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On Positive Recurrence of $M_n/GI/1/\infty$ Model

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Abstract: Positive recurrence for a single-server queueing system is established under generalised intensity conditions of service which does not assume existence of the density distribution function of service but a certain integral type lower bound as a sufficient condition. Positive recurrence implies existence of the invariant distribution and a guaranteed slow convergence to it in the total variation metric.

Keywords: $M/GI/1/\infty$; positive recurrence; general service distribution function.

MSC: 60K25, 90B22

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1. Introduction

The goal of this paper is to establish **positive recurrence** of the model $M_n/G/1/\infty$ under certain assumptions. The Foster condition will be used for this. Intensity of service is assumed only partially at zero (as a lower left derivative value at zero of the "integrated intensity"); in addition, an integral type condition on the "integrated intensity" over intervals of some length is assumed.

For the recent history of the topic see [1] – [7], [14] and many more; see also [15,18]. One of the reasons – although not the only one – why various versions of this system are so popular may be explained by the fact of their intrinsic links to important topics of mathematical insurance theory, see [3].

In this paper we return to the less involved single–server system $M_n/GI/1/\infty$ where the intensity of arrivals may only depend on the number of customers in the system, with the goal of reviewing conditions of its positive recurrence. An importance of this property may be commented, for example, by the publications [1,12] where the investigation of the model *assumes* that it is in the "steady state", which is the synonym of stationarity. As is well-known, positive recurrence along with some mild mixing or coupling properties guarantees the existence of a stationary regime of the system. One particular aspect of this issue is how to achieve bounds without assuming existence of intensity of service in $M/GI/1$ model and, more generally, in Erlang – Sevastyanov's type systems. Certain results in this direction were recently established in [19] for a slightly different model. Still, in [19] it is essential that the absolute continuous part of the distribution function F (in our notation) were non-degenerate; in the present paper this is not required and the approach is different.

Note that in such a model certain close results may be obtained by the methods of regenerative processes if it is assumed that the same distribution function F (see below) has enough number of moments. However, our conditions and methods are different. The main (moderate) hope of the author is that possibly this approach may also be useful in studying ergodic properties of Erlang – Sevastyanov type models, as it happened with the earlier results and approaches based on the intensity of service as in [15,18] successfully applied in [16]. The present paper is some initial attempt toward the programme of developing tools that could help attack the problem outlined recently in [17].

The paper consists of the introduction in section 1, of the setting and main result in section 2, of two simple auxiliary lemmata in section 3 and of the proof of the main result in section 4, and two simple examples in section 5 for the comparison of sufficient conditions of theorem 1 with conditions in terms of the intensity of service in the case if the latter does exist.

2. The setting and main results

2.1. Definition of the process

The model is as follows. There is one server with an incoming flow of customers or jobs; this flow is Poissonian with intensity λ_n where n is the number of customers in the system. Is the server is idle, it immediately starts the service of the customer who arrived, unless the queue of waiting customers is not empty; in the latter case it starts the service of one of them. If the server is busy, then the newly arrived customer goes to the queue where it waits until the server completes the earlier job(s). The buffer for the queue is unlimited (denumerable). The discipline of how the server chooses the next customer from the queue for serving is FIFO ("first in – first out"). All services are independent with the same distribution function F, and they are independent on the arrivals. A serve is a synonym of a completed job. It is assumed that the mean value $\int_0^\infty x dF(x) = \int_0^\infty (1 - F(x)dx) =: \mu^{-1}$ is finite. It is assumed that

$$\Lambda := \sup_n \lambda_n < \infty. \tag{1}$$

The following state space is convenient for the description of the stochastic process which describes the model. It will be convenient to identify the zero state $\{0\}$ with a zero couple $\{0,0\}$. Then the state space of the process is a union,

$$\mathcal{X} = \{(0,0)\} \cup \bigcup_{n=1}^\infty \{(n,x): \ x \geqslant 0\},$$

and the process itself is described at all times by a two-dimensional vector $X_t = (n_t, x_t)$, with $t \geqslant 0$, where $n_t = 0, 1, \dots$ stands for the number of the customers in the system including both in the server and in the queue; after the identification of $\{0\}$ with $\{0,0\}$ mentioned above, $x_t = 0$ in the case of $n_t = 0$ by definition; the second component x_t stands for the elapsed time of the current service. It is assumed that the initial value X_0 is any pair of non-negative values, $X_0 = (n,x)$, and the process evolves in time according to the provided description. By the construction, it is a Markov process in the state space $\mathcal{X}$.

In some occasions it will be convenient to write $n_t = n(X_t)$ for the first component of X_t and $x_t = x(X_t)$ for the second one. For any $X = (n,x)$ where $F(x) < 1$, the "integrated intensity" of service

$$dH(x) = (1 - F(x))^{-1} dF(x)$$

is defined by a Stiltjes integral

$$H(x) = \int_0^x (1 - F(s))^{-1} dF(s).$$

The integral $\int_0^x (1 - F(s))^{-1} dF(s)$ is assumed to be finite for any $x \geqslant 0$, and a bit more, see the assumptions in the next subsection. For simplicity of setting and proofs, in order to avoid possible singularities it is assumed that

$$F(0) = 0, \quad \& \quad F(x) < 1, \quad \forall\, x \geqslant 0,$$
$$\text{and} \tag{2}$$
$$\tilde{F}(1) := \inf_{x \leqslant 1} (F(x+1) - F(x)) > 0.$$

2.2. Main result for $M_n/GI/1/\infty$

Recall that it is assumed that

$$H(t) < \infty, \quad \forall\, t \geqslant 0. \tag{3}$$

Let us also assume that there exists a constant r such that

$$r > 2(1 + \Lambda) \tag{4}$$

(so that $1/2 > (1 + \Lambda)/r$) where Λ was defined in (1), and such that

$$\forall\, 0 \leqslant t \leqslant 1, \quad \int_0^t (1 + s)dH(s) \geqslant rt, \tag{5}$$

and, moreover,

$$\inf_{x \geqslant 1} \int_x^{x+\Delta} (1 + s)dH(s) \geqslant r\Delta, \quad \forall\, \frac{1}{2} \leqslant \Delta \leqslant 1. \tag{6}$$

Let us hightlight that the latter inequality is not supposed to hold for small values of Δ approaching zero, but only for $\Delta \in [1/2, 1]$. The increase of the integral $\int_x^{x+\Delta} (1 + s)dH(s)$ for a fixed x as $\Delta \uparrow 1$ may be achieved due to a positive intensity $H' > 0$ if this derivative exists, and due to jumps of H, and also due to the increase of this function on sets of the Lebesgue measure zero like for the Cantor function.

