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Relative Orderings of Modified Proportional Hazard Rates and Modified Proportional Reversed Hazard Rates Models

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Abstract: In this paper, we identify several relative ordering properties of the modified proportional hazard rate and modified proportional reversed hazard rate models. For this purpose, we use two well-known relative orderings, namely the relative hazard rate ordering and the relative reversed hazard rate ordering. The investigation is to see how a relative ordering between two possible base distributions for the response distributions in these models is preserved when the parameters of the underlying models are changed. We will give some examples to illustrate the results and the conditions under which they are obtained.

Keywords: proportional hazard rates, proportional reversed hazard rates, relative hazard rate order; relative reversed hazard rate order; stochastic order; relative aging

MSC: 62N05, 90B25, 62E10, 65C20

1. Introduction

The parameters of a distribution are typically considered to be real or perhaps vector values. In the literature, families of distributions are considered that are characterized by having a parameter that is itself a distribution function. These families are called semiparametric because they also contain a real parameter. Choosing the parameter that is first a distribution function is one possible way to use a semiparametric model. The underlying distribution is the formal name for this distribution function. In practice, the selection of an underlying distribution leads to the selection of a parametric model, but the selection is limited to families with the structure of the semiparametric model.

The underlying distribution F might already have one or more parameters, in which case a semiparametric family might provide a way to include a new parameter, extending the family from which F originates. One can imagine that the standard families of the Gamma and Weibull distributions are derived from the exponential distribution via semiparametric families that include a second parameter. The Weibull and Gamma families can both be found as special cases of a three-parameter family using the same technique. The study of semiparametric families is therefore advantageous for two reasons: it provides a new understanding of traditional distribution families, and it offers strategies for extending families to make data fitting more flexible (see, e.g., Marshall and Olkin [18]).

Stochastic orderings of random variables have long been a useful tool for making comparisons between probability distributions (see Müller and Stoyan [19], Shaked and Shanthikumar [23], Belzunce et al. [3], and Li and Li [16]). Some researchers used stochastic orders comparing distributions in terms of the magnitude of random variables to perform stochastic comparisons between semiparametric models, including those presented in equations (1), (5), and (9) in Section 2. To this end, we quantify the effect of varying the parameters of the model on the variation of the response variables and, furthermore, the effect of changing the underlying distribution on changing the distribution of the response variables using several known stochastic orders. For example, in the context of the proportional hazard rate (PHR) model for the case where λ is a random variable (frailty), Gupta and Kirmani [10] and subsequently Xu and Li [24] identified some stochastic ordering properties of the model. Considering the proportional reversed hazard rates (PRHR) model, Di Crescenzo [6] made some stochastic comparisons between two candidate distributions of the model that differ in their parameters. Kirmani and Gupta [13] derived some stochastic ordering results for the model proportional odds rates (POR) model.

Recently, however, many researchers have focused on stochastic orders that compare lifetime distributions according to aging behavior, namely the faster aging stochastic orders. One of the key ideas in reliability theory and survival analysis is stochastic aging. It broadly outlines the pattern of aging/degradation of a system over time. Three different notions of aging are presented in the literature: positive aging, negative aging, and no aging. Positive aging implies a stochastically decreasing remaining lifetime of the system, while negative aging implies just the opposite. The system does not mature with time if there is no aging. To study different characteristics of system aging, various aging classes (including increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate on average (IFRA), decreasing failure rate on average (DFRA), increasing likelihood ratio (ILR), and decreasing likelihood ratio (DLR), to name a few) have been presented in the literature based on these three aging principles. The reader can consult Barlow and Proschan [2] and Lai and Xie [15] for further discussion on this topic. In addition to these ideas about aging, relative aging is a useful concept to use when studying system reliability. Relative aging is used to measure how a system changes over time relative to another system.

In real life, there are many situations where we deal with multiple systems of the same type (e.g., TVs from different manufacturers, CPUs from different brands, etc.). In these circumstances, we often encounter the following problem: how to determine whether one system is aging faster than others over time? The idea of relative aging provides a compelling answer to this problem. When dealing with the crossover hazards/medium remaining life phenomena, another component of relative aging proves helpful. Many real-life situations involve this type of circumstance. For example, when Pocock et al. [20] examined survival data on the effects of two different treatments on breast cancer patients and became aware of the phenomenon of crossover hazards. In addition, Champlin et al. [4] described several cases in which the superiority of one treatment over another lasted only for a short period of time. The above considerations suggest that increasing/decreasing hazard ratio models are a viable option in a variety of real-world scenarios. In fact, Kalashnikov and Rachev [12] have developed a concept of relative aging based on the monotonicity of the ratio of two hazard rate functions called relative hazard rate order. This concept is known as faster hazard rate aging. Sengupta and Deshpande [22] presented another idea in a similar way based on the monotonicity of the ratio of two cumulative hazard rate functions. Rezaei et al. [21] proposed a relative order based on the ratio of the reversed hazard rates of two random lifetimes and called it relative reversed hazard rate order.

The aim of this paper is to perform stochastic comparisons between two newly defined semiparametric models, the modified proportional hazard rate model and the modified proportional reversed hazard rate model, corresponding to the relative hazard rate and the relative reversed hazard rate order.

The rest of the paper is organized as follows. In Section 2 we give some advanced preliminary considerations and auxiliary results. In Section 3, we consider the modified proportional hazard rate model for comparison in terms of relative hazard rate order. In Section 4, we consider the modified proportional reversed hazard rate model to give some ordering properties according to the relative reversed hazard rate order. In Section 5, we conclude the paper with a more detailed summary and provide an outlook on possible future studies.

2. Preliminaries

In this section we give some mathematical definitions of the notions that will be utilized in this paper. In the literature, many semiparametric families of distributions have been introduced

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and studied. Among these models some of them find their applicability in the context of lifetime events. The Cox's PHR model is of the important and frequently used such semiparametric family of distributions (see, Cox [5]). For a review on the PHR model we refer the reader to Kumar and Klefsjö [14]. Let us consider the parameter $\lambda > 0$, called the frailty parameter, then the PHR model is defined as

$$\bar{F}(t;\lambda) = \bar{F}^{\lambda}(t),\ t \geq 0, \tag{1}$$

where $\bar{F}(\cdot;\lambda)$ is the survival function (sf) of the response random variable and $\bar{F}(\cdot)$ is the baseline sf. Let X_0 have an absolutely continuous distribution function (cdf) $F(\cdot)$, with probability density function (pdf) $f(\cdot)$. Then, the hazard rate (hr) of X_0, as important reliability quantity in survival analysis, measures the instantaneous risk for failure of a device with lifetime X_0 at a certain age (t, say). The hr of X_0 for all $t \geq 0$ which fulfills $\bar{F}(t) > 0$ is defined as follows:

$$h(t) := -\frac{d}{dt}\ln(\bar{F}(t)) = \frac{f(t)}{\bar{F}(t)}. \tag{2}$$

It is well-known that h characterizes the underlying sf, $\bar{F}$, as follows:

$$\bar{F}(t) = \exp\{-\int_0^t h(x)\,dx\}. \tag{3}$$

Suppose that $h(\cdot;\lambda)$ is the hr function associated with the sf (1), then, it is plainly seen that for every $t \geq 0$ for which $\min(F(\bar{t}), \bar{F}(t;\lambda)) > 0$,

$$h(t;\lambda) = \lambda.h(t). \tag{4}$$

In contrast to the PHR model, the PRHR model was introduced by Gupta et al. [9]. We refer the reader to Gupta and Gupta [11] for further descriptions of the PRHR model. In the PRHR model, a positive parameter, β, called the resilience parameter, is considered. The PRHR model is then defined as

