclosed within an opaque thermally insulating shell [64,65][7] and thereby insulated from that vacuum: otherwise it will *completely* Hawking- (Tolman!-) evaporate into that vacuum within a *finite* time! This has been corroborated by recent research [12].

Black holes evaporate ever more rapidly and get hotter as they lose mass, hence *completely* evaporating into a vacuum at absolute zero (0 K) within a *finite* time $\Delta t^{\text{evap}}_{\text{BH}}$. For evaporation of a Schwarzschild black hole of initial mass M_{initial} into a 0 K vacuum [43–49]:

$$\begin{aligned}
\frac{dM}{dt} &= \frac{1}{c^2}\frac{dE}{dt} = -\frac{1}{c^2}A\sigma T^4_{H,r\to\infty} \\
&= -\frac{1}{c^2}4\pi r_S^2\sigma\left(\frac{\hbar c^3}{8\pi G k_B M}\right)^4 \\
&= -\frac{1}{c^2}4\pi\left(\frac{2GM}{c^2}\right)^2\sigma\left(\frac{\hbar c^3}{8\pi G k_B M}\right)^4 \\
&= -\frac{\sigma\hbar^4 c^6}{256\pi^3 G^2 k_B^4 M^2} \equiv -\frac{C_1}{M^2} \\
&\implies dt = -\frac{M^2}{C_1}dM \\
&\implies \Delta t^{\text{evap}}_{\text{BH}} = \int dt = \frac{-\int_{M_{\text{initial}}}^0 M^2 dM}{C_1} = \frac{\int_0^{M_{\text{initial}}} M^2 dM}{C_1} = \frac{M^3_{\text{initial}}}{3C_1} \\
&\doteq 8.4115\times10^{-17}\left(\frac{M_{\text{initial}}}{1\,\text{kg}}\right)^3\text{s} \doteq 2.0976\times10^{67}\left(\frac{M_{\text{initial}}}{M_\odot}\right)^3\text{y},
\end{aligned} \tag{11}$$

where the minus signs account for the black hole's mass M *de*creasing during evaporation, $A = 4\pi r_S^2$ is its (decreasing) surface area, $\sigma = 5.670374419\times10^{-8}\,\text{W / m}^2\,\text{K}^4$ is the Stefan-Boltzmann constant [66],

$$C_1 \equiv \frac{\sigma\hbar^4 c^6}{256\pi^3 G^2 k_B^4} \doteq 3.9628\times10^{15}\,\text{kg}^3\,\text{s}^{-1} \doteq 1.5891\times10^{-68}M_\odot^3\,\text{y}^{-1} \doteq \frac{M_\odot^3}{6.2929\times10^{67}\,\text{y}}, \tag{12}$$

and $M_\odot \doteq 1.9891\times10^{30}$ kg is the mass of the Sun [67]. The dot-equal sign ($\doteq$) means very nearly equal to.

Equation (11) correctly predicts that a 1-solar-mass Schwarzschild black hole will completely Hawking-evaporate into a vacuum at absolute zero (0 K) in $\Delta t^{\text{evap}}_{\text{1solarmass BH}} \doteq 6.6198\times10^{74}\,\text{s} \doteq 2.0976\times10^{67}$ y. This is comparable with standard evaluations. Equation (11) also predicts that a 1-Earth-mass black hole will completely Hawking-evaporate into a vacuum at absolute zero (0 K) in $\Delta t^{\text{evap}}_{\text{1Earthmass BH}} \doteq 1.7917\times10^{58}\,\text{s} \doteq 5.6776\times10^{50}$ y. [A black hole formed by stellar gravitational collapse must have an initial mass of at least about $2\frac{1}{2}$ solar masses. But a 1-solar-mass black hole can be construed as a partially-Hawking-evaporated stellar black hole, and a 1-Earth-mass black hole as a more-completely Hawking-evaporated one (assuming that the Universe expands forever and hence that the temperature of the cosmic background radiation eventually drops low enough to allow such evaporation). Moreover, the laws of physics allow the existence of primordial 1-solar-mass and 1-Earth-mass black holes, even though none have yet been discovered [51–61].]

But Equation (11) also predicts that a minimal 1-Planck-mass Schwarzschild black hole $[m^{\text{initial}}_{\text{min BH}} = m_{\text{Planck}} = (\hbar c/G)^{1/2} = 2.176434 \times 10^{-8}\,\text{kg},\ r^{\text{initial}}_{\text{min BH}} = r_{\text{S,min}} = 2Gm_{\text{Planck}}/c^2 = 2G(\hbar c/G)^{1/2}/c^2 = 2\left(\hbar G/c^3\right)^{1/2} = 2l_{\text{Planck}} = 3.23251 \times 10^{-35}\,\text{m}]$ will completely Hawking-evaporate into a vacuum at absolute zero (0 K) in $\Delta t^{\text{evap}}_{\text{min BH}} \doteq 8.6718 \times 10^{-40}\,\text{s}$. This is four orders of magnitude longer than what one would intuitively expect, i.e., that a minimal Schwarzschild black hole would completely Hawking-evaporate into a vacuum at absolute zero (0 K) in a time on the order of the Planck time $t_{\text{Planck}} = \left(\hbar G/c^5\right)^{1/2} = 5.391247 \times 10^{-44}\,\text{s}$. This discrepancy obtains at least partially owing to the numerical factors (including the fundamental physical constants) in Equation (11) [49]. Hence perhaps also at least partially owing to these same numerical factors $\Delta t^{\text{evap}}_{\text{1solarmass BH}} \doteq 6.6198 \times 10^{74}\,\text{s} \doteq 2.0976 \times 10^{67}\,\text{y}$ for a 1-solar-mass black hole and $\Delta t^{\text{evap}}_{\text{1Earthmass BH}} \doteq 1.7917 \times 10^{58}\,\text{s} \doteq 5.6776 \times 10^{50}\,\text{y}$ for a 1-Earth-mass black hole could likewise be similarly longer than what one would intuitively expect. Yet, anticipating Equation (15), which predicts that the time required for a minimal Schwarzschild 1-Planck-mass near-but-non-black hole to Tolman-evaporate, $\Delta t^{\text{evap}}_{\text{min NBH}} \approx 3.2 \times 10^{-44}\,\text{s}$, is on the order of the Planck time $t_{\text{Planck}} = \left(\hbar G/c^5\right)^{1/2} = 5.391247 \times 10^{-44}\,\text{s}$, there seems to be a four-order-of-magnitude discrepancy with respect to minimal black-hole Hawking evaporation time versus minimal near-but-non-black-hole Tolman evaporation time [also anticipating the second paragraph following Equation (16)]. The rationale for *this* discrepancy between *these* two evaporation times (as opposed to the aforementioned discrepancy) is unclear to the author at the time of this writing, but perhaps it may be at least partially related to Equation (15) being a weak-field approximation.

By contrast, *non*-black holes evaporate ever more slowly and get cooler as they lose mass. But the time rate of this slowdown is itself sufficiently slow—they get cooler sufficiently slowly—that they, too, *completely* evaporate into a vacuum at absolute zero (0 K) within a *finite* time $\Delta t^{\text{evap}}_{\text{NBH}}$. For a weak-field ($M_{\text{initial}} \ll r_{\text{S}}c^2/2G \iff r_{\text{initial}} \gg r_{\text{S}} = 2GM/c^2$) Schwarzschild (spherically-symmetrical non-rotating) *non*-black hole, $g_{tt}(r)$ and therefore also $[g_{tt}(r)]^{-1/2}$ is essentially constant at unity, and hence also by Equations (4)–(6) the Tolman [8–10] temperature $T_{\text{T}}(r)$ is essentially constant at $T_{\text{T},r\to\infty}$, as M decreases from M_{initial} to $M_{\text{final}} = 0$ and r decreases from r_{initial} to $r_{\text{final}} = 0$ during the *entire* evaporation process. Moreover, $A = 4\pi r^2 \iff r = (A/\pi)^{1/2}/2$, and, assuming uniform density ρ for simplicity [justified in the weak-field ($r_{\text{NBH}} \gg r_{\text{S}}$) limit because gravity is too weak to significantly compress material with depth], also $M = 4\pi\rho r^3/3 = \rho r A/3 = \rho A^{3/2}/6\pi^{1/2} \iff A = \pi^{1/3}(6M/\rho)^{2/3}$. Hence in the weak-field ($r_{\text{NBH}} \gg r_{\text{S}}$) limit for evaporation of a Schwarzschild (spherically-symmetrical non-rotating) uniform-density non-black hole of initial mass M_{initial} into a 0 K vacuum:

$$\begin{aligned}
\frac{dM}{dt} &= \frac{1}{c^2}\frac{dE}{dt} = -\frac{\sigma A T^4_{\text{T},r\to\infty}}{c^2} = -\frac{\pi^{1/3}\sigma T^4_{\text{T},r\to\infty}}{c^2}\left(\frac{6M}{\rho}\right)^{2/3} \equiv -C_2 M^{2/3} \\
&\implies dt = -\frac{M^{-2/3}dM}{C_2} \\
&\implies \Delta t^{\text{evap}}_{\text{NBH}} = \int dt = \frac{-\int^0_{M_{\text{initial}}} M^{-2/3}dM}{C_2} = \frac{\int^0_{M_{\text{initial}}} M^{-2/3}dM}{C_2} = \frac{3M^{1/3}_{\text{initial}}}{C_2},
\end{aligned} \tag{13}$$

where the minus signs account for the non-black hole's mass M *de*creasing during evaporation, and

$$C_2 \equiv \frac{\pi^{1/3}(6/\rho)^{2/3}\sigma T^4_{\text{T},r\to\infty}}{c^2} \doteq 3.0511 \times 10^{-24}\rho^{-2/3}T^4_{\text{T},r\to\infty}\,\text{kg}^{1/3}\,\text{s}^{-1}. \tag{14}$$

$\Delta t^{\text{evap}}_{\text{NBH}}$ is *finite* because although dM/dt decreases with decreasing M, it does so only proportionately to $M^{2/3}$. (In order to render $\Delta t^{\text{evap}}_{\text{NBH}}$ infinite, dM/dt would have to decrease with decreasing M at least proportionately to M itself.) That $\Delta t^{\text{evap}}_{\text{NBH}}$ is *finite* is corroborated by recent research [12].

Equations (11) and (12) yield an exact numerical value for C_1. By contrast (even though the exact numerical values of all factors in C_2 except $T_{\mathrm{T},r\to\infty}$ *are* known) Equations (4)–(6), (13), and (14) do *not* yield enough information to provide an exact (or even less-than-exact) *numerical* value for C_2 and hence also for $T_{\mathrm{T},r\to\infty}$—albeit, as we showed in the eighth paragraph of this Section 3, they *do* yield enough information to prove that $T_{\mathrm{T},r\to\infty}$ *must be finitely higher than absolute zero* (0 K). However, if our conjecture as per Equations (7)–(10) and the associated discussions is correct, then applying our result for $T_{\mathrm{T},r\to\infty}$ in Equation (9) into Equation (13) *does* yield, at least for weak-field ($r_{\mathrm{NBH}} \gg r_{\mathrm{S}}$) uniform-density Schwarzschild (spherically-symmetrical non-rotating) non-black holes, exact numerical values for C_2, hence also for $T_{\mathrm{T},r\to\infty}$, and thence also for $\Delta t^{\mathrm{evap}}_{\mathrm{NBH}}$, as per:

$$
\begin{aligned}
\Delta t^{\mathrm{evap}}_{\mathrm{NBH}} &= \frac{3M^{1/3}_{\mathrm{initial}}}{C_2} = \frac{3M^{1/3}_{\mathrm{initial}}}{\frac{\pi^{1/3}(6/\rho)^{2/3}\sigma T^4_{\mathrm{T},r\to\infty}}{c^2}} = \frac{3M^{1/3}_{\mathrm{initial}}c^2}{\pi^{1/3}(6/\rho)^{2/3}\sigma T^4_{\mathrm{T},r\to\infty}} \\
&= \frac{3M^{1/3}_{\mathrm{initial}}c^2}{\pi^{1/3}(6/\rho)^{2/3}\sigma T_{\mathrm{H},r\to\infty}\left(\frac{r_{\mathrm{S}}}{r_{\mathrm{NBH}}}\right)^2} = \frac{3M^{1/3}_{\mathrm{initial}}c^2}{\pi^{1/3}(6/\rho)^{2/3}\sigma\left(\frac{\hbar c^3}{8\pi G k_B M}\right)^4\left(\frac{r_{\mathrm{S}}}{r_{\mathrm{NBH}}}\right)^2} \\
&= \frac{3M^{1/3}_{\mathrm{initial}}c^2\left(\frac{8\pi G k_B M}{\hbar c^3}\right)^4}{\pi^{1/3}\sigma}\left(\frac{\rho}{6}\right)^{2/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2 \\
&= \frac{12288\pi^{11/3}G^4k_B^4M^{13/3}}{\sigma c^{10}}\left(\frac{\rho}{6}\right)^{2/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2 \\
&\doteq \frac{2.4751\times 10^5 G^4k_B^4\rho^{2/3}M^{13/3}_{\mathrm{initial}}}{\sigma\hbar^4c^{10}}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2\ \mathrm{s} \\
&\doteq 1.7532\times 10^{-74}\left(\frac{\rho}{1\,\mathrm{kg}\,/\,\mathrm{m}^3}\right)^{2/3}\left(\frac{M_{\mathrm{initial}}}{1\,\mathrm{kg}}\right)^{13/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2\ \mathrm{s} \\
&\doteq 5.5555\times 10^{-82}\left(\frac{\rho}{1\,\mathrm{kg}\,/\,\mathrm{m}^3}\right)^{2/3}\left(\frac{M_{\mathrm{initial}}}{1\,\mathrm{kg}}\right)^{13/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2\ \mathrm{y}\,.
\end{aligned}
\tag{15}
$$