Note that the process X_t may have no explosion on any bounded interval of time with probability one since the arrival intensities are bounded. In what follows $\|\mu_1 - \mu_2\|_{TV} = 2\sup_A(\mu_1(A) - \mu_2(A))$ is the distance in total variation between two probability measures; here the supremum is taken over all Borel measurable sets $A \in \mathcal{X}$.

Theorem 1. *Let assumptions (1) – (6) be satisfied. Then there exists $C > 0$ such that*

$$\mathsf{E}_{n,x}\tau_0 \leqslant C(n + x + 1). \tag{7}$$

Also, there exists a unique stationary measure μ, and, moreover, there exists $C > 0$ such that for any $t \geqslant 0$,

$$\|\mu_t^{n,x} - \mu\|_{TV} \leqslant C\,\frac{(1 + n + x)}{(1 + t)}, \tag{8}$$

where $\mu_t^{n,x}$ is a marginal distribution of the process $(X_t,\ t \geqslant 0)$ with the initial data $X = (n, x)$, and constant C is the same as in (7).

Remark 1. *We highlight that the initial value of the second component here is arbitrary. This is not always the case for papers on the model $M/GI/1$, as some of them concentrate on studying the embedded process at times of jumps down where the second component necessarily vanishes.*

Note that this theorem is not a pure existence type result because constant C may be evaluated, as can be seen from the proof.

NB: The following convention will be used, $dH(x) = 0$ if $(n, x) = (0, 0)$.

3. Lemmata

Lemma 1. *Under assumption (2), for any $T > 0$ and for any $m \leqslant n$,*

$$\mathsf{P}_{n,0}(\text{no less than } m \text{ jobs completed on } [0, T]) \leqslant F(T)^m,$$
$$\&$$
$$\mathsf{P}_{n,x}(\text{no less than } m \text{ jobs completed on } [0, T]) \leqslant F(T)^{m-1}. \tag{9}$$

Proof. The probability of no less than m completed jobs over time T for $m \leqslant n$ is given by the repeated integral

$$P_{n,x}(\text{no less than } m \text{ jobs completed over time } T)$$

$$= \int_0^T dF(x+t_1) \int_0^{T-t_1} dF(t_2) \dots \int_0^{T-t_1-\dots-t_{m-2}} dF(t_{m-1}) \int_0^{T-t_1-\dots-t_{m-1}} dF(t_m)$$

$$\leqslant (F(x+T) - F(x)) F(T)^{m-1}.$$

Hence, we obtain both inequalities in (9), as required. Note that assumption (2) implies that $\sup_{x \geqslant 0}(F(x+T) - F(x)) < 1$ for any $T > 0$, otherwise the distribution dF would be concentrated on a finite interval, which would contradict the assumption. $\square$

Recall the notation $\Lambda = \sup_n \lambda_n$ and assumption (1).

Lemma 2. *Under assumption (2), for any n*

$$\inf_{x \leqslant 1} P_{n,x}(\tau_0 \leqslant n) \geqslant p^n \exp(-n\Lambda) > 0, \quad \text{with } p = \tilde{F}(1). \tag{10}$$

Proof. Firstly, for any $n = 0, 1, \dots$, and any x,

$$P_{n,x}(\text{there are no arrivals on } [0, n]) \geqslant \exp(-n\Lambda).$$

Secondly, with probability no less than $p = \inf_{x \leqslant 1}(F(x+1) - F(x))$ there is at least one completed job on $[0, 1]$. The value p is positive according to assumption (2). After each jump down on $[0, 1]$ – which is a stopping time – this argument may be repeated by induction n times. Note that since (n, x) is the initial value of the process, the server may not become idle until n jobs are completed. So, this gives the lower bound

$$\inf_{x \leqslant 1} P_{n,x}(\text{at least } n \text{ jobs completed on } [0, n]) \geqslant p^n.$$

By the multiplication of the two values $\exp(-n\Lambda)$ and p^n, due to the independence of the services and arrivals, the proof is completed. $\square$

4. Proof of Theorem 1

0. The proof will be split into several steps. We shall consider the embedded Markov chain, namely, the process X_t at times $t = 0, 1, \dots$, and it will be shown that *this* process hits some suitable compact around "zero state" $(0, 0)$ in time which admits a finite expectation. From this property the main result will follow. The reader is warned that after this first hit the definition of the embedded Markov chain will change, as further times may become random and possibly non-integer, see step 4 of this proof.

The main goal is to establish the bound (7), from which the estimate (8) follows as a corollary.

1. Let us choose ε so that

$$\varepsilon < \frac{1}{2} - \frac{(1+\Lambda)}{r} \tag{11}$$

(see condition (4)). NB: We highlight that this value ε will be fixed for what follows and will not tend to zero.