$$F(t;\beta) = F^{\beta}(t),\ t \geq 0, \tag{5}$$

in which $F(\cdot;\beta)$ is the cdf of the response random variable and $F(\cdot)$ is the baseline cdf or the underlying distribution function in the model. The reversed hazard rate (rhr) of X_0, as another reliability quantity, measures the risk for failure of a device (with original lifetime X_0) in the past at a certain time point t at which the device is found to be inactive. The rhr of X_0 for all $t \geq 0$ which $F(t) > 0$ is derived via the following relation:

$$\widetilde{h}(t) := \frac{d}{dt}\ln(F(t)) = \frac{f(t)}{F(t)}. \tag{6}$$

It has been verified that $\widetilde{h}$ characterizes the underlying cdf, F, as below:

$$F(t) = \exp\{-\int_t^{+\infty} \widetilde{h}(x)\,dx\}. \tag{7}$$

Let us now assume that $\widetilde{h}(\cdot;\beta)$ is the rhr function of the distribution with the cdf (5). Then, it is readily realized for all $t \geq 0$ for which $\min\{F(t), F(t;\beta)\} > 0$ that

$$\widetilde{h}(t;\beta) = \beta.\widetilde{h}(t). \tag{8}$$

Another reputable semiparametric family of distributions is the POR model (see, e.g., Marshall and Olkin [17]). This model is defined with cdf

$$F(t;\alpha) = \frac{F(t)}{1-\bar{\alpha}\bar{F}(t)};\ \ t,\alpha \in R^{+}, \bar{\alpha} = 1-\alpha, \tag{9}$$

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In some situations the following model is alternatively utilized:

$$F(t;\alpha) = \frac{\alpha F(x)}{1-\bar{\alpha}F(t)};\ \ t,\alpha \in R^+, \bar{\alpha} = 1-\alpha. \tag{10}$$

The odds rate function of X_0, measures the relative odd of the event $\{X_0 > t\}$ in terms of the event $\{X_0 \leq t\}$ where t is some point of time. The odds rate function of X_0 for all $t \geq 0$ which $F(t) > 0$ defined as follows:

$$OR_0(t) := \frac{\bar{F}(t)}{F(t)}. \tag{11}$$

We assume that $OR(t;\alpha) = \frac{\bar{F}(t;\alpha)}{F(t;\alpha)}$ is the odds rate function of the distribution with the cdf (9). Then, it is easily verified for all $t \geq 0$ for which $\min\{F(t), F(t;\alpha)\} > 0$ that

$$OR(t;\alpha) = \alpha.OR_0(t). \tag{12}$$

Balakrishnan et al. [1], utilized the PHR (resp. PRHR) model as baseline model in (9) (resp. (10)) to propose two new models, referred to as modified proportional hazard rates (MPHR) and modified proportional reversed hazard rates (MPRHR) models.

Suppose that X_0 is a baseline random variable with survival function $\bar{F}$. Let $X_{11},\cdots,X_{1n}$ are independent and identically distributed (i.i.d.) lifetimes of n components of a system with a common distribution function $F(\cdot;\alpha,\lambda)$. Then, $X_{11},\cdots,X_{1n}$ are said to follow the MPHR model with tilt parameter α, modified proportional hazard rate λ and baseline survival function $\bar{F}$ (denoted as $MPHR(\alpha;\lambda;\bar{F})$) if, and only if,

$$F(x;\alpha,\lambda) = \frac{1-\bar{F}^{\lambda}(x)}{1-\bar{\alpha}\bar{F}^{\lambda}(x)};\ \ x,\lambda,\alpha \in R^+, \bar{\alpha} = 1-\alpha. \tag{13}$$

For the case $\alpha = 1$, (13) simply reduces to the PHR model. The MPHR model in (13) includes some well-known distributions such as extended exponential and extended Weibull distributions (Marshall and Olkin [18]), extended Pareto distribution (Ghitany [7]) and extended Lomax distribution (Ghitany et al. [8]).

On the other hand, suppose $X_1,\cdots,X_n$ are i.i.d. lifetimes of n components of a system with a common distribution functions F. Then, $X_1,\cdots,X_n$ are said to follow the MPRHR model with tilt parameter α, modified proportional reversed hazard rate β and baseline distribution function F (denoted as $MPRHR(\alpha;\beta;F)$) if and only if

$$F(x;\alpha,\beta) = \frac{\alpha F^{\beta}(x)}{1-\bar{\alpha}F^{\beta}(x)};\ \ x,\beta,\alpha \in R^+, \bar{\alpha} = 1-\alpha. \tag{14}$$

Note that the PRHR model is a sub-model of (14) when $\alpha = 1$.

We assume that the random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1-F$ and $\bar{G} = 1-G$, density functions f and g, hazard rate functions $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ and reversed hazard rate functions $\widetilde{h}_X = f/F$ and $\widetilde{h}_Y = g/G$, respectively. To compare the magnitude of random variables some notions of stochastic orders are introduced below.

Definition 1. *Suppose that X and Y are two non-negative random variables which denote the lifetime of two systems. The random variable X is then said to be smaller than the random variable Y in the*

(i) usual stochastic order (denoted by $X \preceq_{st} Y$) if,

$$\bar{F}(x) \leq \bar{G}(x) \text{ for all } x \geq 0.$$

(ii) *hazard rate order (denoted by $X \preceq_{hr} Y$) if,*

$$\frac{\bar{G}(x)}{\bar{F}(x)} \text{ is non-decreasing in } x \geq 0,$$

or equivalently, if $h_X(x) \geq h_Y(x)$ for all $x \geq 0$.

(iii) *reversed hazard rate order (denoted by $X \preceq_{rh} Y$) if,*

$$\frac{G(x)}{F(x)} \text{ is non-decreasing in } x > 0$$

or equivalently, if $\widetilde{h}_X(x) \leq \widetilde{h}_Y(x)$ for all $x > 0$.

(iv) *likelihood ratio order (denoted by $X \preceq_{lr} Y$) if,*

$$g(x)/f(x) \text{ is non-decreasing in } x \geq 0.$$

(v) *relative hazard rate order (denoted by $X \preceq_c Y$) if,*

$$\frac{h_X(x)}{h_Y(x)} \text{ is non-decreasing in } x \geq 0$$

.

(vi) *relative reversed hazard rate order (denoted by $X \preceq_b Y$) if,*

$$\frac{\widetilde{h}_X(x)}{\widetilde{h}_Y(x)} \text{ is non-increasing in } x \geq 0.$$

Some stochastic orders in Definition 1 are connected to each other. In this regard, $X \preceq_{lr} Y$ implies $X \preceq_{hr} Y$ and also $X \preceq_{lr} Y$ implies $X \preceq_{rh} Y$. Furthermore, $X \preceq_{hr} Y$ gives $X \preceq_{st} Y$ and also $X \preceq_{rh} Y$ yields $X \preceq_{st} Y$. For further relations and properties of the stochastic orders $\preceq_{lr}, \preceq_{hr}, \preceq_{rh}$ and $\preceq_{st}$ we refer the reader to Shaked and Shanthikumar [23]. For more descriptions of the relative order $\preceq_c$ we refer the reader to Kalashnikov and Rachev [12] and also Sengupta and Deshpande [22]. For further properties of the relative order $\preceq_b$ the reader can see Rezaei et al. [21].

3. Results on relative orderings of MPHR distributions

In this section, we obtain a relative ordering property in the MPHR model according to the relative hazard rate order. We will consider the MPHR model in two settings where two sets of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ which are possibly different are assigned and also possibly different baseline sfs $\bar{F}$ and $\bar{G}$ are taken into account. Finding conditions on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and also conditions on $\bar{F}$ and $\bar{G}$ to establish the preservation of the relative hazard rate ordering property in the MPHR model is the main objective of this section.