Comparing Equations (11) and (15),

$$
\begin{aligned}
\frac{\Delta t^{\mathrm{evap}}_{\mathrm{NBH}}}{\Delta t^{\mathrm{evap}}_{\mathrm{BH}}} &\doteq \frac{1.7532\times 10^{-74}\left(\frac{\rho}{1\,\mathrm{kg}\,/\,\mathrm{m}^3}\right)^{2/3}\left(\frac{M_{\mathrm{initial}}}{1\,\mathrm{kg}}\right)^{13/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2\ \mathrm{s}}{8.4115\times 10^{-17}\left(\frac{M_{\mathrm{initial}}}{1\,\mathrm{kg}}\right)^3\ \mathrm{s}} \\
&\doteq 2.0843\times 10^{-58}\left(\frac{\rho}{1\,\mathrm{kg}\,/\,\mathrm{m}^3}\right)^{2/3}\left(\frac{M_{\mathrm{initial}}}{1\,\mathrm{kg}}\right)^{4/3}\left(\frac{r_{\mathrm{NBH}}}{r_{\mathrm{S}}}\right)^2.
\end{aligned}
\tag{16}
$$

A 1-solar-mass (1.9891×10^{30} kg) black dwarf is of average density $\approx 2.7\times 10^9\ \mathrm{kg}\,/\,\mathrm{m}^3$ [see Reference [24], Chapter 16 (especially Section 16.4); also the Wikipedia articles "White dwarf" and "Sirius", the latter in regards to Sirius B]. Hence, by Equation (15), it would (in the approximation of assuming uniform and unchanging density) completely Tolman-evaporate into a vacuum at absolute zero (0 K) in $\approx 2.4\times 10^{70}$ s $\approx 7.6\times 10^{62}$ y, four orders of magnitude less time than $\Delta t^{\mathrm{evap}}_{\mathrm{1solarmass\,BH}} \doteq 6.6198\times 10^{74}$ s $\doteq 2.0976\times 10^{67}$ y required as per Equation (11) for a 1-solar-mass black hole to completely Hawking-evaporate into a vacuum at absolute zero (0 K). Also, by Equation (15), Earth, of radius 6.371×10^6 m, mass 5.97219×10^{24} kg, average density $5.515\times 10^3\ \mathrm{kg}\,/\,\mathrm{m}^3$, and Schwarzschild radius 8.8701×10^{-3} m, would (in the approximations of assuming uniform and unchanging density, and also neglecting Earth's rotation) completely Tolman-evaporate into a vacuum at absolute zero (0 K) in $\approx 6.5\times 10^{53}$ s $\approx 2.1\times 10^{46}$ y. This is, similarly, four orders of magnitude less time than $\Delta t^{\mathrm{evap}}_{\mathrm{1Earthmass\,BH}} \doteq 1.7917\times 10^{58}$ s $\doteq 5.6776\times 10^{50}$ y required as per Equation (11) for a 1-Earth-mass black hole to completely Hawking-evaporate into a vacuum at absolute zero (0 K). [The approximation of assuming uniform and unchanging density of a 1-solar-mass black dwarf is perhaps somewhat justified because, even though *initially* its density increases with depth, this increase with

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depth diminishes as it Tolman-evaporates and hence loses mass. It is more justified for Earth, because Earth's initial density gradient with depth is much weaker than that of a 1-solar-mass black dwarf, and it also diminishes as Earth Tolman-evaporates and hence loses mass. [The nuclear-burning lifetimes of stars whose mass is comparable to that of the Sun ($\sim 10^{10}$ y), the nuclear-burning lifetimes of the least massive red-dwarf stars that can sustain fusion of hydrogen to helium ($\sim 10^{13}$ y)—and indeed even the entire pre-black-dwarf lifetimes of both ($\sim 10^{14}$ y)—are of course negligible compared to 10^{46} y, let alone 10^{62} y or 10^{67} y.]

A minimal Schwarzschild near-but-non-black hole, of initial mass $m^{\text{initial}}_{\text{min NBH}} = m^{\text{initial}}_{\text{min BH}} = m_{\text{Planck}} = (\hbar c/G)^{1/2} = 2.176434 \times 10^{-8}\,\text{kg}$ [19] and of initial radius $r^{\text{initial}}_{\text{min NBH}} = r^{\text{initial}}_{\text{min BH}} + \epsilon = r_{\text{S,min}} + \epsilon = (2Gm_{\text{Planck}}/c^2) + \epsilon = \left[2G(\hbar c/G)^{1/2}/c^2\right] + \epsilon = 2(\hbar G/c^3)^{1/2} + \epsilon = 2l_{\text{Planck}} + \epsilon = 2 \times 1.616255 \times 10^{-35}\,\text{m} + \epsilon = 3.232510 \times 10^{-35}\,\text{m} + \epsilon$ [19], i.e., of initial radius $r^{\text{initial}}_{\text{min NBH}}$ marginally greater than $r^{\text{initial}}_{\text{min BH}} = r_{\text{S,min}} = 2l_{\text{Planck}}$ by $\epsilon \ll 2l_{\text{Planck}}$ and hence with $\left(r^{\text{initial}}_{\text{min NBH}}/r^{\text{initial}}_{\text{min BH}}\right)^2 = \left(r^{\text{initial}}_{\text{min NBH}}/r_{\text{S,min}}\right)^2$ marginally greater than unity by $(1+\epsilon)^2 \doteq 1 + 2\epsilon$, would by Equation (15) completely Tolman-evaporate into a vacuum at absolute zero (0 K) in $\Delta t^{\text{evap}}_{\text{min NBH}} \approx 3.2 \times 10^{-44}\,\text{s} \sim t_{\text{Planck}} = (\hbar G/c^5)^{1/2} = 5.391247 \times 10^{-44}\,\text{s}$ [19]. Such comparability to (even if not exactitude with) the Planck time $t_{\text{Planck}} = (\hbar G/c^5)^{1/2} = 5.391247 \times 10^{-44}\,\text{s}$ [19] is what one would expect intuitively. This contrasts with the $\Delta t^{\text{evap}}_{\text{min BH}} \doteq 8.6718 \times 10^{-40}\,\text{s}$ Hawking-evaporation time for a minimal Schwarzschild black hole [49]. As mentioned in the second paragraph following that containing Equations. (11) and (12), the rationale for the discrepancy between *these* two evaporation times is unclear to the author at the time of this writing, but perhaps it may be at least partially related to Equation (15) being a weak-field approximation.

The important point is (even in the face of both aforementioned discrepancies, especially the latter one) that *all* gravitators—*black holes and non-black holes*—are enveloped by atmospheres of equilibrium blackbody radiation. Because both $T_{\text{H},r\to\infty} > 0\,\text{K}$ [16,43–49] and $T_{\text{T},r\to\infty} > 0\,\text{K}$ [8–10], neither a black hole nor a non-black hole can be in thermodynamic equilibrium with a surrounding vacuum at (or sufficiently close to) absolute zero (0 K), but *must* radiate into that vacuum and *completely* evaporate into that vacuum within a *finite* time, unless shielded from that vacuum by enclosure within an opaque thermally-insulating shell [64,65].[7]

The Tolman-Hawking evaporation of a Schwarzschild black hole into a vacuum colder than $T_{\text{H},r\to\infty}$ and of a non-black hole into a vacuum colder than $T_{\text{T},r\to\infty}$ is in accordance with the Second Law of Thermodynamics. The entropy of a black hole is large [43–49], but the entropy of the radiation dispersed into a vacuum colder than $T_{\text{H},r\to\infty}$ by its Hawking-evaporation is even larger. The entropy of a non-black hole is not as large as that of a black hole of the same mass, affording even more scope for entropy to increase as it Tolman-evaporates into a vacuum colder than $T_{\text{T},r\to\infty}$.

Tolman was aware of the concept of black holes (even if not of the moniker "black hole"): see, for example, the last paragraph of Section 96 of Ref. [9]. Yet nowhere does this enter into Tolman's [8,9] derivations, bolstered with Ehrenfest [10], that at thermodynamic equilibrium temperature increases downwards[6] in *any* static, or even stationary, gravitational field [63]. Indeed, despite early contemplations of the concept of black holes [68–72], this concept [68–72] (and the moniker "black hole" [68–72]) was not mainstream until the 1960s [68–72]. Hence if Tolman's [8,9] discovery, bolstered with Ehrenfest [10], had borne fruit circa 1930 (or shortly thereafter), it would have (i) initially been construed with respect to *non*-black holes and (ii) dubbed Tolman radiation rather than Hawking radiation: Hawking radiation would then initially have been construed as emanating from *non*-black holes—and dubbed Tolman radiation rather than Hawking radiation!

A brief remark pertaining to Tolman-Hawking evaporation in general and to Tolman evaporation of Schwarzschild non-black holes in particular: Our results in this paper in general and in this Section 3 in particular relate, respectively, to gravitators in general and Schwarzschild non-black holes in particular that are *bound solely by their own gravity*. This entails no loss of generality: Black holes are bound *solely* by their own gravity. With respect to non-black holes in general and Schwarzschild (spherically-symmetrical, non-rotating) non-black holes in particular: any *additional*, *non*-gravitational, e.g., chemical, bonding, can be construed as incorporated within gravitational bonding as a negative contribution to the mass of the gravitator, equal to the energy required to break the non-gravitational bonding divided by c^2.

At this point, it is worthwhile to note the similarities[8]—owing to the equivalence principle [73,74][8]—between Tolman-Hawking radiation and Unruh radiation; but also a caveat.[8]

These topics, and related ones, will be further discussed in Sections 4, 5, 6, and 7.

4. All Firewalls are at the Planck Temperature

In Section 4, it may be helpful to envision a Schwarzschild black hole enclosed concentrically within an opaque thermally-insulating spherical shell at $r_{\text{shell}} \to \infty$ [64,65].[7] Hawking radiation at temperature $T_{\text{H},r\to\infty}$ as per Equation (2) is reradiated and/or reflected downwards[6] from the inner surface of this spherical shell, suffering increasing gravitational blueshift with decreasing r in accordance with Equations (3)–(6) [8–11,16–18,43–49,64,65]. Since thermodynamic equilibrium obtains *perfectly* within the shell [64,65],[7] the caveat "(at least essentially)" can be deleted from the sentence containing Equation (2): radiation within the shell is *exactly* blackbody [64,65]. Indeed, enclosure of *any* radiation—whether emanating from a source or freely existing in space[9]—within an opaque thermally-insulating shell ensures *perfect* thermodynamic equilibrium and hence an *exactly* Planckian blackbody spectrum [64,65] (even if not immediately upon enclosure, then after the relaxation time). For example, if the Sun was so enclosed, the currently *approximately* blackbody radiation [67,70–78] at its photosphere would become *exactly* blackbody [64,65]. Without enclosure within an opaque thermally-insulating shell, radiation *can* be *exactly* blackbody; with enclosure, it *must* be *exactly* blackbody [64,65]. (Of course, *exactly* blackbody radiation incorporates the cutoff of the Planckian blackbody spectrum for wavelengths exceeding the size of an enclosure or cavity [79,80]. But this is not a consideration for our spherical shell, because it is at $r_{\text{shell}} \to \infty$ [64,65].[7]) Moreover, it should be noted that the Planckian form of *any exactly*-blackbody spectrum, and thus its having an *exactly* well-defined temperature, survives gravitational frequency shifting [81]—and also motional Doppler frequency shifting [81], cosmological frequency shifting [81], and any combination of any two or all three types of frequency shifting [81].

Prima facie, by Equation (3), it might seem that arbitrarily close to the Schwarzschild radius r_S (but still at $r > r_S$) $T_{\text{H},r\to r_S} \to \infty$. But this is *not* so. Thus far, we have *not* taken into account that, if at $r > r_S$, it is *not* possible, even in principle (let alone in practice) to be arbitrarily close to r_S, because owing to quantum fluctuations spacetime breaks down as ruler distance [13] on the order of the Planck length [14–16,19]

$$l_{\text{Planck}} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616255(18) \times 10^{-35}\,\text{m} \tag{17}$$

is approached. (The standard uncertainty in l_{Planck} is 0.000018×10^{-35} m [16,19].) Thus, even in principle (let alone in practice), it is *not* possible, if at $r > r_S$, to be any closer to r_S than at minimum radial ruler distance [14–16]

$$(\delta l)_{\text{min}} = l_{\text{Planck}} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616255(18) \times 10^{-35}\,\text{m} \tag{18}$$

beyond r_S.