Once ε is chosen, let

$$M = M_\varepsilon := \left\lceil \frac{|\ln \varepsilon|}{|\ln \tilde{F}(1)|} \right\rceil, \tag{12}$$

(see (2) for the definition of $\tilde{F}(1)$) and let us choose $x(\varepsilon) \geqslant 2$ such that

$$\sup_{x \geqslant x(\varepsilon)} (F(x+1) - F(x)) \leqslant \varepsilon. \tag{13}$$

Since $F(x) \uparrow 1$ as $x \uparrow \infty$, this is possible for any $\varepsilon > 0$. Let us introduce an auxiliary stopping time

$$\tau = \tau_\varepsilon := \inf(t = 0, 1, \ldots : \ n_t \leqslant M_\varepsilon \ \& \ x_t \leqslant x(\varepsilon)) \quad (\inf \varnothing = \infty). \tag{14}$$

Denote

$$K_\varepsilon = ((n, x) : n \leqslant M_\varepsilon \ \& \ x \leqslant x(\varepsilon)) \quad \& \quad L(n, x) = n + x + 1.$$

Function L will serve as a Lyapunov function outside the compact set K_ε.

First of all, we are going to estimate the first moment of τ, namely, to prove that there exists $C > 0$ such that

$$\mathsf{E}_{n,x}\tau \leqslant CL(X), \quad X = (n, x). \tag{15}$$

Recall that

$$\tau_0 := \inf(t \geqslant 0 : \ X_t = (0,0)),$$

and highlight that the definition of $\tau = \tau_\varepsilon$ is quite different from that of τ_0.

Let $X_0 = X$. We have for $X_t = (n_t, x_t) \neq (0,0)$,

$$dL(X_t) = \lambda_{n_t} \left((n_t + 2 + x_t) - (n_t + 1 + x_t) \right) dt$$

$$+ \left((n_t + 1 + x_t + dt) - (n_t + 1 + x_t) \right) + dM_t$$

$$+ 1(n_t > 0) \left(n_t - (n_t + 1 + x_t) \right) dH(x_t)$$

$$\equiv (I_1 + I_2)dt - \underbrace{1(n_t > 0)(1 + x_t)dH(x_t)}_{=:dJ^3_{n_t, x_t}} + dM_t, \tag{16}$$

where M_t is a local martingale (see, e.g., [13]), and

$$dJ^1 = I_1 \, dt := \lambda_{n_t} dt \left((n_t + 2 + x_t) - (n_t + 1 + x_t) \right) = \lambda_{n_t} dt,$$

$$dJ^2 = I_2 \, dt := \left((n_t + 1 + x_t + dt) - (n_t + 1 + x_t) \right) = dt,$$

$$dJ^3 := 1(n > 0) \left((n_t + 1 + x_t) - n_t \right) dH(x_t) = 1(n_t > 0)(1 + x_t)dH(x_t).$$

It is assumed that $J^1(0) = J^2(0) = J^3(0) = 0$ by definition. The martingale M_t is, in fact, a "normal", that is, non-local one because due to the assumptions expectations of all terms in the integral version of (16) are finite. The following bound will be established:

$$\mathsf{E}_{n,x}(J^1(1) + J^2(1) - J^3(1)) \leqslant -C, \tag{17}$$

with some $C > 0$, if $(n, x) \notin K_\varepsilon$.

Firstly, we have,

$$J^1(1) + J^2(1) = \int_0^1 (1 + \lambda_{n_t})dt \leqslant 1 + \Lambda. \tag{18}$$

In order to evaluate J^3 let us introduce a sequence of stopping times. Let

$$\gamma = \gamma_{n,x} := \inf(0 \leqslant t \leqslant 1 : x_t = 0) \quad (\inf(\varnothing) = 1).$$

To evaluate J^3, using the identity $1(t < \gamma)1(n_t > 0) = 1(t < \gamma)$ which holds provided that $n > 0$, let us estimate,

$$\mathsf{E}_{n,x} J^3_{n,x}(1) = \mathsf{E}_{n,x} \int_0^1 1(n_t > 0)(1 + x_t)dH(x_t)$$

$$= \mathsf{E}_{n,x} \int_0^\gamma \overset{=1}{\overbrace{1(n_t > 0)}}(1 + x_t)dH(x_t) + \mathsf{E}_{n,x} \int_\gamma^1 1(n_t > 0)(1 + x_t)dH(x_t)$$

$$= \mathsf{E}_{n,x} \int_0^\gamma (1 + x + t)dH(x + t) + \mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t > 0) \int_\gamma^1 (1 + x_t)dH(x_t)$$

$$+ \underbrace{\mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t = 0) \int_\gamma^1 1(n_t > 0)(1 + x_t)dH(x_t)}_{\geqslant 0}$$

$$\geqslant \mathsf{E}_{n,x} \int_0^\gamma (1 + x + t)dH(x + t)$$

$$+ \mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t > 0) \int_\gamma^1 (1 + x_t)dH(x_t) =: F^1 + F^2. \tag{19}$$

Let us estimate the term F^1. We have (recall that $\gamma \leqslant 1$ by definition),

$$F^1 = \mathsf{E}_{n,x} \int_0^\gamma (1 + x + t)dH(x + t) \tag{20}$$

$$\geqslant \mathsf{E}_{n,x}1(\gamma \geqslant 1/2) \underbrace{\int_0^\gamma (1 + x + t)dH(x + t)}_{\geqslant r\gamma} \geqslant r\mathsf{E}_{n,x}1(\gamma \geqslant 1/2)\gamma.$$

Here the complementary part of the integral $\mathsf{E}_{n,x}1(\gamma < 1/2) \int_0^\gamma (1 + x + t)dH(x + t) \geqslant 0$ was just dropped.