Before stating next result we introduce some notation. Let X_0 and Y_0 have pdfs f and g, and sfs $\bar{F}$ and $\bar{G}$, respectively, and, further, $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$. Then, using (13), the sfs of X_1 and Y_1 which are denoted by $\bar{F}(x; \alpha_1, \lambda_1)$ and $\bar{G}(x; \alpha_2, \lambda_2)$, respectively, can be written as follows:

$$\bar{F}(x; \alpha_1, \lambda_1) = \frac{\alpha_1 . \bar{F}^{\lambda_1}(x)}{1 - \bar{\alpha}_1 \bar{F}^{\lambda_1}(x)} \text{ and } \bar{G}(x; \alpha_2, \lambda_2) = \frac{\alpha_2 . \bar{G}^{\lambda_2}(x)}{1 - \bar{\alpha}_2 \bar{G}^{\lambda_2}(x)}. \tag{15}$$

Now, let us denote by $f(\cdot; \alpha_1, \lambda_1)$ and $g(\cdot; \alpha_2, \lambda_2)$ the pdfs of X_1 and Y_1, which can be obtained by taking derivatives of cdfs in (15) as follows:

$$f(x; \alpha_1, \lambda_1) = \frac{\lambda_1 \alpha_1 \bar{F}^{\lambda_1 - 1}(x)}{(1 - \bar{\alpha}_1 \bar{F}^{\lambda_1}(x))^2} f(x) \text{ and } g(x; \alpha_2, \lambda_2) = \frac{\lambda_2 \alpha_2 \bar{G}^{\lambda_2 - 1}(x)}{(1 - \bar{\alpha}_2 \bar{G}^{\lambda_2}(x))^2} g(x). \tag{16}$$

Appealing to (15) together with (16) the hazard rate function of X_1 and the hazard rate function of Y_1 are acquired as:

$$h(x;\alpha_1,\lambda_1)=\frac{h(x)}{\Phi(\bar{F}(x);\alpha_1,\lambda_1)} \text{ and } s(x;\alpha_2,\lambda_2)=\frac{s(x)}{\Phi(\bar{G}(x);\alpha_2,\lambda_2)} \tag{17}$$

where $h(\cdot)$ and $s(\cdot)$ are the hazard rate functions of X_0 and Y_0, respectively, and the function $\Phi(u;\alpha,\lambda)$ is given by

$$\Phi(u;\alpha,\lambda)=\frac{1}{\lambda}(1-\bar{\alpha}u^{\lambda}),\ \ u\in[0,1].$$

We define here two measures of relative hazard rates of Y_0 and X_0 with hazard rate functions $s(\cdot)$ and $h(\cdot)$, respectively. Let us denote two limiting points of hazard rates ratio $\frac{s(t)}{h(t)}$ as follows:

$$\eta_0:=\lim_{t\to 0^+}\frac{s(t)}{h(t)} \text{ and } \eta_1:=\lim_{t\to+\infty}\frac{s(t)}{h(t)}.$$

Theorem 1. *Let X_0 and Y_0 have sfs $\bar{F}$ and $\bar{G}$, respectively. Let $X_1\sim MPHR(\alpha_1;\lambda_1;\bar{F})$ and $Y_1\sim MPHR(\alpha_2;\lambda_2;\bar{G})$, where $\alpha_i\in[0,1]$ and $\lambda_i>0$ for every $i=1,2$. Let $M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0)\geq 0$ be a function of $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)$ and η_0, such that*

$$M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0):=\sup_{u\in[0,1]}\left(\frac{u^{-\lambda_2\eta_0}-\bar{\alpha}_2}{u^{-\lambda_1}-\bar{\alpha}_1}\right).$$

If

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\lambda_2}{\lambda_1}\eta_1\geq M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0)$$

then

$$X_0\preceq_c Y_0\ \Rightarrow\ X_1\preceq_c Y_1. \tag{18}$$

Proof. It suffices to prove that $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ is non-increasing in $t\geq 0$. Since,

$$\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}=\frac{s(t)}{h(t)}.\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}$$

and, by assumption, $\frac{s(t)}{h(t)}$ is non-increasing in $t\geq 0$, thus it is sufficient to to show that $\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}$ is non-increasing in $t\geq 0$, which holds if, and only if,

$$\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right)\leq 0,\ \ \text{for all}\ \ t\geq 0.$$

Denote $\Phi'(u;\alpha,\lambda)=\frac{\partial}{\partial u}\Phi(u;\alpha,\lambda)$. We have:

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\frac{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}\right)&=g(t).\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-f(t).\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\\&=s(t)\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-h(t)\bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\\&\leq h(t).\left(\eta_1.\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}-\bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\right),\end{aligned} \tag{19}$$

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where the last inequality follows from the fact that, for $\alpha_i \in [0,1]$ and $\lambda_i > 0$,

$$\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} \leq 0, \text{ for all } t \geq 0,$$

and that $X_0 \preceq_c Y_0$ yields

$$\frac{s(t)}{h(t)} \geq \lim_{t\to+\infty}\frac{s(t)}{h(t)} = \eta_1,$$

as it implies that $s(t) \geq \eta_1.h(t)$, for all $t \geq 0$. The right hand side of inequality in (19) is negative, if and only if,

$$\eta_1 \geq \frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)} \text{ for all } t \geq 0, \tag{20}$$

in which

$$\begin{aligned}\gamma(u;\alpha,\lambda) &:= \frac{u\Phi'(u;\alpha,\lambda)}{\Phi(u;\alpha,\lambda)}\\ &= \frac{(\alpha-1).\lambda}{u^{-\lambda}-\bar{\alpha}}, \text{ for all } u \in [0,1],\end{aligned}$$

The inequality in (20) is satisfied if

$$\begin{aligned}\eta_1 &\geq \sup_{t\geq 0}\left(\frac{\gamma(\bar{F}(t);\alpha_1,\lambda_1)}{\gamma(\bar{G}(t);\alpha_2,\lambda_2)}\right)\\ &= \sup_{t\geq 0}\left(\frac{\frac{(\alpha_1-1).\lambda_1}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}}{\frac{(\alpha_2-1).\lambda_2}{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}}\right)\\ &= \frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\lambda_1}{\lambda_2}.\sup_{t\geq 0}\left(\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right).\end{aligned} \tag{21}$$

On the other hand, since $X_0 \preceq_c Y_0$ further implies that

$$\frac{s(t)}{h(t)} \leq \lim_{t\to 0^+}\frac{s(t)}{h(t)} = \eta_0,$$

thus $s(t) \leq \eta_0 h(t)$, for all $t \geq 0$. Hence, using (3),

$$\begin{aligned}\bar{G}(t) &= \exp\{-\int_0^t s(x)dx\}\\ &\geq \exp\{-\eta_0\int_0^t h(x)dx\} = \bar{F}^{\eta_0}(t).\end{aligned}$$

So, $\bar{G}^{-\lambda_2}(t) \leq \bar{F}^{-\lambda_2\eta_0}(t)$, for all $t \geq 0$, which further implies that

$$\frac{\bar{G}^{-\lambda_2}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1} \leq \frac{\bar{F}^{-\lambda_2\eta_0}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}, \text{ for all } t \geq 0.$$

Therefore, the inequality in (21) is satisfied if

$$\begin{aligned}\eta_1 &\geq \frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\lambda_1}{\lambda_2}.\sup_{t\geq 0}\left(\frac{\bar{F}^{-\lambda_2\eta_0}(t)-\bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t)-\bar{\alpha}_1}\right)\\ &= \frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\lambda_1}{\lambda_2}.M(\boldsymbol{\alpha},\boldsymbol{\lambda},\eta_0),\end{aligned}$$

or equivalently if

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\lambda_2}{\lambda_1}.\eta_1 \geq M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0).$$

□

In the following example, we show that the result of Theorem 1 is applicable.