We now derive $T_{\text{H},r}$ as a function of radial ruler distance [13] δl beyond r_S, which we denote as $T_{\text{H},r_S+\delta l}$. We focus on regions just barely beyond r_S, i.e., where $\delta l \ll r_S$. (As per the third paragraph of Section 2, we omit the subscript $_r$ on l and on δl. Also, let $\delta r = r - r_S$ be Schwarzschild-coordinate radial distance [13] (which is also radial area distance and radial distance from apparent size [13]) beyond r_S. We focus on regions just barely beyond r_S, i.e., where, also, $\delta r \ll r_S$. Obviously

$$1 - \frac{r_S}{r} = 1 - \frac{r_S}{r_S + \delta r} = \frac{r_S + \delta r - r_S}{r_S + \delta r} = \frac{\delta r}{r_S + \delta r} \overset{\delta r \ll r_S}{=} \frac{\delta r}{r_S}. \tag{19}$$

Applying Equations (1) and (19) [13],

$$\begin{aligned} dl &= \left(1 - \frac{r_S}{r}\right)^{-1/2} dr \\ \Longrightarrow d(\delta l) &= \left(1 - \frac{r_S}{r}\right)^{-1/2} d(\delta r) \overset{\delta r \ll r_S}{=} \left(\frac{\delta r}{r_S}\right)^{-1/2} d(\delta r) = \left(\frac{r_S}{\delta r}\right)^{1/2} d(\delta r). \end{aligned} \tag{20}$$

The last step of Equation (19) and the second-to-last step of Equation (20) are justified because we focus on regions just barely beyond r_S, where $\delta r \ll r_S$. Applying Equation (20), if $\delta r \ll r_S$:

$$\begin{aligned} & d(\delta l) \overset{\delta r \ll r_S}{=} \left(\frac{r_S}{\delta r}\right)^{1/2} d(\delta r) \\ \Longrightarrow\ & \delta l \overset{\delta r \ll r_S}{=} \int_0^{\delta r} \left(\frac{r_S}{\delta r'}\right)^{1/2} d(\delta r') = 2(r_S \delta r)^{1/2} \Longrightarrow \delta r \ll \delta l \ll r_S \\ \Longrightarrow\ & \delta r \overset{\delta r \ll \delta l \ll r_S}{=} \frac{(\delta l)^2}{4 r_S} \\ \Longrightarrow\ & \left(\frac{r_S}{\delta r}\right)^{1/2} \overset{\delta r \ll \delta l \ll r_S}{=} \frac{\delta l}{2\delta r} = \frac{\delta l}{2\frac{(\delta l)^2}{4 r_S}} = \frac{2 r_S}{\delta l}. \end{aligned} \tag{21}$$

Hence, applying Equations (2)–(6), (19), (20), and (21), if $\delta r \ll \delta l \ll r_S$:

$$\begin{aligned} & T_{H,r} = T_{H,r_S+\delta r} \overset{\delta r \ll \delta l \ll r_S}{=} T_{H,r\to\infty} \left(\frac{r_S}{\delta r}\right)^{1/2} \\ \Longrightarrow\ & T_{H,r_S+\delta l} \overset{\delta r \ll \delta l \ll r_S}{=} 2T_{H,r\to\infty} \frac{r_S}{\delta l} = 2 \frac{\hbar c}{4\pi k_B r_S} \frac{r_S}{\delta l} = \frac{\hbar c}{2\pi k_B \delta l}. \end{aligned} \tag{22}$$

As noted in the paragraph containing Equations (17) and (18), even in principle (let alone in practice), δl can be no smaller than $(\delta l)_{\min} = l_{\text{Planck}}$ [14–16]. Thus, minimizing δl at $(\delta l)_{\min} = l_{\text{Planck}}$, by Equations (17), (18), and (22) we obtain

$$\begin{aligned} T_{\text{firewall}} &= T_{H,r_S+(\delta l)_{\min}} = T_{H,r_S+l_{\text{Planck}}} = \frac{\hbar c}{2\pi k_B (\delta l)_{\min}} = \frac{\hbar c}{2\pi k_B l_{\text{Planck}}} = \frac{\hbar c}{2\pi k_B \left(\frac{\hbar G}{c^3}\right)^{1/2}} \\ &= \frac{1}{2\pi k_B}\left(\frac{\hbar c^5}{G}\right)^{1/2} = \frac{1}{2\pi}\left[\frac{1}{k_B}\left(\frac{\hbar c^5}{G}\right)^{1/2}\right] = \frac{T_{\text{Planck}}}{2\pi}, \end{aligned} \tag{23}$$

where

$$T_{\text{Planck}} = \frac{1}{k_B}\left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.416784(16) \times 10^{32}\,\text{K} \tag{24}$$

is the Planck temperature [19]. (The standard uncertainty in T_{Planck} is 0.000016×10^{32} K [19].)

This result is *in*dependent of the mass M and hence also of the Schwarzschild radius $r_S = 2GM/c^2$ of a Schwarzschild black hole. As M and hence also r_S increases, by Equation (2) $T_{H,r\to\infty}$ decreases in inverse proportion. But $r_S/\delta l$ for any given $\delta l \ll r_S$ in general and hence $r_S/(\delta l)_{\min} = r_S/l_{\text{Planck}}$ in particular increases in direct proportion. Hence in accordance with Equations (2) and (22)–(24) these two opposing factors cancel out. Because of quantum fluctuations in the metric at length scales on the order of l_{Planck} [14–16], Equation (23) may be pushing the limit of accuracy of Equation (22), but we should expect Equation (23) to be valid at least in some average sense. Accordingly, perhaps we should not be too adamant about the small numerical factor of $1/2\pi$ in Equation (23), and hence recapitulate Equation (23) as

$$T_{\text{firewall}} = T_{\text{H},r_{\text{S}}+l_{\text{Planck}}} \sim T_{\text{Planck}}. \tag{25}$$

By Equation (22), recapitulated with the help of

$$\frac{\hbar c}{2\pi k_B} = 0.0003644464403\,\text{m K} \tag{26}$$

as

$$T_{\text{H},r_{\text{S}}+\delta l} \overset{\delta r \ll \delta l \ll r_{\text{S}}}{=} \frac{\hbar c}{2\pi k_B \delta l} = \frac{0.0003644464403\,\text{m K}}{\delta l}, \tag{27}$$

$T_{\text{H},r_{\text{S}}+\delta l}$ still has high values in the region $l_{\text{Planck}} \ll \delta l \ll r_{\text{S}}$, hence with quantum fluctuations in the metric of Equation (1) being negligible [14–16]. For example, the temperature of the Sun's core, 1.571×10^7 K [67], is equaled at $\delta l = 2.3198 \times 10^{-11}$ m (within one order of magnitude of typical atomic dimensions); the (effective [67,75–78]) temperature of the Sun's photosphere, 5772 K [67], is equaled at $\delta l = 6.3140 \times 10^{-8}$ m (the dimensions of small microbes); and room temperature, 300 K, is equaled at $\delta l = 1.2148 \times 10^{-6}$ m (only two orders of magnitude below the limit of naked-eye visibility $\sim 10^{-4}$ m).

Now let us consider the ruler-distance [13] wavelength of Hawking radiation in the region only slightly beyond r_{S}, i.e., where $\delta r \ll \delta l \ll r_{\text{S}}$. The ruler-distance [13] wavelength $\lambda^{\text{Wien,max}}_{r_{\text{S}}+\delta l}$ of blackbody radiation in general and of Hawking radiation in particular at the Wien's-Displacement-Law maximum with respect to wavelength [66,82] corresponding to temperature T is [66,82]

$$\lambda^{\text{Wien,max}} = \frac{0.002897771955}{T}\,\text{m}. \tag{28}$$

(Since we now focus on wavelength, we employ the Wien's-Displacement-Law maximum with respect to wavelength [66,82] as opposed to that with respect to frequency [66,82].) Hence by Equations (26) and (28) [66,82]:

$$\begin{aligned}
\lambda^{\text{Wien,max}} &= \frac{\frac{0.002897771955}{T}\,\text{m}}{\frac{\hbar c}{2\pi k_B}} \frac{\hbar c}{2\pi k_B} = \frac{\frac{0.002897771955}{T}\,\text{m}}{0.0003644464403\,\text{m K}} \frac{\hbar c}{2\pi k_B} \\
&= \frac{0.002897771955\,\text{m K}}{0.0003644464403\,\text{m K}} \frac{\hbar c}{2\pi k_B T} = 1.265466416 \frac{\hbar c}{k_B T} \\
\implies \lambda^{\text{Wien,max}}_{r_{\text{S}}+\delta l} &\overset{\delta r \ll \delta l \ll r_{\text{S}}}{=} 1.265466416 \frac{\hbar c}{k_B T_{\text{H},r_{\text{S}}+\delta l}} = 1.265466416 \frac{\hbar c}{k_B \frac{\hbar c}{2\pi k_B \delta l}} \\
&= 1.265466416 \times 2\pi \delta l = 7.951159991 \delta l.
\end{aligned} \tag{29}$$

The numerical factor $1.265466416 \times 2\pi = 7.951159991$ is dimensionless and hence is valid in any self-consistent system of units. In the third line of Equation (29) we applied the second line of Equation (22). In accordance with the reasoning concerning quantum fluctuations in the metric in the paragraph ending with Equation (25) [14–16], perhaps we should not be too adamant about the small numerical factor of $1.265466416 \times 2\pi = 7.951159991$ in the last term of Equation (29), and hence recapitulate Equation (29) as

$$\lambda^{\text{Wien,max}}_{r_{\text{S}}+\delta l} \overset{\delta r \ll \delta l \ll r_{\text{S}}}{\sim} \delta l. \tag{30}$$

Thus the ruler-distance [13] wavelength $\lambda^{\text{Wien,max}}_{r_{\text{S}}+\delta l}$ of Hawking radiation in the region $\delta r \ll \delta l \ll r_{\text{S}}$ is on the order of the ruler distance [13] δl itself. In particular, at $(\delta l)_{\text{min}} = l_{\text{Planck}}$ [14–16]

$$\lambda^{\text{Wien,max}}_{r_{\text{S}}+(\delta l)_{\text{min}}} = \lambda^{\text{Wien,max}}_{r_{\text{S}}+l_{\text{Planck}}} \sim l_{\text{Planck}}. \tag{31}$$

Hawking-radiation photons for which Equations (23)–(25) and (31) apply, and consequently for which $T_{\text{firewall}} \sim T_{\text{Planck}}$ and thus $E = h\nu \sim k_B T_{\text{firewall}} \sim k_B T_{\text{Planck}} \sim E_{\text{Planck}} = m_{\text{Planck}} c^2$ [14–16,83–86], are

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themselves Planck-mass black holes [14–16,87–89], specifically, Planck-mass geons [87–89], and thereby *themselves* contribute to the breakdown of spacetime as the Planck scale is approached, i.e., as $\delta l \to (\delta l)_{\rm min} = l_{\rm Planck}$ [14–16,87–89].

In accordance with the three immediately preceding paragraphs, and for consistency with Equation (29) keeping the numerical factor 7.951159991 [66,82], by Equations (28) and (29) [66,82]

$$\lambda^{\rm Wien,max} = \frac{0.002897771955}{T}\,{\rm m} \overset{\delta r \ll \delta l \ll r_{\rm S}}{=} 7.951159991\delta l. \tag{32}$$

Thus Hawking radiation with $\lambda^{\rm Wien,max}$ corresponding to values of T that are still high occurs in the region $l_{\rm Planck} \ll \delta l \ll r_{\rm S}$, hence with quantum fluctuations in the metric of Equation (1) being negligible [14–16]. For example, $\lambda^{\rm Wien,max}$ corresponding to the temperature of the Sun's core, 1.571×10^7 K [67], is equaled at $\delta l = 2.3198 \times 10^{-11}$ m (within one order of magnitude of typical atomic dimensions); $\lambda^{\rm Wien,max}$ corresponding to the (effective [67,75–78]) temperature of the Sun's photosphere, 5772 K [67], is equaled at $\delta l = 6.3140 \times 10^{-8}$ m (the dimensions of small microbes); and $\lambda^{\rm Wien,max}$ corresponding to room temperature, 300 K, is equaled at $\delta l = 1.2148 \times 10^{-6}$ m (only two orders of magnitude below the limit of naked-eye visibility $\sim 10^{-4}$ m).

Of course, the last two lines of Equation (29), and Equations (30) and (32) [let alone Equation (31)], do *not* apply in the region $r \gg r_{\rm S}$. For, as $r \to \infty$, $\delta l \to r - r_{\rm S} + (r_{\rm S}/2)\ln(r/r_{\rm S})$ [90], whilst applying Equation (2) and the first two lines of Equation (29) [66,82]:

$$\begin{aligned}\lambda^{\rm Wien,max}_{r\to\infty} &= 1.265466416\frac{\hbar c}{k_B T_{{\rm H},r\to\infty}} = 1.265466416\frac{\hbar c}{k_B\frac{\hbar c}{4\pi k_B r_{\rm S}}} = 1.265466416 \times 4\pi r_{\rm S} \\ &= 15.90231998 r_{\rm S} = \text{ constant.}\end{aligned} \tag{33}$$

The numerical factor $1.265466416 \times 4\pi = 15.90231998$ is dimensionless and hence is valid in any self-consistent system of units.

We have considered Schwarzschild black holes whose *only* energy source is their own Hawking radiation. This may eventually be the case for actual black holes if the Universe expands forever. But in the current Universe, black holes are bathed by photons emanating from $r \gg r_{\rm S}$—effectively from $r \to \infty$—far more energetic than thermal photons at temperature $T_{{\rm H},r\to\infty}$ as per Equation (2): photons from the $T_{\rm CBR} = 2.725$ K [50] cosmic background radiation [50], from starlight, etc. [50]. Radiation comprised of these far more energetic photons will be blueshifted to $T_{\rm firewall} \sim T_{\rm Planck}$ as given by Equations (23)–(25) at $r_{\rm S} + \delta l$ with $\delta l \gg l_{\rm Planck}$. But photons corresponding to $T_{\rm firewall} \sim T_{\rm Planck}$, i.e., for which $E = h\nu \sim k_B T_{\rm firewall} \sim k_B T_{\rm Planck} \sim E_{\rm Planck} = m_{\rm Planck}c^2$ [14–16,87–89], are *themselves* Planck-mass black holes [14–16,87–89], specifically, Planck-mass geons [87–89], and thereby *themselves* might contribute to the breakdown of spacetime at *this* $r_{\rm S} + \delta l$, i.e., at *this* $\delta l \gg l_{\rm Planck}$, *well before* $(\delta l)_{\rm min} = l_{\rm Planck}$ is approached [14–16,87–89]. Hence in the current Universe we should consider at least the possibility of the breakdown of spacetime at *this* $r_{\rm S} + \delta l$, i.e., at *this* $\delta l \gg l_{\rm Planck}$, *well before* $(\delta l)_{\rm min} = l_{\rm Planck}$ is approached [14–16,87–89]. But this is *not* what we mean by a Schwarzschild black hole's firewall. By a Schwarzschild black hole's firewall we mean that which is *intrinsic* to the black hole itself, i.e., owing *solely* to its own Hawking radiation.

5. The Exponential Nature of the Gravitational Frequency Shift

In Section 5, as in Section 4, it may be helpful to envision a Schwarzschild black hole enclosed concentrically within an opaque thermally-insulating spherical shell at $r_{\rm shell} \to \infty$ [64,65].[7] Hawking radiation at temperature $T_{{\rm H},r\to\infty}$ as per Equation (2) is reradiated and/or reflected downwards[6] from the inner surface of this spherical shell, suffering increasing gravitational blueshift with decreasing r in accordance with Equations (3)–(6) [8–11,16–18,43–49,64,65].