Further, it will be shown that

$$F^2 \geqslant r\mathsf{E}_{n,x}(1 - \gamma)1(\inf_{0 \leqslant t \leqslant 1} n_t > 0). \tag{21}$$

For this aim, let us introduce by induction the sequence of stopping times,

$$\gamma^1 = \gamma, \quad \gamma^{n+1} := \inf(\gamma^n < t \leqslant 1 : x_{\gamma^n} = 0), \quad n \geqslant 1 \quad (\inf(\varnothing) = 1).$$

Note that the component x_t might only have finitely many jumps down on any finite interval. The times of jumps on the interval $[0, 1]$ are exactly the times $\gamma^n < 1$, and possibly the last jump down on this interval may or may not occur at $t = 1$. In any case, clearly,

$$\lim_{n \to \infty} \gamma^n = 1.$$

So, we have,

$$\int_\gamma^1 (1 + x_t) dH(x_t) = \sum_{n=1}^\infty \int_{\gamma^n}^{\gamma^{n+1}} (1 + x_t) dH(x_t),$$

where for each outcome this series is almost surely a finite sum. On each interval $[\gamma^n, \gamma^{n+1}]$ we may write down

$$\int_{\gamma^n}^{\gamma^{n+1}} (1 + x_t) dH(x_t) = \int_0^{\gamma^{n+1} - \gamma^n} (1 + x_{\gamma^n} + s) dH(x_{\gamma^n} + s)$$

$$= \int_0^{\gamma^{n+1} - \gamma^n} 1(n_{x_{\gamma^n} + s} > 0)(1 + x_{\gamma^n} + s) dH(x_{\gamma^n} + s)$$

$$= \int_0^{\gamma^{n+1} - \gamma^n} 1(n_s > 0)(1 + s) dH(s) \geq r(\gamma^{n+1} - \gamma^n) 1(\inf_{0 \leq t \leq 1} n_t > 0).$$

This is by virtue of assumption (5) and because at each stopping time γ^n which is less than 1 we have $x_{\gamma^n} = 0$. If $\gamma^n \geq 1$, then both sides in the latter inequality equal zero, so that the inequality still holds true. Therefore, taking a sum over n, we obtain (21), just without $1(\inf_{0 \leq t \leq 1} n_t > 0)$ in the left hand side. This miltiplier $1(\inf_{0 \leq t \leq 1} n_t > 0)$ in the right hand side guarantees that its presence in the left hand side of (21) still leads to a valid inequality, which means that the bound (21) is justified.

It follows from (20) and (21) that

$$J^3 = F^1 + F^2 \geq r\mathsf{E}_{n,x} 1(\gamma \geq 1/2)\gamma + r\mathsf{E}_{n,x}(1 - \gamma) 1(\inf_{0 \leq t \leq 1} n_t > 0)$$

$$= r\mathsf{E}_{n,x} 1(\gamma \geq 1/2)\gamma + r\mathsf{E}_{n,x}(1 - \gamma) - r\mathsf{E}_{n,x}(1 - \gamma) 1(\inf_{0 \leq t \leq 1} n_t = 0)$$

$$\geq r - r\mathsf{E}_{n,x} 1(\gamma < 1/2)\gamma - r\mathsf{E}_{n,x} 1(\inf_{0 \leq t \leq 1} n_t = 0) \geq \frac{r}{2} - r\mathsf{P}_{n,x}(\inf_{0 \leq t \leq 1} n_t = 0).$$

Due to lemma 1, if $n > M$ then (see (12))

$$\mathsf{E}_{n,x} 1(\inf_{0 \leq t \leq 1} n_t = 0) = \mathsf{P}_{n,x}(\text{at least } n-1 \text{ jobs completed on } [0,1]) \leq F(1)^{n-1} \leq \varepsilon.$$

(NB: In fact, at least n jobs should be completed, the first one on $[x, x + 1]$; however, we prefer to have a bound independent of x. In any case, this does not change the scheme of the proof.) Likewise, if $x > x(\varepsilon)$, then

$$\mathsf{E}_{n,x} 1(\inf_{0 \leq t \leq 1} n_t = 0) \leq \mathsf{P}_{n,x}(\text{at least 1 completed job on } [0,1])$$

$$\leq F(x + 1) - F(x) \leq \varepsilon,$$

due to the choise of $x(\varepsilon)$, see (13).

Recall that ε was chosen so that $r\varepsilon < (r/2) - (1 + \Lambda)$, see (11). Hence,

$$J^3 \geqslant \frac{r}{2} - r\varepsilon.$$

Denote

$$(0 <) \, \Delta := \frac{r}{2} - \varepsilon - 1 - \Lambda.$$

Thus, for any $(n, x) \notin K_\varepsilon$ we obtain,

$$\mathsf{E}_{n,x}(J^3_{n,x}(1) - J^1(1) - J^2(1)) \geqslant \Delta. \tag{22}$$

The bound (17) follows with a constant C which may be evaluated via r, Λ, ε.

2. So, with the choice of $M = M_\varepsilon$ and $x(\varepsilon)$ according to (12) and (13), respectively, we may write,

$$1((n, x) \notin K_\varepsilon)\mathsf{E}_{n,x}L(X_1) \leqslant 1((n, x) \notin K_\varepsilon)(L(X_0) - \Delta).$$

The event $((n, x) \notin K_\varepsilon)$ may be equivalently expressed as $(\tau > 0)$. Hence, the latter inequality may also be rewritten in the form suitable for the induction:

$$\boxed{1(0 < \tau)\mathsf{E}_{n,x}L(X_1) \leqslant 1(0 < \tau)(L(X_0) - \Delta).}$$

Similarly, $((n_1, x_1) \notin K_\varepsilon)$ may be equivalently expressed as $(\tau > 1)$. So, we get

$$1(1 < \tau)\mathsf{E}_{n_1, x_1}L(X_2) \leqslant 1(1 < \tau)(L(X_1) - \Delta).$$

Hence, by taking expectations we obtain,

$$\mathsf{E}_{n,x}1(1 < \tau)\mathsf{E}_{n_1, x_1}L(X_2) \leqslant \mathsf{E}_{n,x}1(1 < \tau)(L(X_1) - \Delta).$$

Similarly, by induction (in what follows the notation $n_k = n_t|_{t=k}$ is used),

$$\mathsf{E}_{n,x}1(k - 1 < \tau)\mathsf{E}_{n_{k-1}, x_{k-1}}L(X_k) \leqslant \mathsf{E}_{n,x}1(k - 1 < \tau)(L(X_{k-1}) - \Delta), \; k \geqslant 1.$$