Example 1. *Let us write $X \sim W(c,d)$ when X follows Weibull distribution with shape parameter c and scale parameter d, with $c, d > 0$, having sf $\bar{F}_X(t) = \exp(-(dt)^c), t \geq 0$. Suppose that $X_0 \sim W(3,1)$ and $Y_0 \sim W(3,2)$. Assume that $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$ with $\alpha_1 = 0.8, \alpha_2 = 0.1, \lambda_1 = 10$ and $\lambda_2 = 1$ and, further, $\bar{F}(t) = \exp(-t^3)$ and $\bar{G}(t) = \exp(-8t^3)$. We can observe that X_0 and Y_0 have hrs $h(t) = 3t^2$ and $s(t) = 24t^2$. Therefore,*

$$\frac{s(t)}{h(t)} = 8, \ \eta_0 = \eta_1 = 8.$$

Hence, $X_0 \preceq_c Y_0$. We can observe that

$$M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0) := \sup_{u \in [0,1]} \left(\frac{u^{-8} - 0.9}{u^{-10} - 0.2} \right) = 0.56812,$$

and, on the other hand, we can see that

$$\frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\lambda_2}{\lambda_1}\eta_1 = 3.6.$$

Thus, obviously, $\frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\lambda_2}{\lambda_1} > M(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_0)$, and using Theorem 1 we conclude that $X_1 \preceq_c Y_1$. In Figure 1, the graph of $\frac{s(t;\alpha_2,\lambda_2)}{s(t;\alpha_1,\lambda_1)}$ is plotted to exhibit that it is non-increasing in $t \in (0,4)$.

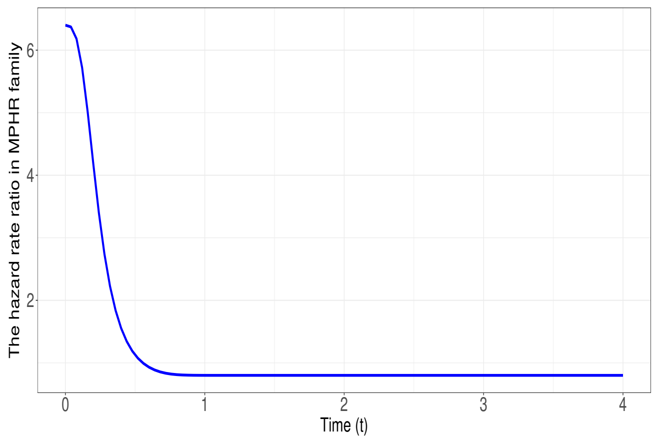

Figure 1. Plot of the hazard rate ratio $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ in Example 1 for $\alpha_1 = 0.8, \alpha_2 = 0.1, \lambda_1 = 10$ and $\lambda_2 = 1$ when $t \in (0,4)$.

Remark 1. *In the context of Theorem 1, the obtained result is immediately followed when $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$. To prove this claim, note that if $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$, then, for all $t \geq 0$ and for every $\lambda_i > 0, i = 1,2$, one has*

$$\bar{G}(t)\frac{\Phi'(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{G}(t);\alpha_2,\lambda_2)} \leq 0, \ \ and \ \ \bar{F}(t)\frac{\Phi'(\bar{F}(t);\alpha_1,\lambda_1)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)} \geq 0.$$

Therefore, the parenthetical statement in right hand side of inequality given in (19) is non-positive. Thus, it is straightforward that if $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$ and $X_0 \preceq_c Y_0$ then $X_1 \preceq_c Y_1$. In this case the additional supremum condition in Theorem 1 could be omitted.

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The following theorem states another setting for the parameters of two MPHR distributions so that the result of Theorem 1 is obtained under a different condition.

Theorem 2. *Let X_0 and Y_0 have sfs $\bar{F}$ and $\bar{G}$, respectively. Let $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$, where $\alpha_i \in (1, +\infty)$ and $\lambda_i > 0$ for every $i = 1, 2$. Let $m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1) \geq 0$ be a function of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and also η_1 such that*

$$m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1) := \inf_{u \in [0,1]} \left(\frac{u^{-\lambda_2 \eta_1} - \bar{\alpha}_2}{u^{-\lambda_1} - \bar{\alpha}_1} \right).$$

If

$$\frac{\alpha_2 - 1}{\alpha_1 - 1} . \frac{\lambda_2}{\lambda_1} \eta_0 \leq m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1)$$

then

$$X_0 \preceq_c Y_0 \Rightarrow X_1 \preceq_c Y_1. \tag{22}$$

Proof. Similarly, as in the proof of Theorem 1, we need to demonstrate that $\frac{\Phi(\bar{F}(t); \alpha_1, \lambda_1)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)}$ is non-increasing in $t \geq 0$, which holds if, and only if,

$$\frac{\partial}{\partial t} \ln \left(\frac{\Phi(\bar{F}(t); \alpha_1, \lambda_1)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)} \right) \leq 0, \quad \text{for all} \quad t \geq 0.$$

Analogously as in the proof of Theorem 1, one has

$$\begin{aligned} \frac{\partial}{\partial t} \ln \left(\frac{\Phi(\bar{F}(t); \alpha_1, \lambda_1)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)} \right) &= s(t) \bar{G}(t) \frac{\Phi'(\bar{G}(t); \alpha_2, \lambda_2)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)} - h(t) \bar{F}(t) \frac{\Phi'(\bar{F}(t); \alpha_1, \lambda_1)}{\Phi(\bar{F}(t); \alpha_1, \lambda_1)} \\ &\leq h(t). \left(\eta_0 . \bar{G}(t) \frac{\Phi'(\bar{G}(t); \alpha_2, \lambda_2)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)} - \bar{F}(t) \frac{\Phi'(\bar{F}(t); \alpha_1, \lambda_1)}{\Phi(\bar{F}(t); \alpha_1, \lambda_1)} \right), \end{aligned} \tag{23}$$

in which the last inequality follows because, for $\alpha_i > 1$ and $\lambda_i > 0$,

$$\bar{G}(t) \frac{\Phi'(\bar{G}(t); \alpha_2, \lambda_2)}{\Phi(\bar{G}(t); \alpha_2, \lambda_2)} \geq 0, \quad \text{for all} \quad t \geq 0,$$

and moreover that $X_0 \preceq_c Y_0$ gives

$$\frac{s(t)}{h(t)} \leq \lim_{t \to 0^+} \frac{s(t)}{h(t)} = \eta_0,$$

which implies that $s(t) \leq \eta_0 . h(t)$, for all $t \geq 0$. The right hand side of the inequality in (23) is negative, if and only if,

$$\eta_0 \leq \frac{\gamma(\bar{F}(t); \alpha_1, \lambda_1)}{\gamma(\bar{G}(t); \alpha_2, \lambda_2)} \quad \text{for all} \quad t \geq 0, \tag{24}$$

in which $\gamma(u; \alpha, \lambda) = \frac{(\alpha - 1) . \lambda}{u^{-\lambda} - \bar{\alpha}}$. The inequality in (24) stands valid if

$$\begin{aligned} \eta_0 &\leq \inf_{t \geq 0} \left(\frac{\gamma(\bar{F}(t); \alpha_1, \lambda_1)}{\gamma(\bar{G}(t); \alpha_2, \lambda_2)} \right) \\ &= \frac{\alpha_1 - 1}{\alpha_2 - 1} . \frac{\lambda_1}{\lambda_2} . \inf \left(\frac{\bar{G}^{-\lambda_2}(t) - \bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t) - \bar{\alpha}_1} \right). \end{aligned} \tag{25}$$

Moreover, since $X_0 \preceq_c Y_0$ provides that

$$\frac{s(t)}{h(t)} \geq \lim_{t \to +\infty} \frac{s(t)}{h(t)} = \eta_1,$$

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so consequently $s(t) \geq \eta_1.h(t)$, for all $t \geq 0$. Therefore, using (3), we obtain

$$\bar{G}(t) = \exp\{-\int_0^t s(x)dx\}$$
$$\leq \exp\{-\eta_1 \int_0^t h(x)dx\} = \bar{F}^{\eta_1}(t).$$

Thus, $\bar{G}^{-\lambda_2}(t) \geq \bar{F}_{-\lambda_2\eta_1}(t)$, which in turn gives

$$\frac{\bar{G}^{-\lambda_2}(t) - \bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t) - \bar{\alpha}_1} \geq \frac{\bar{F}^{-\lambda_2\eta_1}(t) - \bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t) - \bar{\alpha}_1}, \quad \text{for all } \ t \geq 0.$$

Therefore, the inequality in (25) is fulfilled if

$$\eta_0 \leq \frac{\alpha_1 - 1}{\alpha_2 - 1}.\frac{\lambda_1}{\lambda_2}.\inf_{t\geq 0}\left(\frac{\bar{F}^{-\lambda_2\eta_1}(t) - \bar{\alpha}_2}{\bar{F}^{-\lambda_1}(t) - \bar{\alpha}_1}\right)$$
$$= \frac{\alpha_1 - 1}{\alpha_2 - 1}.\frac{\lambda_1}{\lambda_2}.m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1),$$

which holds if, and only if,

$$\frac{\alpha_2 - 1}{\alpha_1 - 1}.\frac{\lambda_2}{\lambda_1}.\eta_0 \leq m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1).$$

□

The following example provides a situation where the result of Theorem 2 is applicable.

