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Article

The Structural Properties of $(2,6)$-Fullerenes †

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Abstract: A $(2,6)$-fullerene F is a 2-connected cubic planar graph whose faces are only 2-length and 6-length. Furthermore, it consists of exactly three 2-length faces by Euler's formula. The $(2,6)$-fullerene comes from Došlić's $(k,6)$-fullerene, a 2-connected 3-regular plane graph with only k-length faces and hexagons. Došlić showed that the $(k,6)$-fullerenes only exist for $k = 2, 3, 4$ or 5, and some of the structural properties of $(k,6)$-fullerene for $k = 3, 4$ or 5 are studied. In this paper, we study the properties of $(2,6)$-fullerene. We obtain that the edge-connectivity of $(2,6)$-fullerenes is 2. Every $(2,6)$-fullerene is 1-extendable, but not 2-extendable (F is called *n-extendable* ($|V(F)| \geq 2n+2$) if any matching of n edges is contained in a perfect matching of F.). F is said to be *k-resonant* ($k \geq 1$) if the deleting of any i ($0 \leq i \leq k$) disjoint even faces of F results in a graph with at least one perfect matching. We have that every $(2,6)$-fullerene is 1-resonant. An edge set S of F is called an anti-Kekulé set if $F - S$ is connected and has no perfect matchings, where $F - S$ denotes the subgraph obtained by deleting all edges in S from F. The anti-Kekulé number of F, denoted by $ak(F)$, is the cardinality of a smallest anti-Kekulé set of F. We have that every $(2,6)$-fullerene F with $|V(F)| > 6$ has anti-Kekulé number 4. Further we mainly prove that there exists a $(2,6)$-fullerene F having f_F hexagonal faces, where f_F is related to the two parameters n, m.

Keywords: $(2,6)$-fullerene; edge-connectivity; anti-Kekulé number; resonance

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1. Introduction

A $(k,6)$-fullerene is a 2-connected cubic planar graph whose faces are only k-length and 6-length. Došlić showed that all $(k,6)$-fullerenes only exist for $k = 2, 3, 4$ or 5 and are 1-extendable [1]. A $(5,6)$-fullerene is the usual fullerene as the molecular graph of a sphere carbon fullerene. A $(4,6)$-fullerene is the molecular graph of a boron-nitrogen fullerene. The structural properties, such as connectivity, extendability, resonance, anti-Kekulé number, are very useful for studying the number of perfect matchings in a graph[2,3]. And the number of perfect matchings is closely related to the stability of molecular graphs[4–8]. Therefore, many articles have studied the structural properties of graphs in both mathematics and chemistry [9–11]. Fullerene graphs are bicritical, cyclically 5-edge-connected, 2-extendable and 1-resonant [12–15]; Boron-nitrogen fullerene graphs are bipartite, 3-connected, 1-extendable, 2-resonant, and have the forcing number at least two [16,17]; A $(3,6)$-fullerene is 1-extendable, 1-resonant and has the connectivity 2 or 3 [18,19]. This paper is mainly concerned with the structural properties of $(2,6)$-fullerenes.

A $(2,6)$-fullerene F is a cubic planar graph such that every face is either 2-length or 6-length. A graph with two vertices and n parallel edges joining them is denoted by $n \times K_2$. The smallest $(2,6)$-fullerene is $3 \times K_2$. A *plane graph* is a graph that can be embedded in the plane such that its edges intersect only at their ends. Any such embedding divides the plane into connected regions called *faces*. Two different faces f_1, f_2 are *adjacent* if their boundaries have an edge in common. A face is said to be *incident with* the vertices and edges in its boundary, and vice versa. An edge is said to be *incident with* the ends of the edge, and vice versa. Two vertices which are incident with a common edge are *adjacent*, and two distinct adjacent vertices are *neighbours*. If S is a set of vertices in a graph

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F, the set of all neighbours of the vertices in S is denoted by $N(S)$, and $|N(S)|$ denotes the number of neighbours of S.