Due to the elementary bound $1(k - 1 < \tau) \geqslant 1(k < \tau)$, this implies,

$$\boxed{\Delta \times \mathsf{E}_{n,x}1(k - 1 < \tau) \leqslant \mathsf{E}_{n,x}1(k - 1 < \tau)L(X_{k-1}) - \mathsf{E}_{n,x}1(k < \tau)L(X_k).}$$

Summing up and dropping the negative term in the right hand side, we obtain

$$\Delta \sum_{k=1}^{N} \mathsf{E}_{n,x}1(k - 1 < \tau) \leqslant 1((n, x) \notin K_\varepsilon)L(X_0),$$

for any $N > 0$. So, by the monotone convergence theorem,

$$\Delta \, 1((n, x) \notin K_\varepsilon) \sum_{k=1}^{\infty} \mathsf{E}_{n,x}1(k - 1 < \tau) \leqslant 1((n, x) \notin K_\varepsilon)L(X_0). \tag{23}$$

By virtue of the well-known relation

$$\sum_{k=1}^{\infty} \mathsf{E}_{n,x}1(k - 1 < \tau) = \mathsf{E}_{n,x} \sum_{k=1}^{\infty} 1(k \leqslant \tau) = \mathsf{E}_{n,x} \sum_{k=1}^{\tau} 1 = \mathsf{E}_{n,x}\tau,$$

for the expectation of τ the bound (23) implies the following inequality with $X_0 = (n, x)$,

$$1((n, x) \notin K_\varepsilon)\mathsf{E}_{n,x}\tau \leqslant \Delta^{-1}1((n, x) \notin K_\varepsilon)L(X_0). \tag{24}$$

In particular, this bound signifies that

$$\mathsf{P}_{n,x}(\tau < \infty) = 1.$$

3. Now, once the bound for the expected value of τ is established, we are ready to explain the details how to get a bound for $\mathsf{E}_{n,x}\tau_0$. The rest of the proof is devoted to this implication, with the last sentences related to the corollary about the invariant measure and convergence to it.

At τ, the process X_k attains the set $(X : X \in K_\varepsilon)$, while $X_0 \notin K_\varepsilon$; hence, both

$$n_\tau \leqslant M \quad \& \quad x_\tau \leqslant x(\varepsilon).$$

By definition, random variable τ is the first integer k where simultaneously

$$n_k \leqslant M \quad \& \quad x_k \leqslant x(\varepsilon).$$

Therefore, at $k - 1$ we have either $n_k > M$, or $x_k > x(\varepsilon)$, or both. If there are no completed jobs on $[k - 1, k]$, then n_t may only increase, or, at least, stay equal on this interval, while x_t certainly increases. Therefore, $\tau = k$ may only be achieved by at least one completed job; it would mean a jump down by one of the n-component and simultaneously a jump down to zero of the x-component. Then at k we obtain $x_k \leqslant 1$, which certainly makes it less than $x(\varepsilon)$, irrespectively of whether or not there were other arrivals or completed jobs on $[k - 1, k]$ (recall that in addition to inequality (13) we assumed that $x(\varepsilon) \geqslant 2$).

Now, given $n \leqslant M$ and $x \leqslant x(\varepsilon)$, by virtue of lemma 2, for any $T > 0$ we have with any $x \leqslant 1$,

$$\mathsf{P}_{n,x}(\text{no arrivals on } [0, T], n(X_T) = 0) \geqslant p(T) > 0, \tag{25}$$

where

$$p(T) := p^T \exp(-T\Lambda) = \tilde{F}(1)^T \exp(-T\Lambda).$$

Here T is any positive integer and

$$\tilde{F}(1) = \inf_{x \leqslant 1}(F(x + 1) - F(x)).$$

Note that, of course, for non-integer values of $T > 0$ there is a likewise bound, but it looks a bit more involved, and using integers T values suffices for the proof. Recall that it was assumed in (2) that $\tilde{F}(1) > 0$, and it follows from the first line of (2) that $\tilde{F}(1) < 1$. Inequality (25) implies that

$$\inf_{x \leqslant 1} \mathsf{P}_{n_\tau, x_\tau}(\text{no arrivals on } [0, T], n(X_T) = 0) \geqslant p(T) > 0. \tag{26}$$

NB: Here the standard notations for homogeneous Markov processes are used, which means that X_T after stopping time τ is, actually, the value $X_{\tau+T}$.

Note that for any $T > 0$ the event $(n(X_{\tau+T}) = 0)$ implies that

$$\tau_0 \leqslant \tau + T.$$

Hence, we conclude,

$$\mathsf{P}_{n_\tau, x_\tau}(\text{no arrivals on } [0, T], \tau_0 \leqslant T) \geqslant p(T) > 0. \tag{27}$$

and, therefore, for any x,

$$P_{n,x}(\text{no arrivals on } [\tau, \tau + T], \tau_0 \leqslant \tau + T) \geqslant p(T) > 0. \qquad (28)$$

4. Consider now the process X started at time τ from state (n_τ, x_τ) with $x_\tau \leqslant 1$ and $n_\tau \leqslant M$.

Let $T := \lceil x(\varepsilon) \rceil$ and let us stop the process either at $\tau + T$, or at

$$\chi := \inf(t \geqslant \tau : n_t \geqslant M + 1, \text{ or } x_t \geqslant x(\varepsilon) + 1),$$

whatever happens earlier. In other words, consider the stopping time

$$\chi^1 := \chi \wedge (\tau + T).$$

The event $(\chi^1 = \tau + T)$ implies that the process $L(X_t)$ does not exceed the level $M + 2 + T$ on the interval $[\tau, \tau + T]$. On the other hand, according to the arguments of step 3 we have,

$$P_{n_\tau, x_\tau}(\tau_0 \leqslant \chi^1) \geqslant p(T) > 0. \qquad (29)$$

Let

$$\chi^0 := 0, \quad \tau^1 := \tau,$$

and further, let us define two sequences of stopping times by induction,

$$\chi^{k+1} := \inf(t \geqslant \tau^{k+1} : n_t \geqslant M + 1, \text{ or } x_t \geqslant x(\varepsilon) + 1) \wedge (\tau^{k+1} + T),$$

$$\tau^{k+1} := \inf(\chi^k + i, i \geqslant 0 : n_{\chi^k + i} > M, \text{ or } x_{\chi^k + i} \geqslant x(\varepsilon) + 1), \quad k \geqslant 0.$$