Example 2. *Suppose that X_0 follows gamma distribution with sf $\bar{F}(t) = (1+3t)\exp(-3t), t \geq 0$ and Y_0 has sf $\bar{G}(t) = (1+3t)^2\exp(-6t), t \geq 0$. It is easily seen that the hrs of X_0 and Y_0 are $h(t) = \frac{9t}{1+3t}$ and $s(t) = \frac{18t}{1+3t}$, respectively. Therefore,*

$$\eta_0 = \eta_1 = \frac{s(t)}{h(t)} = 2.$$

Now, since $\frac{s(t)}{h(t)}$ is non-increasing in t, thus $X_0 \preceq_c Y_0$. We assume that $X_1 \sim MPHR(\alpha_1; \lambda_1; \bar{F})$ and $Y_1 \sim MPHR(\alpha_2; \lambda_2; \bar{G})$ with $\alpha_1 = 10, \alpha_2 = 3, \lambda_1 = 4$ and $\lambda_2 = 2$. It is observable that

$$m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1) := \inf_{u\in[0,1]}\left(\frac{u^{-4}+2}{u^{-4}+9}\right) = 0.3,$$

and, in parallel, it is seen that

$$\frac{\alpha_2 - 1}{\alpha_1 - 1}.\frac{\lambda_2}{\lambda_1}\eta_0 = \frac{2}{9}.$$

Therefore, clearly, $\frac{\alpha_2-1}{\alpha_1-1}.\frac{\lambda_2}{\lambda_1}\eta_0 < m(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \eta_1)$, and thus an application of Theorem 2 concludes that $X_1 \preceq_c Y_1$. In Figure 2, the graph of $\frac{s(t;\alpha_2,\lambda_2)}{s(t;\alpha_1,\lambda_1)}$ is plotted to indicate that this ratio is non-increasing in $t \in (0,4)$.

4. Results on relative orderings of MPRHR distributions

In this section, we investigate the relative reversed hazard rate ordering property in two MPRHR models with possibly different sets of parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and also under possibly different baseline distributions F and G.

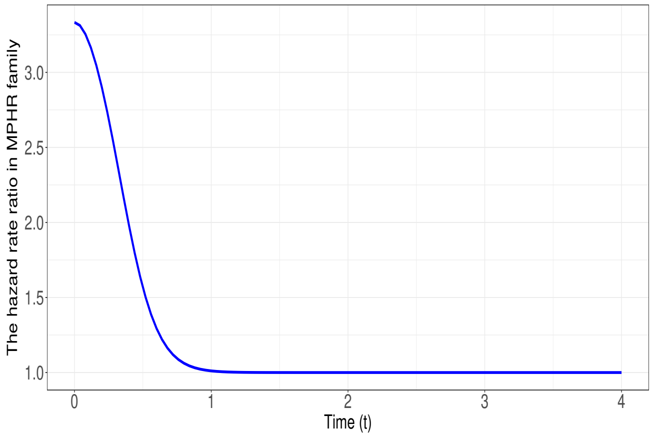

Figure 2. Plot of the hazard rate ratio $\frac{s(t;\alpha_2,\lambda_2)}{h(t;\alpha_1,\lambda_1)}$ in Example 2 for $\alpha_1 = 10, \alpha_2 = 3, \lambda_1 = 4$ and $\lambda_2 = 2$ when $t \in (0,4)$.

We start with introducing some notation. Let X_0 and Y_0 have pdfs f and g, with underlying cdfs F and G, respectively. Let us assume that $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$ and denote by $F^\star(\cdot;\alpha_1,\beta_1)$ and $G^\star(\cdot;\alpha_2,\beta_2)$. Then, using (14), we have

$$F^\star(x;\alpha_1,\beta_1) = \frac{\alpha_1.F^{\beta_1}(x)}{1-\bar{\alpha}_1 F^{\beta_1}(x)} \text{ and } G^\star(x;\alpha_2,\beta_2) = \frac{\alpha_2.G^{\beta_2}(x)}{1-\bar{\alpha}_2 G^{\beta_2}(x)}. \quad (26)$$

Using (26), the pdfs of $X_1^\star$ and $Y_1^\star$ (signified by $f^\star(\cdot;\alpha_1,\beta_1)$ and $g^\star(\cdot;\alpha_2,\beta_2)$) are acquired as below:

$$f^\star(x;\alpha_1,\beta_1) = \frac{\beta_1.\alpha_1.F^{\beta_1-1}(x)}{(1-\bar{\alpha}_1 F^{\beta_1}(x))^2} f(x) \text{ and } g^\star(x;\alpha_2,\beta_2) = \frac{\beta_2.\alpha_2.G^{\beta_2-1}(x)}{(1-\bar{\alpha}_2 G^{\beta_2}(x))^2} g(x). \quad (27)$$

By dividing the pdfs in (27) into the cdfs given in (26), the reversed hazard rate function of X_1 and the reversed hazard rate function of Y_1 are derived as follows:

$$\widetilde{h}(x;\alpha_1,\beta_1) = \frac{\widetilde{h}(x)}{\Psi(F(x);\alpha_1,\beta_1)} \text{ and } \widetilde{s}(x;\alpha_2,\beta_2) = \frac{\widetilde{s}(x)}{\Psi(G(x);\alpha_2,\beta_2)} \quad (28)$$

where $\widetilde{h}(\cdot)$ and $\widetilde{s}(\cdot)$ are the reversed hazard rate functions of X_0 and Y_0, respectively, and further the function $\Psi(u;\alpha,\beta)$ is defined as

$$\Psi(u;\alpha,\beta) = \frac{1}{\beta}(1-\bar{\alpha}u^\beta), \quad u \in [0,1].$$

Now, let us define two measures of relative reversed hazard rates of Y_0 and X_0 having reverenced hazard rate functions $\widetilde{s}(\cdot)$ and $\widetilde{h}(\cdot)$, respectively. The limiting points of reversed hazard rates ratio $\frac{\widetilde{h}(t)}{\widetilde{s}(t)}$ as follows:

$$\eta_0^\star := \lim_{t\to 0^+} \frac{\widetilde{h}(t)}{\widetilde{s}(t)} \text{ and } \eta_1^\star := \lim_{t\to+\infty} \frac{\widetilde{h}(t)}{\widetilde{s}(t)}.$$

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Theorem 3. *Let X_0 and Y_0 have cdfs F and G, respectively. Let $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$, where $\alpha_i \in [0,1]$ and $\beta_i > 0$ for every $i = 1,2$. Suppose that $m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$ which is a non-negative function of $(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$ is defined as*

$$m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star) := \inf_{u\in[0,1]}\left(\frac{u^{-\frac{\beta_1}{\eta_1^\star}} - \bar{\alpha}_1}{u^{-\beta_2} - \bar{\alpha}_2}\right).$$