Expressed in terms of r, at $r \geq r_{\rm S}$ the relativistic gravitational scalar potential Φ of a Schwarzschild black hole and its magnitude $|\Phi|$ are [13,17,18]

$$\Phi = \frac{c^2}{2}\ln\left(1-\frac{2GM}{rc^2}\right) = \frac{c^2}{2}\ln\left(1-\frac{r_S}{r}\right)$$
$$\Longrightarrow |\Phi| = \frac{c^2}{2}\left|\ln\left(1-\frac{2GM}{rc^2}\right)\right| = \frac{c^2}{2}\left|\ln\left(1-\frac{r_S}{r}\right)\right|. \quad (34)$$

Applying Equations (2), (3), (19), (20), and (21) [especially Equation (19) and the last two lines of Equation (21)], if $\delta r \ll \delta l \ll r_S$, expressing Φ and $|\Phi|$ in terms of δr and δl [13,17,18]:

$$\Phi \overset{\delta r\ll\delta l\ll r_S}{=} \frac{c^2}{2}\ln\frac{\delta r}{r_S} = \frac{c^2}{2}\ln\frac{(\delta l)^2}{4r_S^2} = \frac{c^2}{2}\ln\left(\frac{\delta l}{2r_S}\right)^2 = c^2\ln\frac{\delta l}{2r_S}$$
$$\Longrightarrow |\Phi| \overset{\delta r\ll\delta l\ll r_S}{=} \frac{c^2}{2}\ln\frac{r_S}{\delta r} = \frac{c^2}{2}\ln\frac{4r_S^2}{(\delta l)^2} = \frac{c^2}{2}\ln\left(\frac{2r_S}{\delta l}\right)^2 = c^2\ln\frac{2r_S}{\delta l}. \quad (35)$$

It may be interesting to note that corresponding to minimum-definable ruler distance [13] $\delta l_{\text{min}} = l_{\text{Planck}}$ [14–16] beyond r_S

$$|\Phi|_{r_S+(\delta l)_{\text{min}}} = |\Phi|_{r_S+l_{\text{Planck}}} = c^2\ln\frac{2r_S}{\delta l_{\text{min}}} = c^2\ln\frac{2r_S}{l_{\text{Planck}}} = c^2\ln\frac{2r_S}{\left(\frac{\hbar G}{c^3}\right)^{1/2}} = c^2\ln\left[r_S\left(\frac{4c^3}{\hbar G}\right)^{1/2}\right]$$
$$= c^2\ln\left(\frac{4r_S^2c^3}{\hbar G}\right)^{1/2} = \frac{c^2}{2}\ln\frac{4r_S^2c^3}{\hbar G} = \frac{c^2}{2}\ln\frac{Ac^3}{\pi\hbar G} = \frac{c^2}{2}\ln\frac{16GM^2}{\hbar c} = \frac{c^2}{2}\ln\frac{4S}{\pi k_B}, \quad (36)$$

where A is the surface area of a black hole and S is its entropy [43–49].

We re-emphasize that a relativistic gravitational scalar potential Φ and hence also its magnitude $|\Phi|$ [17,18], and the relation thereof to gravitational potential energy [17,18], are valid concepts for *all* static, and even stationary, spacetimes [63] (not just Schwarzschild spacetime [91–93]). And that the spacetime engendered at $r \geq r_S$ by any Schwarzschild black hole and at *all* $r \geq 0$ by any Schwarzschild non-black hole (we focus on Schwarzschild metrics) is static [91–93], not merely stationary [63].

The blueshift of *any* photon (Tolman/Hawking-radiation photon or otherwise) whose frequency, energy, and mass [83–86] at $r \to \infty$ are $\nu_{r\to\infty}$, $E_{r\to\infty} = h\nu_{r\to\infty}$, and $m_{r\to\infty} = E_{r\to\infty}/c^2 = h\nu_{r\to\infty}/c^2$ [83–86], respectively, upon falling radially inwards from $r \to \infty$, increases *exponentially* rather than merely linearly with decreasing Φ—or, equivalently, with increasing $|\Phi|$) [17,18]—in accordance with [17,18]

$$\nu(|\Phi|) = \nu_{r\to\infty}e^{|\Phi|/c^2}$$
$$\Longrightarrow E(|\Phi|) = h\nu(|\Phi|) = E_{r\to\infty}e^{|\Phi|/c^2} = h\nu_{r\to\infty}e^{|\Phi|/c^2}$$
$$\Longrightarrow m(|\Phi|) = \frac{E(|\Phi|)}{c^2} = \frac{h\nu(|\Phi|)}{c^2} = m_{r\to\infty}e^{|\Phi|/c^2} = \frac{E_{r\to\infty}e^{|\Phi|/c^2}}{c^2} = \frac{h\nu_{r\to\infty}e^{|\Phi|/c^2}}{c^2}. \quad (37)$$

This obtains because as a photon falls and gets blueshifted its mass [83–86] $m = E/c^2 = h\nu/c^2$ [which of course is solely its (kinetic energy)$/c^2$ [83–86], because a photon's rest mass is zero [83–86]] increases: the photon gets more massive as it falls. Thus as a photon falls through successive ruler-distance [13] increments dl, a Schwarzschild black hole's gravitational field at $r \geq r_S$, a Schwarzschild non-black hole's gravitational field at *all* $r \geq 0$—indeed, the gravitational field $\mathcal{G} = -d\Phi/dl$ in *any* static, or even stationary, spacetime [63,91–93]—does successive increments of (positive) work [92,93]

$$dW = -m\,d\Phi = -m\frac{d\Phi}{dl}dl = m\mathcal{G}dl \quad (38)$$

not on a fixed mass m but on an *ever-increasing* mass m. [The minus sign in $\mathcal{G} = -d\Phi/dl$ obtains because $\mathcal{G}$ acts *downwards*,[6] i.e., in the direction of *decreasing* l. dW in Equation (38) is positive, because

$\mathcal{G}$ is negative, and both $d\Phi$ and dl are negative during infall.] $dW/d\Phi$ and thus the rate of increase of $m = E/c^2 = h\nu/c^2$ with decreasing Φ is proportional to $m = E/c^2 = h\nu/c^2$ itself: consequently the *exponential* form of Equation (37). (At r smaller than the radius $r_{\rm NBH}$ of a Schwarzschild non-black hole, a photon can be construed as falling through a borehole.)

Hence also, in accordance with Equations (3), (22), (23), and (34)–(37), the temperature T of any Planckian blackbody distribution of photons increases *exponentially* rather than merely linearly with decreasing Φ (or, equivalently, with increasing $|\Phi|$) [16–18,43–49,64,65,81,83–86].

Of course, the same reasoning also applies in reverse: as a photon rises, a Schwarzschild black hole's gravitational field at $r > r_S$—indeed, the gravitational field $\mathcal{G} = -d\Phi/dl$ in *any* static, or even stationary, spacetime [63,91–93]—does negative work on, or equivalently receives positive work from, *not* a fixed mass m but an *ever-decreasing* mass m. $dW/d\Phi$ and thus the rate of decrease of $m = E/c^2 = h\nu/c^2$ is proportional to $m = E/c^2 = h\nu/c^2$ itself: consequently as per Equations (37) and (38) a rising photon's mass [83–86] $m = E/c^2 = h\nu/c^2$ decreases *exponentially* rather than merely linearly with increasing Φ (or, equivalently, with decreasing $|\Phi|$) [16–18,43–49,64,65,81,83–86]. Hence also, in accordance with Equations (3), (22), (23), and (34)–(37), the temperature T of any Planckian blackbody distribution of photons decreases *exponentially* rather than merely linearly with increasing Φ (or, equivalently, with decreasing $|\Phi|$) [16–18,43–49,64,65,81,83–86]. (At r smaller than the radius $r_{\rm NBH}$ of a Schwarzschild non-black hole, a photon can be construed as rising through a borehole.)

By contrast, for a slowly radially-moving (slow *physical*—*not* necessarily slow coordinate—radial velocity $V_{\rm phys} = dl/d\tau \ll c$) nonzero-rest-mass particle (of rest mass $m > 0$), the increase of total mass in free fall (and its decrease in free rise from an upwards[6] flying start) is on a pro rata basis much smaller than for a photon—a linear rather than exponential function of Φ (or $|\Phi|$). This obtains because its (kinetic energy)$/c^2 = mV_{\rm phys}^2/2c^2$ is only a *negligibly small fraction* of its total mass $m\left(1 + \frac{V_{\rm phys}^2}{2c^2}\right)$—*not* the *entirety* [83–86] of its total mass as is the case for a photon (or other zero-rest-mass particle) [83–86].

Of course, the First Law of Thermodynamics (energy conservation) always obtains. The kinetic energy that any entity gains (loses) by falling (rising) in a gravitational field is *exactly offset* by the energy of the gravitational field itself becoming more (less) strongly negative. This point will be discussed more thoroughly in Section 6.

In wrapping up Section 5, we note that for static, and even stationary, spacetimes [63], the relativistic gravitational scalar potential Φ is related to the time-time component of the metric in accordance with [94]

$$g_{tt} = e^{2\Phi/c^2} \iff \Phi = \frac{c^2}{2}\ln g_{tt}. \tag{39}$$

Hence in static, and even stationary, spacetimes [63], the substitution $\Phi \to \frac{c^2}{2}\ln g_{tt}$ can be made in Equations (34)–(38).

6. Negative Gravitational Mass-energy and Birkhoff's Theorem Versus Massiveness of Firewalls

Thus far, we have taken for granted that a firewall does not contribute (at a maximum, not more than negligibly) to the mass M of a Schwarzschild black hole. But this has been seriously questioned [3]. It has been averred that this can*not* be even approximately true for any Schwarzschild black hole whose mass M appreciably exceeds the Planck mass $m_{\rm Planck} = (\hbar c/G)^{1/2}$—a minimum-possible-mass $M = M_{\rm min} = m_{\rm Planck} = (\hbar c/G)^{1/2}$ Schwarzschild black hole [3]. This is the firewall-mass problem [3]. [The four-order-of-magnitude discrepancy between the second paragraph following that containing Equations (11) and (12) and the second paragraph following Equation (16) does not affect our discussions in this Section 6.]

There is not universal agreement concerning the firewall-mass problem [3]. Counter-arguments resolving this problem have been proposed [4].

In Section 6, we do not make any assumption about what the mass of a firewall might be: small, large, or perhaps annulled to zero (except that it is *not* negative) [3,4]. However, we consider the firewall-mass problem [3], and propose an at least *prima facie* tentative resolution thereto. Our tentative resolution is based on: (i) the mass of a firewall (whatever it might be, if not otherwise annulled to zero [4]) being *exactly canceled* by the *negative* gravitational mass [17,18] = (*negative* gravitational energy)$/c^2$ [17,18] accompanying its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem [20–31] (actually first discovered by Jørg Tofte Jebsen [25–28]). This is in addition to, and perhaps may complement, other lines of reasoning [4] disputing massiveness [3] of firewalls. (There is a caveat [28–31][1] with respect to Birkhoff's Theorem [20–31], but it [28–31][1] is *not* relevant with respect to our considerations.[1])

The viewpoint [3] that formation of a firewall imparts a *huge net* increase to the mass of a Schwarzschild black hole [3] seems to overlook the *negative* gravitational mass [17,18] = (*negative* gravitational energy)$/c^2$ [17,18] contribution to the black-hole/firewall system. The *negativity* of gravitational energy [17,18] is the perhaps the central aspect of our tentative resolution of the firewall-mass problem [3]. We hope to show that the *negative* gravitational energy [17,18] accompanying *formation* of a firewall *exactly*—not merely approximately—cancels the firewall mass, so that the mass M of a black hole remains *exactly*—not merely approximately—*un*changed if a firewall forms. Is this, at least *prima facie*, what Ref. [3] overlooks? Reference [3] derives the mass of an *already-extant* firewall of an *already-collapsed* black hole, but seems to overlook the increased negativity of gravitational mass-energy accompanying *formation* of the firewall *during collapse*.

We note that the negative gravitational mass-energy accompanying formation of a firewall should *not* be confused with considerations regarding negative energy states of the firewall *itself* [3]. We do not make any assumption about what the mass of a firewall might be—except that in accordance with the first two paragraphs of the section entitled "Discussion" in Ref. [3], we *always* construe its mass (if not annulled to zero [4]) to be positive—*even if* there exist negative energy states: the squares of both positive and negative numbers are positive: see the term E_F^2 in Equation (10) of Ref. [3]. We show that, whatever the mass of a firewall might be, the *negative* gravitational mass [17,18] = (*negative* gravitational energy)$/c^2$ [17,18] accompanying its formation annuls it (*even if* it is not otherwise annulled [4])—effecting *zero net change* in the mass of a Schwarzschild black hole.

Consider a spherically-symmetrical non-rotating gravitator of mass M but of sufficiently large radius r that it is a Schwarzschild *non*-black hole ($r > r_S = 2GM/c^2 \iff M < r_S c^2/2G$), surrounded by a vacuum at absolute zero (0 K). As shown in Section 3, this gravitator will completely *Tolman*-radiation [8–10] evaporate within a *finite* time (see also Garrod [11]), yielding energy $E = Mc^2$ to a distant observer at $r_{obs} \gg r > r_S$. We re-emphasize that this has been corroborated by recent research [12].

Now instead consider another *identical* spherically-symmetrical non-rotating gravitator of mass M and radius r. But this time let the structural strength of the gravitator be annulled, so that it gravitationally collapses *radially* to a Schwarzschild black hole. *This* gravitator will then completely *Hawking*-radiation evaporate within a *finite* time, *also* yielding the *same* energy $E = Mc^2$ to a distant observer at $r_{obs} \gg r > r_S$. Indeed, this is required not only by the First Law of Thermodynamics (energy conservation), but also by Birkhoff's Theorem [20–31]. (There is a caveat [28–31][1] with respect to Birkhoff's Theorem [20–31], but it [28–31][1] is *not* relevant with respect to our considerations.[1]) For Birkhoff's Theorem [20–31] states that *any* purely radial gravitational collapse (or *any* purely radial dispersion against gravity from a flying start) of a spherically-symmetrical non-rotating gravitator can*not* cause *any* change detectable by a distant observer [not even gravitational waves, because radial collapse (or radial dispersion) does not generate them [20–31]]: Birkhoff's Theorem [20–31] *authorizes no exception* for gravitational collapse of the innermost shell of a gravitator's Tolman-Hawking [8–11] radiation atmosphere to a firewall. This is possible *if and only if* the mass of the gravitator does *not* change during collapse—*even if* a firewall forms. And *this*, in turn, is possible *if and only if* the mass of the firewall is *exactly counterbalanced* by the increased negativity of gravitational mass-energy accompanying its formation.