Note that all integers here in expressions like χ^k and τ^k stand for upper indices, not for power functions. Let us highlight that stopping time τ^{k+1} equals χ^k plus some integer, but χ^k may or may be not integer itself. All these stopping times are finite with probability one, and, moreover,

$$1(\tau_0 > \tau^k) P_{X_{\tau^k}}(\tau_0 \leqslant \chi^{k+1}) \geqslant 1(\tau_0 > \tau^k) p(T).$$

Also, almost surely

$$\liminf_{k \to \infty} \tau^k \geqslant \tau_0. \qquad (30)$$

NB: If $\inf_n \lambda_n = 0$, then, in general, we may not claim that $\lim_{k \to \infty} \tau^k = \infty$, but it is not necessary for our aims.

Using strong Markov property at time χ^i (see [9, Section 4]), we obtain by induction,

$$P_{X_0}(\tau_0 > \chi^k) \leqslant (1 - p(T))^k, \quad k \geqslant 1. \qquad (31)$$

5. Denote

$$d^k := \chi^k - \tau^k, \quad \delta^k := \tau^{k+1} - \chi^k.$$

Also by induction it follows from (24) and from the elementary bound by definition

$$d^k \leqslant T,$$

that there exist a constant C such that

$$E_{X_0}(\underbrace{(\chi^{k+1} - \chi^k)}_{=: \delta^k + d^k} | X_{\chi^k}) \leqslant C, \quad \forall k \geqslant 1. \qquad (32)$$

Using the representation

$$\chi^{k+1} = \overbrace{\tau^1 + d^1}^{=\chi^1} + \sum_{i=2}^{k+1}(\delta^{i-1} + d^i),$$

and due to (30), we estimate,

$$\mathsf{E}_{X_0}\tau_0 = \sum_{k=0}^{\infty} \mathsf{E}_{X_0}\tau_0 1(\chi^k < \tau_0 \leqslant \chi^{k+1}) \leqslant \sum_{k=0}^{\infty} \mathsf{E}_{X_0}\chi^{k+1}1(\chi^k < \tau_0)1(\chi^{k+1} \geqslant \tau_0)$$

$$= \sum_{k=0}^{\infty} \mathsf{E}_{X_0}\left(\tau^1 + d^1 + \sum_{i=2}^{k+1}(\delta^{i-1} + d^i)\right)1(\chi^k < \tau_0)1(\chi^{k+1} \geqslant \tau_0). \tag{33}$$

Now we are going to estimate this sum by some geometric type series in a combination with bounds (31) and (32). Some little issue is that we are not able to use Hölder's, or Cauchy – Buniakowskii – Schwarz' inequality here because we only possess a first moment bound for τ^k, while higher moments are not available. This minor obstacle is resolved in the next step of the proof by the following arguments using conditional expectation with respect to suitable sigma-algebras.

6. We have,

$$\mathsf{E}_{X_0}\left(\tau^1 + d^1 + \sum_{i=2}^{k+1}(\delta^{i-1} + \underbrace{d^i}_{\leqslant T})\right)1(\chi^k < \tau_0)1(\chi^{k+1} \geqslant \tau_0)$$

$$= \mathsf{E}_{X_0}\left(\tau^1 + d^1 + \sum_{i=1}^{k}(\delta^{i-1} + \underbrace{d^i}_{\leqslant T})\right)\prod_{j=1}^{k}1(\chi^j < \tau_0)$$

$$+ \mathsf{E}_{X_0}(\delta^k + \underbrace{d^{k+1}}_{\leqslant T})\prod_{j=1}^{k}1(\chi^j < \tau_0).$$

Let us investigate the last term. Since each random variable $1(\chi^j \leqslant \tau_0)$ is $\mathcal{F}_{\chi^k}$-measurable for any $j \leqslant k$, and because, due to the strong Markov property and the bound (31),

$$\mathsf{E}_{X_0}\prod_{j=1}^{k}1(\chi^j < \tau_0) \leqslant (1 - p(T))^k. \tag{34}$$

Moreover, by virtue of the inequality (15) and since by definition all $d^k \leqslant T$, we have for $k \geqslant 1$,

$$\mathsf{E}_{X_0}(\delta^k + d^{k+1})\prod_{j=1}^{k}1(\chi^j < \tau_0) = \mathsf{E}_{X_0}\mathsf{E}_{X_0}\left((\delta^k + d^{k+1})\prod_{j=1}^{k}1(\chi^j < \tau_0)|\mathcal{F}_{\chi^k}\right)$$

$$= \mathsf{E}_{X_0}\prod_{j=1}^{k}1(\chi^j < \tau_0)\overbrace{\mathsf{E}\left((\delta^k + d^{k+1})|\mathcal{F}_{\chi^k}\right)}^{\leqslant C} \leqslant C(1 - p(T))^k$$

with some finite constant C.