If

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\beta_1}{\beta_2}.\eta_0^\star \le m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star)$$

then

$$X_0 \preceq_b Y_0 \;\Rightarrow\; X_1^\star \preceq_b Y_1^\star. \tag{29}$$

Proof. To prove (29), it is sufficient to establish that $\frac{\tilde{s}(t;\alpha_2,\beta_2)}{\tilde{h}(t;\alpha_1,\beta_1)}$ is non-decreasing in $t>0$. Following the equations (28), one has:

$$\frac{\tilde{s}(t;\alpha_2,\beta_2)}{\tilde{h}(t;\alpha_1,\beta_1)} = \frac{\tilde{s}(t)}{\tilde{h}(t)}.\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}$$

and, due to assumption, $\frac{\tilde{s}(t)}{\tilde{h}(t)}$ is non-decreasing in $t>0$, thus it is enough to prove that $\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}$ is non-decreasing in $t>0$. The latter statement is valid if, and only if,

$$\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) \ge 0, \quad \text{for all} \quad t>0.$$

We use the notation $\Psi'(u;\alpha,\beta) := \frac{\partial}{\partial u}\Psi(u;\alpha,\beta)$. We get

$$\begin{aligned}
\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) &= f(t).\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - g(t).\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)} \\
&= \tilde{h}(t)F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - \tilde{s}(t)G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)} \\
&\ge \tilde{s}(t).\left(\eta_0^\star.F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\right),
\end{aligned} \tag{30}$$

in which the last inequality is due to the fact that, for $\alpha_i \in [0,1]$ and $\beta_i > 0$ whenever $i=1,2$,

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} \le 0, \quad \text{for all} \quad t>0,$$

and, further, that $X_0 \preceq_b Y_0$ provides that

$$\frac{\tilde{h}(t)}{\tilde{s}(t)} \le \lim_{t\to 0^+}\frac{\tilde{h}(t)}{\tilde{s}(t)} = \eta_0^+,$$

which further implies that $\tilde{h}(t) \le \eta_0^\star.\tilde{s}(t)$, for all $t>0$. Note that the right hand side in (30) is non-negative, if and only if,

$$\eta_0^\star \le \frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)} \quad \text{for all} \quad t>0, \tag{31}$$

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where the function $\gamma^\star(\cdot;\alpha,\beta)$ is defined as below:

$$\gamma^\star(u;\alpha,\beta) := \frac{u\Psi'(u;\alpha,\beta)}{\Psi(u;\alpha,\beta)} = \frac{(\alpha-1).\beta}{u^{-\beta}-\bar{\alpha}}, \quad \text{for all } \ u \in [0,1],$$

Now, it is sufficient to observe that the inequality in (31) is fulfilled if

$$\eta_0^\star \leq \inf_{t\geq 0}\left(\frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}\right) = \inf_{t\geq 0}\left(\frac{\frac{(\alpha_2-1).\beta_2}{G^{-\beta_2}(t)-\bar{\alpha}_2}}{\frac{(\alpha_1-1).\beta_1}{F^{-\beta_1}(t)-\bar{\alpha}_1}}\right) = \frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\beta_2}{\beta_1}.\inf_{t\geq 0}\left(\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right). \tag{32}$$

Note that, since $X_0 \preceq_b Y_0$ yields

$$\frac{\widetilde{s}(t)}{\widetilde{h}(t)} \leq \lim_{t\to+\infty}\frac{\widetilde{s}(t)}{\widetilde{h}(t)} = \eta_1^\star,$$

hence, $\widetilde{s}(t) \leq \eta_1^\star.\widetilde{h}(t)$, for all $t > 0$. Consequently, for all $t > 0$, using the characterization relation (7) one gets:

$$F(t) = \exp\{-\int_t^{+\infty}\widetilde{h}(x)dx\} \leq \exp\{-\frac{1}{\eta_1^\star}.\int_t^{+\infty}\widetilde{s}(x)dx\} = G^{\frac{1}{\eta_1^\star}}(t).$$

Thus, $F^{-\beta_1}(t) \geq G^{-\frac{\beta_1}{\eta_1^\star}}(t)$, for all $t > 0$, which leads to

$$\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2} \geq \frac{G^{-\frac{\beta_1}{\eta_1^\star}}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}, \quad \text{for all } \ t > 0.$$

As a result, the inequality in (32) stands valid if

$$\eta_0^\star \leq \frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\beta_2}{\beta_1}.\inf_{t\geq 0}\left(\frac{G^{-\frac{\beta_1}{\eta_1^\star}}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right) = \frac{\bar{\alpha}_2}{\bar{\alpha}_1}.\frac{\beta_2}{\beta_1}.m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star),$$

or, equivalently if

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\beta_1}{\beta_2}.\eta_0^\star \leq m^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_1^\star).$$

□

Example 3. *Let us assume $X \sim IW(c,d)$ whenever X has Inverse Weibull distribution with shape parameter c and scale parameter d, where $c > 0$ and also $d > 0$. Then, X has cdf $F_X(t) = \exp(-\left(\frac{d}{t}\right)^c)$ for $t > 0$. We assume that $X_0 \sim IW(2,1)$ and $Y_0 \sim IW(2,3)$. Further, we suppose that $X_1^* \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^* \sim MPRHR(\alpha_2;\beta_2;G)$ with $\alpha_1 = 0.25, \alpha_2 = 0.5, \beta_1 = 2$ and $\beta_2 = 18$ so that $F(t) = \exp(-\left(\frac{3}{t}\right)^2)$ is the*

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cdf of X_0 and $G(t) = \exp(-\left(\frac{1}{t}\right)^2)$. It can be readily show that X_0 and Y_0 have rhrs $\widetilde{h}(t) = \frac{2}{t^3}$ and $\widetilde{s}(t) = \frac{18}{t^3}$, respectively. Thus,

$$\frac{\widetilde{h}(t)}{\widetilde{s}(t)} = \frac{1}{9}, \ \eta_0^\star = \eta_1^\star = \frac{1}{9}.$$

Consequently, $X_0 \preceq_b Y_0$. On can easily check that

$$m^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_1^\star) := \inf_{u\in[0,1]} \left(\frac{u^{-18} - 0.75}{u^{-18} - 0.5}\right) = 0.5,$$

and, simultaneously, one has

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\beta_1}{\beta_2}\eta_0^* \simeq 0.00823.$$

Therefore, one realizes that $\frac{\bar{\alpha}_1}{\bar{\alpha}_2}.\frac{\beta_1}{\beta_2}\eta_0^* < m^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_1^\star)$, and using Theorem 3 we deduce that $X_1^\star \preceq_b Y_1^\star$. In Figure 3, the graph of $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ is exhibited to indicate that it is non-decreasing in $t \in (0,4)$.

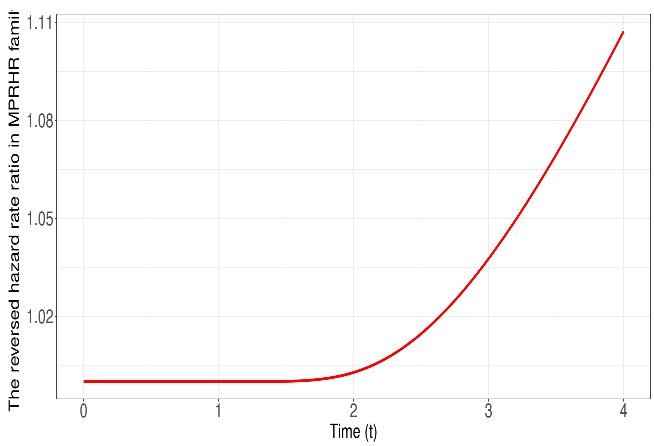

Figure 3. Plot of the reversed hazard rate ratio $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ in Example 3 for $\alpha_1 = 0.25, \alpha_2 = 0.5, \beta_1 = 2$ and $\beta_2 = 18$ when $t \in (0,4)$.