Thus there *must* be *zero net change* in the mass of the gravitator. Any increase in mass—whether due to formation of a firewall and/or otherwise—accompanying collapse *must be exactly counterbalanced by a negative contribution*. Gravitational mass = (gravitational energy)/c^2 *is always a negative contribution to mass*. And the only possible counterbalancing negative contribution is the gravitational mass-energy of the gravitator becoming *more strongly negative* during collapse. This *must* be true whether or not a firewall forms. If a firewall does *not* form, the increase in mass of the collapsing gravitator's Tolman-Hawking [8–11] radiation atmosphere will be less than if one *does* form—but so will the increase in the negativity of gravitational mass-energy.

It may be helpful to expound on Tolman-Hawking [8–11] radiation atmospheres. Consider a spherically-symmetrical non-rotating entity (Schwarzschild black hole or Schwarzschild non-black hole) enclosed concentrically within an opaque thermally-insulating spherical shell at $r_{\text{shell}} \to \infty$ [64,65].[7] Such an entity is enveloped by a Tolman-Hawking [8–11] radiation atmosphere. Because the entity is enclosed within an opaque thermally-insulating spherical shell, its Tolman-Hawking [8–11] radiation atmosphere is at thermodynamic equilibrium throughout. Hence photons of radiation emanate from *anywhere* in this radiation atmosphere. To be specific, if this entity *is* a black hole, it is equally valid to construe photons emanating either (i) from $r_S + l_{\text{Planck}}$ and then suffering gravitational redshift upon streaming outwards towards the inner surface of our spherical shell at $r \to \infty$ or (ii) from the inner surface of our spherical shell at $r \to \infty$ and then suffering gravitational blueshift upon falling inwards. Thus either we can construe Hawking radiation as suffering maximal gravitational redshift at $r \to \infty$ and no gravitational redshift at $r_S + l_{\text{Planck}}$, or we can construe it as suffering no gravitational blueshift at $r \to \infty$ and maximal gravitational blueshift at $r_S + l_{\text{Planck}}$. This viewpoint is valid because: (a) the *entire region* within our spherical shell is at *thermodynamic equilibrium throughout*. And at thermodynamic equilibrium, the principles of microscopic reversibility and detailed balance obtain [95]: hence it is equally valid to consider *any* microscopic process occurring in either the "forward" or "reverse" direction [95]. Indeed, at *thermodynamic equilibrium*, which direction (i) or (ii) immediately above is construed as "forward" or "reverse" is arbitrary [95]. (There are caveats [96,97],[10] but they are *not* relevant with respect to our considerations.[10]) (b) Curved spacetime is hot [8–11]. Thus—if the gravitational frequency shift and hence temperature increasing downwards[6] in gravitational fields is taken into account [8–11,81]—it is equally valid to construe Tolman-Hawking [8–11] radiation as emanating from *any* $r > r_S$ [8–11,81,95]. A Tolman-Hawking [8–11] radiation photon of mass $m_{r\to\infty} = E_{r\to\infty}/c^2 = h\nu_{r\to\infty}/c^2 \sim k_B T_{\text{H},r\to\infty}/c^2$ [83–86] at the inner surface of our spherical shell at $r \to \infty$ does indeed gain mass $m_{\text{Planck}} - m_{r\to\infty}$ during its infall to $r_S + l_{\text{Planck}}$, i.e., to $(\delta l)_{\text{min}} = l_{\text{Planck}}$ [14–16,83–86], attaining mass $m_{\text{Planck}} = E_{\text{Planck}}/c^2 = h\nu_{r_S+l_{\text{Planck}}}/c^2 \sim k_B T_{\text{firewall}}/c^2 \sim k_B T_{\text{Planck}}/c^2$ after having fallen to $r_S + l_{\text{Planck}}$, i.e., to $(\delta l)_{\text{min}} = l_{\text{Planck}}$ [14–16,83–86]. But the increase $m_{\text{Planck}} - m_{r\to\infty}$ in the photon's mass [14–16,83–86] that occurs during its infall is *exactly counterbalanced* by the increased negativity of the gravitational mass-energy [17,18] of the black-hole/photon system [83–86] that, by the First Law of Thermodynamics (energy conservation), *also* occurs during the photon's infall. Thus the *net* contribution to the mass of the black-hole/photon system [83–86] continues to be *only* $m_{r\to\infty}$—it does *not* increase by $m_{\text{Planck}} - m_{r\to\infty}$ to m_{Planck}, indeed, it does not increase *at all—exactly as if the photon had not suffered infall!*

If this is true with respect to any *one* infalling photon, then it must also be true with respect to *all* of the infalling photons combined required to produce a spherical shell of equilibrium blackbody radiation with inner boundary at r_S, of ruler-distance [13] radial thickness l_{Planck}, and at temperature T_{Planck}—i.e., to produce a firewall. Hence at least *prima facie* it seems that a large increase in the mass [3]—indeed *any* increase in mass at all—of the black hole attributable to firewall formation [3] is *exactly canceled out to zero*.

We re-emphasize that the downwards[6] increase in the temperature of Tolman-Hawking [8–11] radiation in the gravitational fields of Schwarzschild gravitators (black holes and non-black holes) is a special case of the general result of relativistic thermodynamics that at thermodynamic equilibrium temperature increases downwards[6] in *any* gravitational field [8–11] (at least, in *any* static, or even stationary, one [63,91–93]). Tolman-Hawking [8–11] radiation should be construed as emanating *not only* from $r_S + l_{Planck}$—indeed *not only* from *any* $r \geq r_S + l_{Planck}$—in the gravitational field of a Schwarzschild black hole—but from *anywhere* in *any* gravitational field whatsoever. This was very well conveyed by a seminar given by Dr. James H. Cooke at the Department of Physics at the University of North Texas in the 1980s—and most succinctly expressed by the title of this seminar: "Curved spacetime is hot"—confirming Tolman [8–10] (see also Garrod [11], and recall our Sections 3, 4, and 5). Of course, by "hot" it is meant hotter than absolute zero (0 K)—in even the weakest gravitational fields. Tolman-Hawking [8–11] radiation emanates from *every* location in *any* gravitational field however weak *in general*—not only from black holes, but also from *non*-black holes: Curved spacetime is hot [at least, hotter than absolute zero (0 K)] *in general*. This is *required* for consistency with temperature increasing downwards[6] given thermodynamic equilibrium in *any* gravitational field, however weak [8–11]. In this regard we re-emphasize, as Dr. James H. Cooke pointed out, that not only black holes, but also *non*-black holes, Tolman-Hawking [8–11]) radiate: In this regard, it may at this point be worthwhile to again recall Section 3. Indeed, as we noted in Section 3, when Tolman [8,9], bolstered with Ehrenfest [10], anticipated Hawking radiation (see also Garrod [11]), if that anticipation had borne fruit circa 1930 (or shortly thereafter), it would have (i) initially been construed with respect to *non*-black holes and (ii) dubbed Tolman radiation rather than Hawking radiation!

Generalizing, the free fall of *any* entity in *any* gravitational field can*not* result in *any* change in the mass of the gravitator/entity system, because by the First Law of Thermodynamics (energy conservation) the gain in the falling entity's kinetic energy [via increased frequency if it is a photon, or via increased *physical* downwards[6] velocity $V_{phys} = dl/d\tau$ (*not* necessarily increased *coordinate* downwards[6] velocity V_{coor}) if it is of nonzero rest mass] *must be exactly counterbalanced* by the gravitational mass-energy [17,18] of the gravitator/entity system becoming more strongly negative. And likewise the free rise (from an upwards[6] flying start) of *any* entity in *any* gravitational field can*not* result in *any* change in the mass of the gravitator/entity system, because by the First Law of Thermodynamics (energy conservation) the loss in the rising entity's kinetic energy [via decreased frequency if it is a photon, or via decreased *physical* upwards[6] velocity $V_{phys} = dl/d\tau$ (*not* necessarily decreased *coordinate* upwards[6] velocity) if it is of nonzero rest mass] *must be exactly counterbalanced* by the gravitational mass-energy [17,18] of the gravitator/entity system becoming less strongly negative. Furthermore this remains true even if the fall or rise is *not* free but retarded by friction [21], because friction merely thermalizes the entity's kinetic energy within the gravitator/entity system. For example, a landslide on Earth (whether or not retarded by friction) does *not* change Earth's total mass-energy $E_{Earth} = M_{Earth}c^2$ (which includes the negative contribution from Earth's gravitational energy), because the kinetic energy of the landslide (whether or not thermalized by friction [21]) is *exactly counterbalanced* by the gravitational mass-energy [17,18] of the Earth/landslide system becoming more strongly negative.

We briefly remark that Earth's negative gravitational mass-energy [17,18] reduces Earth's mass by a fraction on the order of $V_{escape}^2/c^2 \sim 10^{-9}$, where V_{escape} is the escape velocity from Earth's surface ($\approx 1.1 \times 10^4$ m / s). While this fraction is small in relative terms, in absolute terms it is a substantial negative contribution to Earth's mass, on the order of the mass of an asteroid ~ 10 km in diameter ($\sim 10^{-3}$ of Earth's diameter)—e.g., the K–T boundary asteroid [98] that was the major factor (even if not the only one) that ended the dinosaurs' reign [98].[11]

We close Section 6 with this speculative paragraph. It has been speculated [3] that owing to a firewall perhaps an infalling particle "burns up at the horizon" [3]. So we are steered in the direction of asking the following four admittedly speculative questions: (i) Might the particle be saved from falling through the horizon, i.e., through the Schwarzschild radius r_S of a black hole, by burning up? (ii) If so, does this at least *prima facie* seem to suggest the possibility that a collapsing *near*-black hole might be saved from falling through *its own* Schwarzschild radius r_S by beginning to burn up mass as soon as its surface approaches a ruler distance [13] of one Planck length beyond r_S ($r_S + l_{Planck}$), i.e., that black holes can thus come within this gnat's eyelash of forming, but can*not completely* form? This gnat's eyelash would of course *not* be sufficient to result in any *measurable or observable* astronomical

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or astrophysical dissimilarity from *completely*-formed black holes. (iii) And, for example, given (ii) immediately above, that as Hawking evaporation of a gnat's-eyelash *near*-black hole proceeds into a vacuum whose temperature is at (or sufficiently close to) absolute zero (0 K), its surface always remains a ruler distance [13] of one Planck length beyond r_S, this being maintained until Tolman-Hawking evaporation is complete? (iv) Might this be relevant, for example, with respect to solving the black-hole information paradox? For, if black holes *can* thus come within a gnat's eyelash of fully forming but can*not* fully form, no information can ever fall into a fully-formed black hole and hence there is no need for it to be retrieved from one. Of course, various (hopefully, at least to some extent, mutually compatible) resolutions of the black-hole information paradox have been proposed [99–108][12]. We note that if black holes can thus come within a gnat's eyelash—but no further—of forming, the *maximum* possible depth $|\Phi|_{max}$ of their gravitational wells is *finite*. For then, $|\Phi|_{max} = |\Phi|_{r_S+(\delta l)_{min}} = |\Phi|_{r_S+l_{Planck}}$ as per Equations (17), (18), and (36).

7. Equilibrium Relativistic Gravitational Temperature Gradients Can *Not* Defy the Second Law of Thermodynamics

We now show that equilibrium gravitational temperature gradients that exist [8–11]—indeed that are *required* [8–11]—by relativistic thermodynamics [8–11] can*not* be exploited to violate the Second Law of Thermodynamics.

First, consider a gravitator enclosed concentrically within an opaque thermally-insulating spherical shell.[7] Now consider a heat engine trying to exploit the equilibrium relativistic gravitational temperature gradient, via a hot reservoir at a lower altitude at temperature T_{hot} and a cold reservoir at a higher altitude at temperature T_{cold}.

Macroscopic consideration: Thermodynamic equilibrium [8–11,109,110] exists within the shell, and thermodynamic equilibrium [8–11,109,110] necessarily implies hydrostatic equilibrium [110–115] (but not necessarily vice versa [8–11,109–115]). Owing to hydrostatic equilibrium [110–115] that thermodynamic equilibrium [8–11,109,110] necessarily implies, the weight Eg/c^2 of a parcel of thermal energy E where the gravitational acceleration is g [8–11] *exactly counterbalances* its tendency to flow from higher temperatures at lower altitudes to lower temperatures at higher altitudes, so macroscopically there is staticity and hence no flow of heat that a heat engine can utilize. [Likewise at hydrostatic equilibrium—even without, let alone with, thermodynamic equilibrium, and either relativistically or non-relativistically—the weight mg of a parcel of fluid (gas or liquid) of mass m where the gravitational acceleration is g [8–11] *exactly counterbalances* its tendency to flow from higher pressures at lower altitudes to lower pressures at higher altitudes, so macroscopically there is staticity and hence no flow of fluid that a pneumatic engine can utilize.]

Microscopic consideration: While *macro*scopically thermodynamic equilibrium is, or at least can be construed as, *static*, by contrast, *micro*scopically, thermodynamic equilibrium is *dynamic*. At thermodynamic equilibrium, individual blackbody-radiation photons move up and down in any gravitational field. But: *Even without* our heat engine trying to convert any heat whatsoever from the hot reservoir into work, the gravitational redshift diminishes the temperature of equilibrium blackbody photons radiated at T_{hot} from the lower altitude of the hot reservoir to T_{cold} upon them reaching the higher altitude of the cold reservoir—thus diminishing the Carnot efficiency $\epsilon_{Carnot} = 1 - (T_{cold}/T_{hot})$ to $\epsilon_{Carnot} = 1 - (T_{cold}/T_{cold}) = 0$. What the gravitational temperature gradient giveth, the gravitational redshift taketh away [8–11]: after the gravitational redshift has taken its cut, there is *nothing* left over to be converted into work [8–11]. (Similarly, in accordance with either relativistic or non-relativistic hydrodynamics and thermodynamics [109–115], even though *micro*scopically at thermodynamic equilibrium individual fluid molecules comprising a gas or liquid move up and down in any gravitational field, gravitational pressure gradients can*not* be exploited by a pneumatic engine: at hydrostatic equilibrium—even without, let alone with, thermodynamic equilibrium [109–115], and either relativistically or non-relativistically—what the gravitational pressure gradient giveth, the weight taketh away [110–115].)