Let us inspect the previous term. Using that δ^{k-1} and all random variables $1(\chi^j < \tau_0)$ are $\mathcal{F}_{\tau^k}$-measurable for any $j \leqslant k-1$, we obtain by induction,

$$\mathsf{E}_{X_0}\left(\left(\delta^{k-1} + \underbrace{d^k}_{\leqslant T}\right)\right)\prod_{j=1}^{k}1(\chi^j < \tau_0) \leqslant T\mathsf{E}_{X_0}\prod_{j=1}^{k}1(\chi^j < \tau_0) + \mathsf{E}_{X_0}\delta^{k-1}\prod_{j=1}^{k}1(\chi^j < \tau_0)$$

$$\leqslant T(1-p(T))^k + \mathsf{E}_{X_0}\mathsf{E}_{X_0}\left(\delta^{k-1}\prod_{j=1}^{k}1(\chi^j < \tau_0)|\mathcal{F}_{\tau^k}\right)$$

$$= T(1-p(T))^k + \mathsf{E}_{X_0}\delta^{k-1}\prod_{j=1}^{k-1}1(\chi^j < \tau_0)\underbrace{\mathsf{E}_{X_0}\left(1(\chi^k < \tau_0)|\mathcal{F}_{\tau^k}\right)}_{\leqslant 1-p(T)}$$

$$\leqslant T(1-p(T))^k + (1-p(M))\mathsf{E}_{X_0}\mathsf{E}_{X_0}\left(\delta^{k-1}\prod_{j=1}^{k-1}1(\chi^j < \tau_0)|\mathcal{F}_{\chi^{k-1}}\right)$$

$$= T(1-p(T))^k + (1-p(M))\mathsf{E}_{X_0}\prod_{j=1}^{k-1}1(\chi^j < \tau_0)\underbrace{\mathsf{E}_{X_0}\left(\delta^{k-1}|\mathcal{F}_{\chi^{k-1}}\right)}_{\leqslant C}$$

$$\leqslant T(1-p(T))^k + C(1-p(T))^k = C(1-p(T))^k. \tag{35}$$

Also by induction we find that a similar upper bound with the multiplier $(1-p(T))^k$ holds for each term in the sum in the right hand side of (33), for $2 \leqslant i < k$, and $k \geqslant 3$. Indeed, using the identity $1(\chi^k < \tau_0) = \prod_{j=1}^{k} 1(\chi^j < \tau_0)$ we have for $2 \leqslant i < k$,

$$\mathsf{E}_{X_0}\left(\left(\delta^{i-1} + \underbrace{d^i}_{\leqslant T}\right)\right)\left(\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)\right)1(\chi^k < \tau_0)$$

$$= \mathsf{E}_{X_0}\mathsf{E}_{X_0}\left(1(\chi^k < \tau_0)(\delta^{i-1} + \underbrace{d^i}_{\leqslant T})\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)|\mathcal{F}_{\tau^k}\right)$$

$$\leqslant \mathsf{E}_{X_0}(\delta^{i-1} + T)\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)\ \underbrace{\mathsf{E}_{X_0}\left(1(\chi^k < \tau_0)|\mathcal{F}_{\tau^k}\right)}_{\leqslant 1-p(T)}$$

$$\leqslant (1-p(T))\mathsf{E}_{X_0}(\delta^{i-1} + T)\prod_{j=1}^{k-1} 1(\chi^j < \tau_0).$$

It was used that the random variable $(\delta^{i-1} + d^i)\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)$ is $\mathcal{F}_{\tau^k}$-measurable.

Further, by virtue of the bound (31)

$$(1-p(T))\mathsf{E}_{X_0}(\delta^{i-1} + T)\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)$$

$$= (1-p(T))\mathsf{E}_{X_0}\mathsf{E}_{X_0}\left((\delta^{i-1} + T)\prod_{j=1}^{k-1} 1(\chi^j < \tau_0)|\mathcal{F}_{\tau^i}\right)$$

$$= (1-p(T))\mathsf{E}_{X_0}(\delta^{i-1} + T)\prod_{j=1}^{i-1} 1(\chi^j < \tau_0)\ \underbrace{\mathsf{E}_{X_0}\left(\prod_{j=i}^{k-1} 1(\chi^j < \tau_0)|\mathcal{F}_{\tau^i}\right)}_{\leqslant (1-p(T))^{k-i}}$$

$$\leqslant (1-p(T))^{k-i+1}\mathsf{E}_{X_0}(\delta^{i-1} + T)\prod_{j=1}^{i-1} 1(\chi^j < \tau_0)$$

$$\leqslant (1-p(T))^{k-i+1}C(1-p(T))^{i-1} = C(1-p(T))^k. \tag{36}$$

We used the bound (35) with d^k replaced by its upper bound T (as in the calculus leading to (35)) and with k replaced by i.

The first term is estimated similarly with the only change that instead of the constant C we obtain a multiplier $C(x + n + 1)$ which is a function of the initial data $X_0 = (n, x)$ and which makes the resulting bound non-uniform with respect to the initial data:

$$(1 - p(T))\mathsf{E}_{n,x}(\tau^1 + d^1) \prod_{j=1}^{k-1} 1(\chi^j < \tau_0) \leqslant C(n + x + 1)(1 - p(T))^k. \tag{37}$$

Overall, collecting the bounds (35) – (37), we get,

$$\mathsf{E}_{X_0} \left(\tau^1 + d^1 + \sum_{i=2}^{k+1} (\delta^{i-1} + \underbrace{d^i}_{\leqslant T}) \right) 1(\chi^k < \tau_0) 1(\chi^{k+1} \geqslant \tau_0) \leqslant kCL(X_0)(1 - p(T))^k.$$

Therefore, it follows that

$$\mathsf{E}_{X_0} \tau_0 \leqslant CL(X_0) + \sum_{k=1}^{\infty} kC(1 - p(T))^k \leqslant CL(X_0),$$

with some new constant C, as required.

Existence of the invariant measure for the model and the inequality (8) follow straightforwardly from the established positive recurrence (7) and from the coupling technique in a usual way, see, for example, [18]. Theorem 1 is proved.   □

5. Two examples

Let us provide two examples for a comparison to "local" conditions in terms of the intensity of service if the latter exists.