Let F be a $(2,6)$-fullerene graph with vertex-set $V(F)$ and edge-set $E(F)$. We denote the number of vertices and edges in F by $|V(F)|$ and $|E(F)|$. For $H \subseteq H$, we let $F - H$ be the subgraph of F obtained from F by removing the elements in H. A *matching* of F is a set of disjoint edges M of F. A *perfect matching* of F is a matching M that covers all vertices of F. A perfect matching of a graph coincides with a Kekulé structure of some molecular graph in organic chemistry. A set $\mathscr{H}$ of disjoint even faces of a graph F is a *resonant pattern* if F has a perfect matching M such that the boundary of each face in $\mathscr{H}$ is an M-alternating cycle. F is said to be *k-resonant* ($k \geq 1$) if any i ($0 \leq i \leq k$) disjoint even faces of F form a resonant pattern. Moreover, F is called *n-extendable* ($|V(F)| \geq 2n+2$) if any matching of n edges is contained in a perfect matching of F. F is *bicritical* if F contains an edge and $F - u - v$ contains a perfect matching, for every pair of distinct vertices $u, v \in V(F)$. In this paper, we show that every $(2,6)$-fullerene is 1-extendable, 1-resonant but not 2-extendable, bicritical.

The anti-Kekulé set of a $(2,6)$-fullerene F with perfect matchings is an edge set $S \subseteq E(F)$ such that $F - S$ is connected and has no perfect matchings. The anti-Kekulé number of F, denoted by $ak(F)$, is the cardinality of a smallest anti-Kekulé set of F. It is NP-complete to find the smallest anti-Kekulé set of a graph. Moreover, it has been shown that the anti-Kekulé set of a graph significantly affects the whole molecule structure by the valence bond theory. We have known the $(5,6)$, $(4,6)$, and $(3,6)$-fullerenes have the anti-Kekulé numbers 4, 4 and 3 respectively. In this paper, We show that every $(2,6)$-fullerene F has the anti-Kekulé number 4 with $|V(F)| > 6$.

2. Main results

An *edge-cut* of F is a subset of edges $E' \subseteq E(F)$ such that $F - E'$ is disconnected. An *k-edge-cut* is an *edge-cut* with k edges. The *edge-connectivity* of F, denoted by $\kappa'(F)$, is equal to the minimum cardinality of edge-cuts. F is *k-edge-connected* if F cannot be separated into at least two components by removing less than k edges.

Lemma 1. *The* $(2,6)$*-fullerene* F *has edge-connectivity 2, where* $|V(F)| > 2$.

Proof. Since every edge of F is incident with a 2-length face or a 6-length face, there is no cut edge in F. Therefore, F is 2-edge-connected. For one 2-length face C in F, denoted by $C = xyx$. Then either $F \cong 3 \times K_2$ or the two edges incident with x and y respectively other than xy form an 2-edge-cut of F. Therefore, $\kappa'(F) = 2$, where $|V(F)| > 2$. □

We call an edge e is *incident to* a subgraph H if $|V(e) \bigcap V(H)| = 1$.

Lemma 2. *Every 2-edge-cut of a* $(2,6)$*-fullerene isolates a 2-length face.*

Proof. Let F be a $(2,6)$-fullerene. If $|V(F)| = 2$, then $F \cong 3 \times K_2$, and the conclusion holds as F has no 2-edge-cut. So next we suppose $|V(F)| > 2$. By Lemma 1, F has an 2-edge-cut. Let $E = \{e_1, e_2\}$ be an 2-edge-cut whose deletion separates F into two components, F', F''. Then E is a matching of F as F is 3-regular and has edge-connectivity 2. Let every edge e_i has one endpoint, say x_i, on F', the other endpoint, say y_i, on F'', $i = 1,2$. Suppose the outer face of F'' is exactly the outer face of F, thus F' lies in some inner face of F''. Then there are two hexagons, denoted by f_1, f_2, such that both f_1 and f_2 are incident with x_1, x_2, y_1, y_2. If one of F' and F'' contains a cut edge, without losing generality, assume that F' contains a cut edge $e = uv$, then $F' - e$ has two connected components, say F_1, F_2. Then both e_1 and e_2 cannot be incident to the same component F_i ($i = 1,2$), otherwise, there exists a cut edge e in F, a contradiction. Then $V(f_1) = \{u, v, x_1, y_1, x_2, y_2\}$, $V(f_2) = \{u, v, x_1, y_1, x_2, y_2\}$. That is, all of F_1, F_2 and F'' are 2-length faces and we get a $(2,6)$-fullerene with six vertices, thus the conclusion holds. If both F' and F'' contain cut edges, then there is a face with length more than 6, a contradiction. If neither F' nor F'' has a cut edge, then F' and F'' are 2-edge-connected, and in each of them there is only one face