Next, consider a gravitator *not* enclosed concentrically within an opaque thermally-insulating spherical shell, but instead surrounded by a vacuum at (or sufficiently close to) absolute zero (0 K). Such a gravitator is *not* at thermodynamic equilibrium. *Any* gravitator's Tolman-Hawking equilibrium blackbody radiation will disperse into *any* sufficiently cold surrounding vacuum. Hence with*out* enclosure within such a shell, a heat engine *can* operate—but only at the expense of the increase in entropy owing to dispersal of the radiation into the vacuum. [Similarly, with*out* enclosure within a shell, if a gravitator has a gaseous atmosphere and/or liquid hydrosphere, it will evaporate into a surrounding vacuum. Hence with*out* enclosure within a shell, a pneumatic engine also *can* operate—but only at the expense of the increase in entropy owing to dispersal of the atmosphere and/or hydrosphere into the vacuum.]

Hence either with or without enclosure by an opaque thermally-insulating spherical shell, the Second Law of Thermodynamics is obeyed.

To re-emphasize, thermodynamic equilibrium [8–11,109,110] necessarily implies hydrostatic equilibrium [110–115], but not necessarily vice versa [8–11,109–115]. The terms "hydrostatic equation [110–113]" or "barometric equation [115]" are sometimes employed to denote hydrostatic equilibrium [110–113] but not necessarily thermodynamic equilibrium [8–11,109–115]. Earth's atmosphere and oceans are typically at hydrostatic equilibrium (or at least very nearly so). But, of course, because they are impelled by the large temperature difference between the hot solar disk and the cold rest of the sky, they are not at thermodynamic equilibrium.

8. Conclusion

Following introductory remarks in Section 1, in Section 2 we reviewed basic concepts pertaining to Schwarzschild black holes and Hawking radiation. In Section 3 we discussed the anticipation of Hawking radiation—albeit from *non*-black holes—initially by R. C. Tolman alone and shortly thereafter with P. Ehrenfest (see also Garrod [11]). This has been corroborated by recent research [12]. This *anticipation* culminates in the *proof* [8–12] that *any* gravitator—black hole or *non*-black hole—*must radiate* and hence can*not* be in thermodynamic equilibrium with a surrounding vacuum at (or sufficiently close to) absolute zero (0 K), but *must completely evaporate into that vacuum within a finite time*. The times required for evaporation of black holes and non-black holes were compared. We showed that a 1-solar-mass black dwarf (completely-cooled white dwarf), and also Earth, will completely Tolman-evaporate into a vacuum at absolute zero (0 K) in much less time than required for a 1-solar-mass black hole and a 1-Earth-mass black hole, respectively, to completely Hawking-evaporate into a vacuum at absolute zero (0 K). [A black hole formed by stellar gravitational collapse must have an initial mass of at least about $2\frac{1}{2}$ solar masses. But a 1-solar-mass black hole can be construed as a partially Hawking-evaporated stellar black hole, and a 1-Earth-mass black hole as a more-completely Hawking-evaporated one (assuming that the Universe expands forever and hence that the temperature of the cosmic background radiation eventually drops low enough to allow such evaporation). Moreover, the laws of physics allow the existence of primordial 1-solar-mass and 1-Earth-mass black holes, even though none have yet been discovered [51–61].] We also compared the times required for a minimal (Planck-mass) black hole to Hawking-evaporate and a minimal near-but-non-black hole to Tolman-evaporate into a vacuum at absolute zero (0 K), mentioning but not resolving a four-order-of-magnitude discrepancy related thereto. In Section 4, we showed that (i) if firewalls exist, they can originate via Hawking radiation at the minimum possible ruler distance [13] (the Planck length [14–16]) beyond the Schwarzschild horizon, where it has not suffered any gravitational redshift [17,18], or, alternatively, suffered maximal gravitational blueshift and (ii) the firewall temperature is on the order of the Planck temperature [19], *in*dependently of the mass and hence also of the Schwarzschild radius of a Schwarzschild black hole. In Section 5, we explained the exponential nature of the gravitational frequency shift as a function of the gravitational potential. In Section 6, we considered the firewall-mass problem [3], and provided an at least *prima facie* tentative resolution thereto based on: (i) the mass of a firewall being canceled by the *negative* gravitational mass [17,18] $=$ (*negative* gravitational energy)$/c^2$ [17,18] accompanying

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its formation, (ii) the *un*changed observations of a distant observer upon formation of a firewall, and (iii) Birkhoff's Theorem [20–31] (actually first discovered by Jørg Tofte Jebsen [25–28]). (The basis upon Birkhoff's Theorem is *un*altered by the caveat [28–31][1] thereto.) We showed that the mass of a firewall is *exactly counterbalanced* by the (negative) gravitational mass-energy accompanying its formation. Perhaps this may complement other lines of reasoning [4] disputing massiveness [3] of firewalls. In Section 7, we showed that equilibrium relativistic gravitational temperature gradients can*not* be exploited to violate the Second Law of Thermodynamics. Auxiliary topics are discussed in the Notes.

9. Notes

[1] Based on Birkhoff's Theorem (see Refs. [20–28]), it is usually averred that in General Relativity—as in Newtonian gravitational theory—the gravitational field vanishes and the gravitational potential is negative and *constant* within an evacuated non-rotating spherical shell. This implies that spacetime is Minkowskian within the shell. (See, for example, Ref. [13], Section 12.2B.) However, there is a caveat [28–31]: To the contrary, it has also been averred that, in General Relativity—*un*like in Newtonian gravitational theory—the gravitational field does *not* vanish within an evacuated non-rotating spherical shell, but instead that the field within the shell is directed radially outwards from the center [28–31]. This implies that the gravitational potential within the shell is negative but *not* constant, being least strongly negative at the center of the shell and most strongly negative at the inner surface of the shell. Moreover, contrary to the corresponding usual inference based Birkhoff's Theorem (see Ref. [13], Section 11.2B and Refs. [20–28]), this *further* implies that spacetime is *not* Minkowskian within the shell [28–31]. (The discussion of this caveat in Ref. [28] is in the section entitled "Inside Spherical Shell" under the Talk tab thereof, and is intermediate in viewpoint between the standard interpretation of Birkhoff's Theorem as per Refs. [20–27] on the one hand, and that as per Refs. [29–31] on the other.) This at least helps to resolve a clock paradox in General Relativity [31]: If the gravitational *field* vanishes, the gravitational potential is negative and *constant*, and hence spacetime is Minkowskian within an evacuated non-rotating spherical shell, how is a clock within the shell to *know* that it is within the shell and thus at a *negative gravitational potential*, and hence that it must tick more slowly than a clock at $r/r_{\text{shell}} \to \infty$ and hence at *zero gravitational potential*? For, like a clock at $r/r_{\text{shell}} \to \infty$, it would then see *zero gravitational field* and hence Minkowski spacetime. And according to General Relativity, a clock, like any other entity, interacts *locally* with a gravitational *field—no action at a distance*. A *non*-vanishing gravitational *field* within an evacuated non-rotating spherical shell, which a clock therein can interact with *locally*, thus at least helps to resolve this clock paradox [28–31]: via *local* interaction with a *non*-vanishing gravitational *field* a clock at the center of the shell *knows* that it must tick more slowly than a clock at $r/r_{\text{shell}} \to \infty$ [28–31], and a clock at the inner surface of the shell *knows* that it must tick more slowly yet [28–31]. Nonetheless a non-vanishing gravitational field within an evacuated non-rotating spherical shell does *not* alter any *other* inferences based on Birkhoff's Theorem. If, on the contrary, the gravitational field *does* vanish, the gravitational potential is negative and *constant*, and hence spacetime *is* Minkowskian within an evacuated non-rotating spherical shell, then resolution of this clock paradox would seem to require either (i) *local* interaction of *the clock's gravitational field*—which extends *beyond* the shell—with the *shell's gravitational field* at $r > r_{\text{shell}}$ somehow being communicated to the clock itself [30] or (ii) *local* interaction of the clock itself with the *shell's gravitational potential* within the shell [31]. The Aharonov-Bohm-effect counterpart of Option (i) is interpreting the Aharonov-Bohm effect as due to *local* interaction of an electron's *magnetic field* with the *magnetic field* within a tightly-wound solenoid—the electron's *magnetic field* penetrates *into* the solenoid—even though the electron *itself* sees *only* the solenoid's magnetic vector potential and *not* the solenoid's magnetic field [31]. (The electron *must be moving* relative to the solenoid in order for the Aharonov-Bohm-effect to occur and hence *must* generate a *magnetic* field in the reference frame of the solenoid. If the solenoid is tightly wound, the electron's *electric* field can*not* penetrate into it.) The Aharonov-Bohm-effect counterpart of Option (ii) is the standard interpretation of the Aharonov-Bohm effect: *local* interaction of the electron with the solenoid's *magnetic vector potential*, which does *not* vanish *outside* of the solenoid [31]. (Although not related to the topics discussed in this paper, perhaps as a brief aside it should be noted that the Aharonov-Bohm effect is important in both theoretical and experimental investigations of electromagnetic quantum phenomena. See, for example, Imry, Y. In: Fraser, G., editor, Ref. [15]; Chapter 12 (especially Sections 12.4–12.7). [We note that the first sentence in Ref. [28],"In

general relativity, Birkhoff's theorem states that any spherically-symmetrical solution of the field equations must be static and asymptotically flat" obviously does not apply *within* the Schwarzschild horizon, i.e., at $0 \le r < r_S$, of a Schwarzschild black hole. (See "Something is wrong with the statement" under the Talk tab in Ref. [28].) But this also is *not* relevant with respect to our considerations, which pertain solely to *beyond* the Schwarzschild horizon, i.e., at $r > r_S$, of a Schwarzschild black hole, and to Schwarzschild non-black holes, and specifically with respect to this Note 1, wherein we obviously take $r_{\text{shell}} > r_S = 2GM_{\text{shell}}/c^2$.]

[2] Note, however, that while the *angular* part of the Schwarzschild metric [the last term in the first and second lines of Equation (1) and the sum of the last two terms in the third and fourth lines thereof] is of *identical Euclidean* form at *all* $r \ge 0$, by contrast *radial* ruler distance is $dl_r = \left(\frac{r_S}{r} - 1\right)^{1/2} dt$ at $r < r_S$, as opposed to $dl_r = \left(1 - \frac{r_S}{r}\right)^{-1/2} dr$ at $r \ge r_S$. This obtains because the dt and dr terms of Schwarzschild metric [Equation (1)] switch sign as r_S is crossed. See Ref. [13], Sections 11.1 and 12.C–12.1E (especially Sections 12.1D and 12.1E). Yet also note that in the line immediately following Equation (12.15) in Section 12.1E: At $r < r_S$: r is referred to as a 'time'—quotation marks in the original text—recognizing that while r is time*like*, r is *not* time itself.

[3] The special case discussed in Ref. [34]—the excess (extra-Euclidean) vertical radial ruler distance of $GM/3c^2$ from the center to the surface of a non-rotating sphere of mass M and uniform density (in the weak-field limit, i.e., $M \ll r_S c^2/2G \iff r \gg r_S = 2GM/c^2$)—may help to clarify the vertical stretching of space from the Euclidean by gravity in general. It is a special case of the more general result discussed in Section 11.5 of Ref. [13]. By *vertical* it is of course meant perpendicular to the equipotential surface. The vertical direction does not in general coincide with the geometric center of a gravitator [see Ref. [13], Section 9.6 (especially the last two paragraphs)], but it does so coincide in the special case of a non-rotating spherical gravitator whose density varies at most only radially.

[4] English translations of Ref. [35] are provided in Refs. [36–38]. See also the Editor's Note (Ref. [39]) and Ref. [40], which synopsize and discuss Ref. [35].

[5] Even if the classical vacuum might be construed as nothingness, the quantum-mechanical vacuum—space as it actually exists—certainly can*not*. (See Ref. [13], pp. 418–419 and 480, Section 21.4, and Chapters 43–44; and Refs. [15,16].) If gravity stretches space, can space sustain tension? Since a medium capable of sustaining tension is required for the transmission of transverse waves [by contrast, longitudinal waves, e.g., sound, can travel through any (material, i.e., non-vacuum) medium], and since electromagnetic radiation is comprised of transverse waves, might space be construed as a latter-20th-century and 21st-century interpretation of the ether [sometimes spelled aether (the a is silent)] postulated in 19th-century physics? [See Ref. [13], Chapter 1 (especially Sections 1.6–1.10); Ref. [40], Chapter 1 (especially pp. 8–20), and p. 66; and Ref. [36], pp. 495–496.] The conventional viewpoint is, of course, that electromagnetic waves serve as their own medium—their own ether—via the continual handoff of energy from transverse electric field to transverse magnetic field to transverse electric field ... . See Ref. [41], pp. 450–458 (especially pp. 452–453).

[6] By *downwards* it is of course meant perpendicular to the equipotential surface and towards a gravitator. Downwards is not in general towards the geometric center of a gravitator [see Ref. [13], Section 9.6 (especially the last two paragraphs)], but it is so in the special case of a non-rotating spherical gravitator whose density varies at most only radially. (Of course, upwards is in the opposite direction, i.e., perpendicular to the equipotential surface and away from the gravitator.)