Example 1. *Assume that there exists the derivative function $F'(s)$ and that the hasard function $h = H'$ is no less than a constant,*

$$h(s) = H'(s) = \frac{F'(s)}{1 - F(s)} \geqslant \mu, \quad s \geqslant 0. \tag{38}$$

The upper bound for $J_{n,x}^1 + J_{n,x}^2$ is the same as in the proof of the theorem:

$$J_{n,x}^1 + J_{n,x}^2 \leqslant 1 + \Lambda.$$

To estimate $J_{n,x}^3$, in the case of either initial value n, or initial second component x, or both are large enough, by lemma 1 we have similarly to (19) and (??),

$$J^3_{n,x}(1) := \mathsf{E}_{n,x} \int_0^1 1(n_t > 0)(1 + x_t)dH(x_t)$$

$$= \mathsf{E}_{n,x} \int_0^{1 \wedge \gamma} \overbrace{1(n_t > 0)}^{=1}(1 + x_t)\mu dt + \mathsf{E}_{n,x} \int_{1 \wedge \gamma}^1 1(n_t > 0)(1 + x_t)\mu dt$$

$$= \mathsf{E}_{n,x} \int_0^{1 \wedge \gamma} (1 + x + t)\mu dt + \mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t > 0) \underbrace{\int_{1 \wedge \gamma}^1 (1 + x_t)\mu dt}_{\geqslant \mu(1 - 1 \wedge \gamma)}$$

$$+ \mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t = 0) \underbrace{\int_{1 \wedge \gamma}^1 1(n_t > 0)(1 + x_t)\mu dt}_{\geqslant 0}$$

$$\geqslant \mathsf{E}_{n,x} \int_0^{1 \wedge \gamma} \underbrace{(1 + x + t)}_{\geqslant 1}\mu dt + \mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t > 0)\mu(1 - 1 \wedge \gamma).$$

So, we get,

$$J^3_{n,x}(1) \geqslant \mu\mathsf{E}_{n,x}(1 \wedge \gamma) + \mu\mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t > 0)(1 - 1 \wedge \gamma)$$

$$= \mu\mathsf{E}_{n,x}(1 \wedge \gamma) + \mu\mathsf{E}_{n,x}(1 - 1 \wedge \gamma) - \mu\mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t = 0)(1 - 1 \wedge \gamma)$$

$$\geqslant \mu - \mu\mathsf{E}_{n,x}1(\inf_{0 \leqslant t \leqslant 1} n_t = 0) \geqslant \mu - \mu\varepsilon,$$

where the latter inequality is due to the lemma 1 and to the choice of the values of M and $x(\varepsilon)$, see (12) and (13).

Therefore, the condition $\mu > 1 + \Lambda$ suffices for the claims of the theorem (assuming all its other conditions are met). This should be compared with the assumption (4). The multiplier 2 in (4) may be regarded as a price for non-local, integral type conditions, see (5) − (6). Note, however, that assumption (38) looks clearly stronger than necessary for the bound obtained. □

Example 2. *Assume that there exists the derivative function $F'(s)$ and that the hasard function $h = H'$ satisfies the condition (compare to [16] where similar to (39) condition was assumed, but affine Lyapunov functions were replaced by polynomial ones with higher degrees; respectively, polynomial moments and convergence rate bounds were established for the recurrence times; the larger is μ, the faster convergence towards the invariant measure hods)*

$$(1 + s)h(s) \geqslant \mu, \quad s \geqslant 0, \tag{39}$$

with a constant μ.

Here the upper bound for $J^1_{n,x} + J^2_{n,x}$ is the same as in the proof of the theorem and as in the previous example:

$$J^1_{n,x} + J^2_{n,x} \leqslant 1 + \Lambda.$$

To estimate $J_{n,x}^3$, in the case if either the initial value n, or the initial second component x (or both) is large enough, we have by lemma 1, similarly to (19) and to the previous example,

$$J_{n,x}^3(1) := \mathsf{E}_{n,x} \int_0^1 1(n_t > 0)(1 + x_t)dH(x_t)$$

$$= \mathsf{E}_{n,x} \int_0^{1 \wedge \gamma} \overbrace{1(n_t > 0)}^{=1} \mu dt + \mathsf{E}_{n,x} \int_{1 \wedge \gamma}^1 1(n_t > 0)\mu dt$$

$$= \mathsf{E}_{n,x} \int_0^{1 \wedge \gamma} \mu dt + \mathsf{E}_{n,x} 1(\inf_{0 \leqslant t \leqslant 1} n_t > 0) \underbrace{\int_{1 \wedge \gamma}^1 \mu dt}_{= \mu(1 - 1 \wedge \gamma)}$$

$$+ \underbrace{\mathsf{E}_{n,x} 1(\inf_{0 \leqslant t \leqslant 1} n_t = 0) \int_{1 \wedge \gamma}^1 1(n_t > 0)\mu dt}_{\geqslant 0}$$

$$\geqslant \mu \mathsf{E}_{n,x}(1 \wedge \gamma) + \mu \mathsf{E}_{n,x} 1(\inf_{0 \leqslant t \leqslant 1} n_t > 0)(1 - 1 \wedge \gamma)$$

$$= \mu \mathsf{E}_{n,x}(1 \wedge \gamma) + \mu \mathsf{E}_{n,x}(1 - 1 \wedge \gamma) - \mu \mathsf{E}_{n,x} 1(\inf_{0 \leqslant t \leqslant 1} n_t = 0)(1 - 1 \wedge \gamma)$$

$$\geqslant \mu - \mu \mathsf{E}_{n,x} 1(\inf_{0 \leqslant t \leqslant 1} n_t = 0) \geqslant \mu - \mu \varepsilon,$$

again due to the definition of M and x(ε), see (12) and (13). Hence, here the same condition $\mu > 1 + \Lambda$ as in the previous example suffices for the claims of the theorem (assuming all other conditions of the theorem are met). Condition (39) is clearly more relaxed than (38), but both assume existence of the intensity of service, which is not required in theorem 1. □

6. Discussion

The result may serve as a sufficient condition for the "steady-state" property of the model $M_n/GI/1$ used as a background in [1,12], et al. It is plausible that the method used in this paper admits extensions to more general models. As it was said in the introduction, there is some moderate hope that it may also be applied to Erlang – Sevastyanov's type systems, which could potentially allow to find sufficient conditions for convergence rates in such systems without assuming existence of intensity of service, thus, generalising the results from [16].

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