Remark 2. *In the setting of Theorem 3, the derived result can be acquired when $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$. To verify this claim, one needs to observe that if $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$, then, for all $t > 0$ and for $\beta_i > 0, i = 1,2$, we have*

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} \geq 0 \ \text{ and } \ G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)} \leq 0.$$

Thus, the parenthetical statement in right hand side of inequality given in (30) is clearly non-negative. Hence, it is not hard to see that if $\alpha_2 \leq 1$ and $\alpha_1 \geq 1$ and $X_0 \preceq_b Y_0$ then $X_1^* \preceq_b Y_1^*$. In this setting the additional infimum condition in Theorem 3 can be removed.

In the next theorem, we obtain the result of Theorem 3 under different conditions.

Theorem 4. *Let X_0 and Y_0 follow cdfs F and G, respectively. Suppose that $X_1^\star \sim MPRHR(\alpha_1;\beta_1;F)$ and $Y_1^\star \sim MPRHR(\alpha_2;\beta_2;G)$, where $\alpha_i \in (1,+\infty)$ and $\beta_i > 0$ for every $i = 1,2$. Consider $M^\star(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^\star)$ as a non-negative function of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^\star)$ defined as*

$$M^\star(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^\star) := \sup_{u\in[0,1]} \left(\frac{u^{-\beta_1.\eta_0^\star} - \bar{\alpha}_1}{u^{-\beta_2} - \bar{\alpha}_2}\right).$$

If

$$\frac{\alpha_1-1}{\alpha_2-1}.\frac{\beta_1}{\beta_2}.\eta_1^\star \geq M^\star(\boldsymbol{\alpha},\boldsymbol{\beta},\eta_0^\star)$$

then

$$X_0 \preceq_b Y_0 \;\Rightarrow\; X_1^\star \preceq_b Y_1^\star. \tag{33}$$

Proof. In order to verify the implication in (33), as in the proof of Theorem 3, it suffices to show that

$$\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) \geq 0, \text{ for all } t>0.$$

Analogously as in the proof of Theorem 3, we can get

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\frac{\Psi(F(t);\alpha_1,\beta_1)}{\Psi(G(t);\alpha_2,\beta_2)}\right) &= \widetilde{h}(t)F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - \widetilde{s}(t)G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\\ &\geq \widetilde{s}(t).\left(\eta_1^\star.F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} - G(t)\frac{\Psi'(G(t);\alpha_2,\beta_2)}{\Psi(G(t);\alpha_2,\beta_2)}\right),\end{aligned} \tag{34}$$

where the last inequality is due to the fact that, for $\alpha_i>1$ and $\beta_i>0$ for every $i=1,2$,

$$F(t)\frac{\Psi'(F(t);\alpha_1,\beta_1)}{\Psi(F(t);\alpha_1,\beta_1)} \geq 0, \text{ for all } t>0,$$

and moreover because $X_0 \preceq_b Y_0$, thus

$$\frac{\widetilde{h}(t)}{\widetilde{s}(t)} \geq \lim_{t\to+\infty}\frac{\widetilde{h}(t)}{\widetilde{s}(t)} = \eta_1^+,$$

from which one obtains $\widetilde{h}(t) \geq \eta_1^\star.\widetilde{s}(t)$, for all $t>0$. Now, one can see that the right hand side in (34) is non-negative, if and only if,

$$\eta_1^\star \geq \frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}, \text{ for all } t>0, \tag{35}$$

where the function $\gamma^*(\cdot;\alpha,\beta)$ is as defined in the proof on Theorem 3. It is now enough to see that the inequality in (35) is satisfied when

$$\begin{aligned}\eta_1^\star &\geq \sup_{t\geq 0}\left(\frac{\gamma^\star(G(t);\alpha_2,\beta_2)}{\gamma^\star(F(t);\alpha_1,\beta_1)}\right)\\ &= \frac{\alpha_2-1}{\alpha_1-1}.\frac{\beta_2}{\beta_1}.\sup_{t\geq 0}\left(\frac{F^{-\beta_1}(t)-\bar{\alpha}_1}{G^{-\beta_2}(t)-\bar{\alpha}_2}\right).\end{aligned} \tag{36}$$

Now observe that $X_0 \preceq_b Y_0$ gives

$$\frac{\widetilde{s}(t)}{\widetilde{h}(t)} \geq \lim_{t\to 0^+}\frac{\widetilde{s}(t)}{\widetilde{h}(t)} = \frac{1}{\eta_0^\star}.$$

Therefore, $\widetilde{s}(t) \geq (\eta_0^\star)^{-1}.\widetilde{h}(t)$, for all $t>0$. Hence, for all $t>0$, by appealing to the relationship (7) we can write:

$$\begin{aligned}F(t) &= \exp\{-\int_t^{+\infty}\widetilde{h}(x)dx\}\\ &\geq \exp\{-\eta_0^\star.\int_t^{+\infty}\widetilde{s}(x)dx\} = G^{\eta_0^\star}(t).\end{aligned}$$

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As a result, $F^{-\beta_1}(t) \leq G^{-\eta_0^\star \beta_1}(t)$, for all $t > 0$, providing that

$$\frac{F^{-\beta_1}(t) - \bar{\alpha}_1}{G^{-\beta_2}(t) - \bar{\alpha}_2} \leq \frac{G^{-\eta_0^\star \cdot \beta_1}(t) - \bar{\alpha}_1}{G^{-\beta_2}(t) - \bar{\alpha}_2}, \quad \text{for all} \quad t > 0.$$

The inequality in (36) is, therefore, fulfilled if

$$\begin{aligned} \eta_1^\star &\geq \frac{\alpha_2 - 1}{\alpha_1 - 1} \cdot \frac{\beta_2}{\beta_1} \cdot \sup_{t \geq 0} \left(\frac{G^{-\eta_0^\star \beta_1}(t) - \bar{\alpha}_1}{G^{-\beta_2}(t) - \bar{\alpha}_2} \right) \\ &= \frac{\alpha_2 - 1}{\alpha_1 - 1} \cdot \frac{\beta_2}{\beta_1} . M^\star(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^\star), \end{aligned}$$

or, equivalently, when

$$\frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \frac{\beta_1}{\beta_2} . \eta_1^\star \geq M^\star(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^\star).$$

□

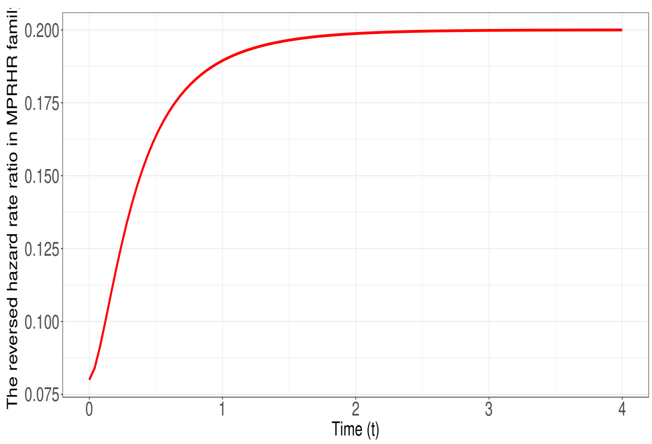

Figure 4. Plot of the reversed hazard rate ratio $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$ in Example 4 for $\alpha_1 = 5, \alpha_2 = 2, \beta_1 = 5, \beta_2 = 2$ and $\theta = 2$ when $t \in (0, 4)$.