[7] Because matter is not a continuum but is comprised of atoms, our opaque thermally-insulating spherical shell cannot be arbitrarily thin and therefore cannot have an arbitrarily small surface mass density ρ_{shell}. Even to exist at all, it must be at least one atom thick. To be thermally-insulating, it must be opaque, and to be opaque it must be many atoms thick. (Opacity is a necessary but not sufficient condition for thermal insulation.) Hence (ignoring our speculations as per the last paragraph of Section 6) our spherical shell's $M_{\text{shell}}/r_{\text{shell}}$ ratio must be within a finite upper limit if it is not to be a black hole itself and suffer gravitational collapse: we must require the inequality $M_{\text{shell}}/r_{\text{shell}} = 4\pi\rho_{\text{shell}}r^2_{\text{shell}}/r_{\text{shell}} = 4\pi\rho_{\text{shell}}r_{\text{shell}} < c^2/2G \implies r_{\text{shell}} < c^2/8\pi G\rho_{\text{shell}}$. But it certainly is feasible for r_{shell} to greatly exceed $\lambda^{\text{Wien,max}}_{r\to\infty} = 15.90231998 r_S$ [see the paragraph containing Equation (33)] while still meeting this inequality and hence without risk of the shell's gravitational collapse: the strong inequality $\lambda^{\text{Wien,max}}_{r\to\infty} \ll r_{\text{shell}} < c^2/8\pi G\rho_{\text{shell}}$, indeed, even the double strong inequality $\lambda^{\text{Wien,max}}_{r\to\infty} \ll$

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$r_{\text{shell}} \ll c^2/8\pi G\rho_{\text{shell}}$, is very easily met. This is sufficient is for $r_{\text{shell}} \to \infty$ to *effectively* obtain for all practical purposes.

[8] Equations (2)–(6), (9), and (10) are in accordance with considerations of Unruh radiation and the equivalence principle (see Refs. [68,69]). An object undergoing acceleration a in Minkowski spacetime experiences Unruh radiation at temperature $T_{\text{U}} = \frac{\hbar a}{2\pi c k_B}$. Force $f = \frac{mc^4}{4GM}\left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2} = \frac{GMm}{r_{\text{S}}^2}\left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2}$ is required to dangle a mass m at r_{S} from a higher altitude $r > r_{\text{S}}$ (with a massless string) above a Schwarzschild black hole of mass M: see Ref. [13], Section 12.2 [especially Equation (12.17)]. The corresponding acceleration is $a = \frac{f}{m} = \frac{c^4}{4GM}\left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2} = \frac{GM}{r_{\text{S}}^2}\left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2}$ and hence the corresponding Unruh-radiation temperature is $T_{\text{U},r} = \frac{\hbar a}{2\pi c k_B} = \frac{\hbar}{2\pi c k_B} \times \frac{c^4}{4GM}\left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2} = \left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2}\frac{\hbar c^3}{8\pi G k_B M} = \left(1-\frac{r_{\text{S}}}{r}\right)^{-1/2} T_{\text{H},r\to\infty} = T_{\text{H},r}$. In accordance with the equivalence principle, $T_{\text{U},r}$ is equal to $T_{\text{H},r}$ as per Equation (3). In the limit $r \to \infty$, in accordance with the equivalence principle, $T_{\text{U},r\to\infty} = \frac{\hbar c^3}{8\pi G k_B M} = T_{\text{H},r\to\infty}$ as per Equation (2). {See Ref. [13], Section 12.2 [especially Equation (12.17)] and Section 12.6.} But a caveat: It is *important* to note that: Hawking-radiation temperature T_{H} is by Equations (2) and (3) a function of the gravitational *potential* Φ. By contrast, Unruh-radiation temperature T_{U} is a function of the motional *acceleration* a in Minkowski spacetime, so *prima facie* the equivalence principle might seem to suggest that it be the *same function* of the magnitude $|\mathcal{G}| = |d\Phi/dl|$ of the gravitational *acceleration*, i.e., the *same function* of the magnitude of the *gradient* of the potential rather than a function of the potential itself (whether of a black hole or a non-black hole). But this *in*correctly implies that T_{U} need *not* in general be equal to T_{H}: e.g., at large enough r away from a Schwarzschild black hole or for a sufficiently weak Schwarzschild non-black hole that the Newtonian approximation $a = f/m = GM/r^2$ is valid with negligible error, this *in*correctly implies that $T_{\text{U},r} = \frac{\hbar a}{2\pi c k_B} = \frac{\hbar GM}{2\pi c k_B r^2}$, and in the limit $r \to \infty$, that $T_{\text{U},r\to\infty} = 0$—in *dis*agreement with $T_{\text{H},r\to\infty} > 0$ as per Equation (2), $T_{\text{H},r} \geq T_{\text{H},r\to\infty} > 0$ as per Equation (3), and Tolman's [8,9] generalization, bolstered with Ehrenfest [10] (see also Garrod [11]), as per Equations (4)–(6) and the associated discussions. The *correct* correlation between T_{H} and T_{U} is that obtained as per Ref. [13], Section 12.2, especially the paragraph containing Equation (12.17): via dangling a mass at r_{S} from a higher altitude $r > r_{\text{S}}$ (with a massless string). With respect to the *correct* correlation between T_{T} and T_{U}, perhaps our plausibility argument as per Equations (7)–(10) and the associated discussions may at least serve as a starting point.

[9] Usually it is assumed that electromagnetic radiation can be tracked to a source, but Maxwell's equations *do not require* this. This was pointed out to me by Dr. James H. Cooke in a private communication in the 1980s.

[10] The concepts of microscopic reversibility and detailed balance require modifications in cases of (i) time-symmetry-violating dynamics and (ii) collisions between *un*symmetrical molecules even given *non*-time-symmetry-violating dynamics. See, for example, Ref. [95] concerning (i) and Ref. [96] concerning (ii). But these modifications do *not* apply with respect to electromagnetic radiation in general and hence with respect to equilibrium blackbody radiation in particular. Hence the analyses provided in Refs. [8–11,64,65] concerning equilibrium blackbody radiation are *completely valid*.

[11] Auxiliary phenomena that might have contributed to the end of the dinosaurs' reign could have included a surge in volcanic activity, the impact of a secondary asteroid, e.g., if the primary impactor had a satellite, etc.

[12] Reference [104] states that owing to quantum gravitational corrections, Hawking radiation is *not exactly* Planckian, i.e., *not exactly* blackbody, and thus *not exactly* maximum-entropy and hence a carrier of information. But this assumes that a black hole radiates into empty space. What if, instead, a black hole is enclosed concentrically by an opaque thermally-insulating spherical shell? *Initially* upon emission from the black hole, Hawking radiation emanating from the black hole would still carry information. But the Hawking radiation emanating from the black hole would then be thermalized to an *exactly* Planckian distribution within the spherical shell. Would its information then be lost? Or would the information be preserved, even if only in latent form, even after thermalization? (Of course, any physically-realizable shell can*not* be *perfectly* insulating, which perhaps may automatically solve this particular paradox.) Also, a few caveats concerning Ref. [104] are quoted in Ref. [105]. References [106] and [107] seem especially pertinent with respect to the last paragraph of our Section 6, because they discuss the possibility—at least in principle, even if not in practice with realizable technology—of *experimentally* determining whether the black-hole information paradox is resolved via firewalls, as we discussed qualitatively in the last paragraph of our Section 6, or via complementarity, according to which the interior of a black hole and Hawking radiation are not independent, but correlated.

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References

1. Polchinski, J. Burning Rings of Fire. *Scientific American Special Edition* "Black Holes". **Winter 2021,** *30* (1), 58–63.
2. Giddings, S.B. Escape from a Black Hole. *Scientific American Special Edition* "Black Holes". **Winter 2021,** *30* (1), 70–77.
3. Abramowicz, M.A.; Kluzniak, W.; Lasota, J.P. Mass of a Black Hole Firewall. *Phys. Rev. Lett.* **2014**, *112*, Article Number 091301, 4 pages. See also references cited therein.
4. Kaplan, D.E.; Rajendran, S. Firewalls in general relativity. *Phys. Rev. D* **2019**, *99* (4), Article Number 044033, 8 pages. See also references cited therein.
5. Harlow, D. Jerusalem lectures on black holes and quantum information. *Reviews of Modern Physics* **2016**, *88*, Article Number 015002, 58 pages. See also references cited therein.
6. Dai, X. The Black Hole Paradoxes and Possible Solutions. *J. Phys. Conf. Ser.* **2020**, *1634*, Article Number 012088, 7 pages. See also references cited therein.
7. Firewall (physics). Wikipedia. https://en.wikipedia.org/wiki/Firewall_(physics) (2023). See also references cited therein. (accessed on 20 November 2025).
8. Tolman, R.C. On the Weight of Heat and Thermal Equilibrium in General Relativity. *Phys. Rev.* **1930**, *35* (8), 904–924.
9. Tolman, R.C.; Ehrenfest, P. Temperature Equilibrium in a Static Gravitational Field. *Phys. Rev.* **1930**, *36* (12), 1791–1798.
10. Tolman, R.C. *Relativity, Thermodynamics, and Cosmology*; Oxford University Press: Oxford, UK, 1934; unabridged and unaltered republication by Dover, New York, NY, USA, 1987; Sections 128 and 129.
11. Garrod, C. *Statistical Mechanics and Thermodynamics*; Oxford University Press: New York, NY, USA, 1995; Exercises 7.29 and 7.30.
12. Mann, A. Disappearing Act. *Sci. Amer.* **September 2023,** *329* (2), 8–10.
13. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Sections 9.3–9.4, 11.1–11.5, and 12.2, the last paragraph on p. 326 and the first paragraph on p. 327, pp. 367, 374, 384–385, and Exercise 3.27.
14. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; pp. 12–13, in Box 17.2 Items 2′c and 2′d on p. 419, and Item 6 on pp. 426–428, the last paragraph of Section 20.6 on p. 480, Chapter 43 (especially Sections 43.1 and 43.4), and Chapter 44 (especially Sections 44.3 and 44.6).
15. Adler, R. Gravity. In: Fraser, G., editor. *The New Physics for the Twenty-First Century*, First paperback edition; Cambridge University Press: Cambridge, UK, 2009; Chapter 2. (See especially Sections 2.9 and 2.10.)
16. Green, M.B. Superstring Theory. In: Fraser, G., editor. *The New Physics for the Twenty-First Century*, First paperback edition; Cambridge University Press: Cambridge, UK, 2009; Chapter 5. (See especially Sections 5.3 and 5.4.)
17. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006, 2017; Section 1.16, Chapter 9 (especially Sections 9.3 and 9.4), and Sections 11.2A, 11.6, and 12.2 (especially Section 12.2).

18. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, Sections 7.2–7.4.
19. CODATA Internationally recommended 2022 values of the Fundamental Physical Constants. National Institute of Standards and Technology (NIST). https://physics.nist.gov/cuu/Constants/ (2022). See the category entitled “Universal constants”. (accessed on 20 November 2025).
20. Tolman, R.C. *Relativity, Thermodynamics, and Cosmology*; Oxford University Press: Oxford, UK, 1934; unabridged and unaltered republication by Dover, New York, NY, USA, 1987; Section 99.
21. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006 2017; Sections 11.2B, 12.2 (especially the last three paragraphs), 14.4, 16.1H, and 17.4, and Exercise 12.12.
22. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Section 32.2, Exercise 32.1, and Section 32.3.
23. Ohanian, H.C.; Ruffini, R. *Gravitation and Spacetime*, 3rd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2013; the 6th- through 3rd-to-last paragraphs of Section 7.4, the second paragraph preceding that containing Equation (8.14), the paragraph containing Equations (9.21)–(9.25) and the immediately following paragraph, and Appendix A.5.
24. Carroll, B.W.; Ostlie D.A. *An Introduction to Modern Astrophysics*, 2nd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2017; the section entitled “Extending Our Simple Model to Include Pressure” on pp. 1160–1162, Problem 10.2, and the section entitled “The Development of Adiabatic Density Fluctuations” on pp. 1248–1251.
25. Jebsen, J.T. Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum. *Arkiv för Matematik, Astronomi och Fysik* **1921**,15 (18), 9 pages.
26. Deser, S. Introduction to Jebsen’s paper. *Gen. Relativ. Gravit.* **2005**, *37* (12), 2251–2252.
27. Jebsen, J.T. (translation from the German original by Antonci, S.; Liebscher, D.). On the general spherically-symmetrical solutions of Einstein’s gravitational equations in vacuuo. *Gen. Relativ. Grav.* **2005**, *37* (12), 2253–2259.
28. Birkhoff’s theorem (relativity). Wikipedia. https://en.wikipedia.org/wiki/Birkhoff%27s_theorem_(relativity) (2023). See the references cited therein, and the caveat discussed in the section entitled “Inside Spherical Shell” under the Talk tab. (accessed on 20 November 2025).
29. Neslušan, L. The second rise of general relativity in astrophysics. *Modern Physics Letters A* **2019**, *34* (30), Article Number 1950244, 24 pages.
30. deLyra, J.L.; Carneiro, C.E.I. Complete Solution of the Einstein Field Equation for a Spherical Distribution of Polytropic Matter. 22 pages. Available online at publica-sbi.if.usp.br/PDFs/pd1731.pdf and portal.if.usp.br/bib/sites/portal.if.usp.br.bib/files/PDFs/pd1731.pdf (accessed on 20 November 2025).
31. A spherical-shell clock paradox in General Relativity? Available online at https://www.researchgate.net/post/a_spherical-shell_clock_paradox_in_General_Relativity (accessed on 20 November 2025).
32. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Chapters 11 and 12.
33. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Chapters 9 and 11 (in Chapter 11 see especially Sections 11.1–11.6).
34. Feynman, R.P.; Leighton, R.B.; Sands, M. *The Feynman Lectures on Physics*, Kindle edition; University of California Press: Berkeley, CA, USA, 2015, Volume II; Chapter 42 (especially Sections 42-2, 42-3, and 42-9).
35. Sakharov, A.D. Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Doklady Akademii Nauk SSSR* **1967**, *177*, 70–71.
36. Sakharov, A.D. Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Soviet Physics Doklady* **1968**, *12*, 1040–1041.
37. Sakharov, A.D. Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Soviet Physics Uspekhi* **1991**, *34* (5), 394.
38. Sakharov, A.D. Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Gen. Relativ. Gravitat.* **2000**, *32* (2), 365–367.
39. Schmidt, Hans-Jürgen. Editor’s Note: Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Gen. Relativ. Gravitat.* **2000**, *32* (2), 361–363.

40. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; pp. 426–428 and 1206–1208.
41. Epstein, L.C. *Relativity Visualized*; Insight Press: San Francisco, CA, USA, 1997; Chapter 1, (especially pp. 8–20), and p. 66.
42. Epstein, L.C. *Thinking Physics: Understandable Practical Reality*, 3rd edition; Insight Press: San Francisco, CA, USA, 2015; pp. 495–496.
43. 43. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Sections 1.12–1.15, 11.2A, 12.2, and 12.6 (especially Section 12.6).
44. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Item 1.B.d on p. xxxvi and Item 4.B.f on p. xxxviii.
45. Adler, R. Gravity. In: Fraser, G., editor. *The New Physics for the Twenty-First Century*, First paperback edition; Cambridge University Press: Cambridge, UK, 2009; Chapter 2. (See especially Section 2.10.)
46. Ohanian, H.C.; Ruffini, R. *Gravitation and Spacetime*, 3rd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2013; Section 8.7.
47. Carroll, B.W.; Ostlie D.A. *An Introduction to Modern Astrophysics*, 2nd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2017; pp. 644–646 and Problem 17.23.
48. Scully, M.; Sokolov, A.; Svidzinsky, A. Virtual Photons: From the Lamb Shift to Black Holes, *Optics & Photonics News*, **February 2018**, 34–40.
49. Hawking radiation. Wikipedia. https://en.wikipedia.org/wiki/Hawking_radiation (2023). See also references cited therein. (accessed on 20 November 2025).
50. Lang, K.R. *Essential Astrophysics*, Kindle edition; Springer-Verlag: Berlin, Germany, 2013; Sections 2.1–2.4 (for background material), Sections 15.1 and 15.2, and p. 608.
51. García-Bellido, S.; Clesse, S. Black Holes from the Beginning of Time. *Scientific American Special Edition* "Black Holes" **Winter 2021**, *30* (1), 26–31.
52. Carr, B.; Kühnel, F. Primordial Black Holes as Dark Matter Candidates: Recent Developments. *Annual Review of Nuclear and Particle Science* **2020**, *70*, 355–394. See also references cited therein.
53. Villanueva-Domingo, P.; Mena, O.; Palomarez-Ruiz, S. A Brief Review on Primordial Black Holes as Dark Matter, *Front. Astron. Space Sci.* **2021**, *8*, Article Number 681084, 10 pages. See also references cited therein.
54. Green, A.M.; Kavanagh, B.J. Primordial black holes as a dark matter candidate, *J. Phys. G. Nucl. Part. Phys.* **2021**, *48*, Article Number 043001, 29 pages. See also references cited therein.
55. Kainulainen, K.; Nurmi, S.; Schaippacasse, E.D.; Yanagida, T.T. Can primordial black holes as all dark matter explain fast radio bursts? *Phys. Rev. D* **2021**, *104*, Article Number 123033, 7 pages. See also references cited therein.
56. Stuart, C. The First Black Holes, *Science Focus* **November 3, 2021**, 70–77.
57. Carr, B. Black holes from a previous universe. *New Scientist* **April 1–7, 2023,** *257* (3432), 46–49.
58. Fumagalli, J.; Renaux-Petel, S.; Ronayne, J.W.; Witkowski, L.T. Turning in the landscape: A new mechanism for generating primordial black holes. *Physics Letters B* **2023**, *841*, Article Number 137921, 8 pages.
59. Sutter, P. Tiny primordial black holes could have created their own Big Bang. https://www.space.com/primordial-black-holes-create-big-bang (accessed on 20 November 2025).
60. Dienes, K.R.; Heurtier, L.; Huang, F.; Kim, D.; Tait, T.M.P.; Thomas, B. Primordial Black Holes Place the Universe in Statis. arXiv: 2212.01369v2 [astro-ph.CO] **28 March 2023**, 25 pages. (accessed on 20 November 2025).
61. Primordial black hole. Wikipedia. https://en.wikipedia.org/wiki/Primordial_black_hole (2023). See also references cited therein. (accessed on 20 November 2025).
62. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Section 12.2, especially the paragraph containing Equation (12.17).
63. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Section 1.16, Chapter 9 (especially Sections 9.3, 9.4, 9.6, and 9.7), and Sections 11.1–11.6, 11.13, 12.1, and 12.2 (especially Sections 11.1, 11.2, and 12.2).
64. Reif, F. *Fundamentals of Statistical and Thermal Physics*, Kindle edition; Waveland Press: Long Grove, IL, 2009; Sections 9.13–9.15 and Problems 9.9–9.13.
65. Baierlein, R. *Thermal Physics*; Cambridge University Press: Cambridge, UK, 1999; introductory material for Chapter 6 on p. 116, Section 6.1, Items 1–7 in Section 6.6, and Problems 1–15 for Chapter 6.

66. CODATA Internationally recommended 2022 values of the Fundamental Physical Constants. National Institute of Standards and Technology (NIST). https://physics.nist.gov/cuu/Constants/ (2022). See the category entitled "Physico-chemical constants". (accessed on 20 November 2025).
67. Williams, D.R. Sun Fact Sheet. National Space Science Data Center **2022**. https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfac (accessed on 20 November 2025).
68. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Item C of Box 24.1, Figure 24.1, and the caption in the title of Chapter 33.
69. Wheeler, J.A.; Ford, K. *Geons, Black Holes, and Quantum Foam: A Life in Physics*; Kindle edition: W. W. Norton, New York, NY, USA, 2000; Chapters 10 and 13.
70. Lang, K.R. *Essential Astrophysics*, Kindle edition; Springer-Verlag: Berlin, Germany, 2013; Sections 13.7.1 and 13.8.1.
71. Carroll, B.W.; Ostlie D.A. *An Introduction to Modern Astrophysics*, 2nd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2017; pp. 633–634 and Problem 17.10.
72. Black hole. Wikipedia. https://en.wikipedia.org/wiki/Black_hole (2023), especially the second paragraph and the section entitled "History". See also references cited therein. (accessed on 20 November 2025).
73. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Sections 1.12–1.15, Exercise 1.7, and Sections 8.4 and 14.1.
74. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Sections 7.4 and 10.1, Chapter 16, and Section 38.6.
75. Fritz, S. Solar Radiant Energy and Its Modification By the Earth and Its Atmosphere. In: Malone, T.F., editor. *Compendium of Meteorology*; American Meteorological Society: Boston, MA, USA, 1951; pp. 13–33. (See especially pp. 13–19.)
76. Wallace, J.M.; Hobbs, P.V. *Atmospheric Science*, 2nd edition; Elsevier: Amsterdam, The Netherlands, 2006; Section 4.3 (especially Subsections 4.3.1–4.3.3).
77. Houghton, H.G. *Physical Meteorology*; MIT Press: Cambridge, MA, USA, 1985; Chapter 3 (especially Sections 3.3 and 3.4).
78. Sunlight. Wikipedia. https://en.wikipedia.org/wiki/Sunlight **2023**. See also references cited therein. (accessed on 20 November 2025).
79. Duley, W.W. Blackbody Radiation in Small Cavities. *Am. Jour. Phys.* **1972**, *40* (9), 1337–1338.
80. Baltes, H.P. Comment on Blackbody Radiation in Small Cavities. *Amer. Jour. Phys.* **1974**, *42* (6), 505–507.
81. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Exercises 22.15–22.17 (especially Exercise 22.17) on pp. 588–589, and Box 29.2.
82. Wien's displacement law. Wikipedia. https://en.wikipedia.org/wiki/Wien%_27s_displacement law(2023). See also references cited therein. (accessed on 20 November 2025).
83. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Sections 1.16, 11.2, and 12.2.
84. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Section 7.2.
85. Rindler, W. *Introduction to Special Relativity*, 2nd (Kindle) edition; Oxford University Press: New York, NY, USA, 1991; Section 33.
86. Aharoni, J. *The Special Theory of Relativity*, 2nd edition; Oxford University Press: Oxford, UK, 1965; unabridged and unaltered republication by Dover, New York, NY, USA, 1985; pp. 158–160 and 324.
87. Wheeler, J.A.; Ford, K. *Geons, Black Holes, and Quantum Foam: A Life in Physics*; Kindle edition: W. W. Norton, New York, NY, USA, 2000; Chapters 10 and 11 (especially pp. 234–239, 252–253, and 256–257).
88. Klimets, A.P. On the fundamental role of massless form of matter in physics **2021**, 33 pages. (See especially Chapter 3.) https://philpapers.org/archive/ALXOTF.pdf (accessed on 20 November 2025).
89. Geon (physics). Wikipedia. https://en.wikipedia.org/wiki/Geon_(physics) (2023). See also references cited therein. (accessed on 20 November 2025).
90. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Section 11.5.
91. Tolman, R.C. *Relativity, Thermodynamics, and Cosmology*; Oxford University Press: Oxford, UK, 1934; unabridged and unaltered republication by Dover, New York, NY, USA, 1987; Sections 82 and 94–96.
92. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Chapter 11 (especially Sections 11.1–11.6), and Sections 12.1 and 12.2.

93. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*, First Princeton (Kindle) edition; Princeton University Press: Princeton, NJ, USA, 2017; Part VII (especially Chapter 31).
94. Rindler, W. *Relativity: Special, General, and Cosmological*, 2nd (Kindle) edition; Oxford University Press: Oxford, UK, 2006; Section 7.2.F., Chapter 9 (especially Sections 9.3, 9.4, 9.6, and 9.7), and Sections 11.1, 11.2, 11.5, 11.6, and 12.2.
95. Reif, F. *Fundamentals of Statistical and Thermal Physics*, Kindle edition; Waveland Press: Long Grove, IL, 2009; Sections 9.15, 15.1, 15.2, and 15.18 (in Section 15.18 see especially pp. 598–600); also, Problem 9.14.
96. Davies, P.C.W. *The Physics of Time Asymmetry*, 1977 edition; University of California Press: Berkeley, CA, USA, 1977; Section 6.4.
97. Tolman, R.C. *The Principles of Statistical Mechanics*; Oxford University Press, Oxford, UK, 1938; unabridged and unaltered republication by Dover, New York, NY, USA, 1979; Sections 40–43.
98. Carroll, B.W.; Ostlie D.A. *An Introduction to Modern Astrophysics*, 2nd (Kindle) edition. Cambridge University Press, Cambridge, UK, 2017, pp. 842–844.
99. Roncaglia, M. On the Conservation of Information in Quantum Physics. *Found. Phys.* **49**, 1278–1286 (2019).
100. Strauch, F.W. Resource Letter QI-1: Quantum Information. *Amer. Jour. Phys.* **84** (7), 495–507 (2016). (See especially Section VII.)
101. Giddings, S.B. Black Holes in the Quantum Universe. *Phil. Trans. Royal Soc. A* **2019**, *377*, Article Number 20190029, 13 pages.
102. Scientific American Special Report: Black Hole Mysteries Solved. *Sci. Amer.* **September 2022**, *327* (3), 28–53: (i) Moskowitz, C. Black Hole Mysteries Solved. *Sci. Amer.* **September 2022**, *327* (3), 28–29. (ii) Musser, G. Paradox Resolved. *Sci. Amer.* **September 2022**, *327* (3) 30–33. (iii) Almheiri, A. Black Holes, Wormholes, and Entanglement. *Sci. Amer.* **September 2022**, *327* (3), 34–41. (iv) Shaghoulian, E. A Tale of Two Horizons. *Sci. Amer.* **September 2022**, *327* (3), 42–47. (v) Fletcher, S. Portrait of a Black Hole. *Sci. Amer.* **September 2022**, *327* (3), 48–53. See also the references cited in From Our Archives on p. 53 [in (v)]. The black-hole information paradox, and its resolution, is discussed mainly in (i)–(iv).
103. Ananthaswamy, A. The Holographic Universe Turns 25. *Sci. Amer.* **March 2023**, *328* (3), 58–61.
104. Calmet, X.; Hsu, S.D.H.; Sebastianutti, M. Quantum gravitational corrections to particle creation by black holes. *Phys. Lett. B* **2023** Article Number 137820.
105. Crane, L. Information can survive a black hole. *New Scientist* **April 1–7, 2023**, *258* (3433), 8.
106. Wilkins, A. A way to solve the black hole paradox. *New Scientist* **January 13–19, 2024**, *261* (3473), 10–11.
107. Bousso, R.; Penington, G. Islands Far Outside the Horizon. arXiv:2312.03078v3 [hep-th] **20 December 2023**, 28 pages.
108. Black hole information paradox. Wikipedia. https://en.wikipedia.org/wiki/Black_hole_information_paradox (2023). See also references cited therein. (accessed on 20 November 2025).
109. Reif, F. *Fundamentals of Statistical and Thermal Physics*, Kindle edition; Waveland Press: Long Grove, IL, 2009; Sections 2.3 and 6.1–6.4 (in Section 6.3 see especially the subsection entitled "Molecule in an ideal gas in the presence of gravity").
110. Wark Jr., K.; Richards, D.E. *Thermodynamics*, 6th edition; WCB/McGraw-Hill: Boston, MA, USA, 1999; p. 11 and Section 6-3-5.
111. Wark Jr., K.; Richards, D.E. *Thermodynamics*, 6th edition; WCB/McGraw-Hill: Boston, MA, USA, 1999; Section 1-5-4.
112. Wallace, J.M.; Hobbs, P.V. *Atmospheric Science*, 2nd edition; Elsevier: Amsterdam, The Netherlands, 2006; Section 3.2.
113. Holton, J.R.; Hakim, G.J. *An Introduction to Dynamic Meteorology*, 5th edition; Elsevier: (Amsterdam, The Netherlands, 2013; Section 1.4.1.
114. Schroeder, D.V. *Thermal Physics*; Addison Wesley Longman: San Francisco, CA, USA, 2000; Section 1.1, and Problems 1.16, 3.37, and 6.14.
115. Schroeder, D.V. *Thermal Physics*; Addison Wesley Longman: San Francisco, CA, USA, 2000; Problem 1.16.