Example 4. *Let X_0 have cdf $F(t) = (1 - \exp(-\theta.t))^{\frac{1}{3}}, t \geq 0$ and let Y_0 have exponential distribution with cdf $G(t) = 1 - \exp(-\theta.t), t \geq 0$ where $\theta > 0$ is a common parameter in F and G. Note that $\widetilde{h}(t) = \frac{1}{3}.\widetilde{s}(t)$, for all $t > 0$ where $\widetilde{h}$ is the rhr of X_0 and $\widetilde{s}$ is the rhr of Y_0, respectively. Hence, $\eta_0^\star = \eta_1^* = \frac{1}{3}$, and also clearly, $X_0 \preceq_b Y_0$. Suppose that $X_1^* \sim MPRHR(\alpha_1, \beta_1, F)$ and $Y_1^* \sim MPRHR(\alpha_2, \beta_2, G)$ such that $\alpha_1 = 5, \alpha_2 = 2, \beta_1 = 5$ and $\beta_2 = 2$. In view of the notations and definitions in Theorem 4, we have*

$$M^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^*) := \sup_{u \in [0,1]} \left(\frac{u^{-\frac{5}{3}} + 4}{u^{-2} + 1} \right) = 2.5,$$

and on the other hand, one has

$$\frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \frac{\beta_1}{\beta_2} . \eta_1^* = \frac{10}{3}.$$

So, it is obvious that $\frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \frac{\beta_1}{\beta_2} . \eta_1^* > M^*(\boldsymbol{\alpha}, \boldsymbol{\beta}, \eta_0^*)$. Therefore, Theorem 4 is applicable which provides that $X_1^* \preceq_b Y_1^*$. In Figure 4, the curve of $\frac{\widetilde{s}(t;\alpha_2,\beta_2)}{\widetilde{h}(t;\alpha_1,\beta_1)}$, when $\theta = 2$, is plotted to verify that it is non-decreasing in $t \in (0, 4)$.

5. Concluding Remarks

In this paper, we have examined two recently proposed semiparametric models, namely the MPHR model and the MPRHR model. As shown by Balakrishnan et al. [1], these models include as special cases three important models in the literature, namely the proportional hazard rate model, the proportional reversed hazard rate model, and the proportional odds ratio model. Because these three models have found many applications in the literature so far and because they are available to the two newly defined semiparametric models, it deserves an analytical study of the latter models because they cover and generalize the previous studies. The study of stochastic orderings for model comparisons has been done in the literature in various contexts, including reliability theory, survival analysis, actuarial analysis, risk theory, biostatistics, and many other areas. Stochastic orderings are very useful potential tools for model analysis. For example, stochastic orders are very useful for detecting underestimation and overestimation problems in models. Stochastic orderings are usually recognized as tools for making inferences about models without data. The ordering properties of probability distributions reveal other aspects of the distribution or a family of distributions that can be used for various purposes.

The study conducted in this paper addresses situations in which there is a relative ordering property between two candidates from the MPHR family and, moreover, two candidates from the MPRHR family of semiparametric distributions. In general, the base distributions were assumed to be unknown but to satisfy a relative ordering property according to either the relative hazard rate order ($\preceq_c$) or the relative reverse hazard rate order ($\preceq_b$). It was assumed that the external parameters of the candidate models were generally different. Sufficient conditions were established for the conservation of the relative hazard rate order in the MPHR model and also for the conservation of the relative reverse hazard rate order in the MPRHR model. In the literature, for the preservation of the stochastic order in some scenarios, some stochastic orders are set as assumptions, which is a very strong condition. However, the conditions we found and presented in our work involve comparisons between two numbers, one of which is the supremum or infimum of a function and the other a function of the parameters of the models. With some examples we have shown that even very well known standard statistical distributions, such as Weibull, Gamma or reversed Weibull distributions, can be used as the basic distribution in the MPHR and MPRHR model.

In the future study we shall consider stochastic comparisons in the MPHR and the MPRHR models according to other stochastic orders such as likelihood ratio order ($\preceq_{lr}$), hazard rate order ($\preceq_{hr}$), reversed hazard rate order ($\preceq_{rh}$) and the usual stochastic order ($\preceq_{st}$). In the context of MPHR model, in view of (16), when X_1 and Y_1 follows pdfs $f(x;\alpha_1,\lambda_1)$ and $g(x;\alpha_2,\lambda_2)$, respectively, then $X_0 \preceq_{lr} Y_0$ implies $X_1 \preceq_{lr} Y_1$ if

$$\frac{\Phi_1(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi_1(\bar{F}(t);\alpha_1,\lambda_1)} \quad \text{is non-decreasing in} \quad t \geq 0,$$

where $\Phi_1(u;\alpha,\lambda) := \frac{\lambda.\alpha.u^{\lambda-1}}{(1-\bar{\alpha}.u^{\lambda})^2}$. In addition, in the context of the MPHR model, when X_1 and Y_1 follow sfs $\bar{F}(x;\alpha_1,\lambda_1)$ and $\bar{G}(x;\alpha_2,\lambda_2)$, respectively, as given in (15), then $X_0 \preceq_{st} Y_0$ implies $X_1 \preceq_{st} Y_1$ if

$$\Phi_2(u;\alpha_1,\lambda_1) \leq \Phi_2(u;\alpha_2,\lambda_2), \quad \text{for all} \quad u \in [0,1],$$

where $\Phi_2(u;\alpha,\lambda) = \frac{\alpha.u^{\lambda}}{1-\bar{\alpha}.u^{\lambda}}$. In parallel, when the MPHR model is under consideration, as X_1 and Y_1 have hrs $h(x;\alpha_1,\lambda_1)$ and $s(x;\alpha_2,\lambda_2)$, respectively, as formulated in (17), then $X_0 \preceq_{hr} Y_0$ yields $X_1 \preceq_{hr} Y_1$ if

$$\inf_{t\geq 0}\left(\frac{\Phi(\bar{G}(t);\alpha_2,\lambda_2)}{\Phi(\bar{F}(t);\alpha_1,\lambda_1)}\right) \geq 0,$$

where Φ is defined as before after (17). On the other hand, concerning the MPRHR model, by appealing to (27), and assumign that X_1^* and Y_1^* have pdfs $f^\star(x;\alpha_1,\beta_1)$ and $g^*(x;\alpha_2,\beta_2)$, respectively, then $X_0 \preceq_{lr} Y_0$ implies $X_1^* \preceq_{lr} Y_1^*$ if

$$\frac{\Psi_1(G(t);\alpha_2,\beta_2)}{\Psi_1(F(t);\alpha_1,\beta_1)} \text{ is non-decreasing in } \; t \geq 0,$$

where $\Psi_1(u;\alpha,\beta) := \frac{\beta.\alpha.u^{\beta-1}}{(1-\bar{\alpha}.u^\beta)^2}$. Moreover, by considering the MPRHR model, as X_1^* and Y_1^* follow cdfs $F^*(x;\alpha_1,\beta_1)$ and $G^*(x;\alpha_2,\beta_2)$, respectively, as provided in (26), then $X_0 \preceq_{st} Y_0$ implies $X_1^* \preceq_{st} Y_1^*$ if

$$\Psi_2(u;\alpha_1,\beta_1) \geq \Psi_2(u;\alpha_2,\beta_2), \;\text{ for all }\; u \in [0,1],$$

where $\Psi_2(u;\alpha,\beta) = \frac{\alpha.u^\beta}{1-\bar{\alpha}.u^\beta}$. Furthermore, when the MPRHR model is regarded, so that X_1^* and Y_1^* have rhrs $\widetilde{h}(x;\alpha_1,\beta_1)$ and $\widetilde{s}(x;\alpha_2,\beta_2)$, respectively, as written in (28), then $X_0 \preceq_{rh} Y_0$ yields $X_1^* \preceq_{rh} Y_1^*$ if

$$\sup_{t\geq 0}\left(\frac{\Psi(G(t);\alpha_2,\beta_2)}{\Psi(F(t);\alpha_1,\beta_1)}\right) \geq 0,$$

in which Ψ is defined earlier after the equations (28). The analogous study can also be carried out in the context of other stochastic orders such as dispersive order, star order, and super-additive order.

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