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that is not 2-length or 6-length, and we denote these two boundaries of the exceptional faces by C' and C'', respectively. Let v', e', and f' be the number of vertices, edges and faces in F', respectively. Let l' be the length of C', and f_2', f_6' be the number of 2-length faces, 6-length faces in F', respectively. By Euler's formula and the structure of F', it follows that

$$\begin{cases} 3v' = 2e' + 2 \\ v' - e' + f_2' + f_6' = 1 \\ 2f_2' + 6f_6' + l' = 2e'. \end{cases} \tag{1}$$

By (1), we obtain that

$$l' = 4f_2' - 2. \tag{2}$$

Since F has no face with length more than 6, the two faces f_1, f_2 each has at most two additional vertices on C'. Hence $2 \le l' \le 6$. By (2), we can get $1 \le f_2' \le 2$. If $f_2' = 1$, we have $l' = 2$, which means that F' is a 2-length face, thus the conclusion holds. If $f_2' = 2$, then $l' = 6$ and there are no additional vertices on C'', which implies that F'' is a 2-length face, thus the conclusion holds. Therefore, every 2-edge-cut of a $(2,6)$-fullerene isolates a 2-length face.

□

In [1], Došlić proved that the $(k,6)$-fullerene is 1-extendable for $k = 3,4,5$. In fact, we may observe that the conclusions remain valid for $k = 2$.

Lemma 3 ([1]). *Let F be a $(2,6)$-fullerene graph. Then F is 1-extendable.*

The resonance of faces of a plane bipartite graph is closely related to 1-extendable property. It was revealed that every face (including the infinite one) of a plane bipartite graph G is resonant if and only if G is 1-extendable [20]. Combing with Lemma 3, we can know every $(2,6)$-fullerene is 1-resonant.

Corollary 1. *Every $(2,6)$-fullerene is 1-resonant.*

Moreover, we can know no $(2,6)$-fullerene is 2-extendable.

Theorem 1. *No $(2,6)$-fullerene is 2-extendable.*

Proof. Let F be a $(2,6)$-fullerene graph. Let f be a 2-length face of F with the boundary $v_1v_2v_1$. By the definition of extendability, we know that $|V(F)| \ge 4$. Then there exist two vertices u_1, u_2 of F which are different from v_1, v_2 such that $u_1v_1 \in E(F)$ and $u_2v_2 \in E(F)$. Since the four vertices u_1, u_2, v_1, v_2 must be contained in the same hexagon of F, there is a vertex $u_3 \neq u_1$, $u_3 \neq v_2$ of F such that $u_2u_3 \in E(F)$. Obviously, u_1v_1, u_2u_3 is a matching and cannot be contained in a perfect matching of F. Thus no $(2,6)$-fullerene is 2-extendable.

□

Similarly, we can show no $(2,6)$-fullerene is bicritical.

Theorem 2. *No $(2,6)$-fullerene is bicritical.*

Proof. Let F be a $(2,6)$-fullerene graph. Let f be a 2-length face of F with the boundary $v_1v_2v_1$. When $F \cong 3 \times K_2$, then $F - v_1 - v_2$ has no perfect matchings. When $F \ncong 3 \times K_2$, then there exists a vertex u of F which is different from v_2 such that $uv_1 \in E(F)$. Then $F - u - v_2$ has a single vertex v_1 as a component. So $F - u - v_2$ has no perfect matchings. That is, F is not bicritical.

□

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Theorem 3 (Tutte's Theorem [21])**.** *A graph G has a perfect matching if and only if $c_o(G-U) \leq |U|$ for any $U \subseteq V(G)$, where $c_o(G-U)$ is the number of odd components of $G-U$.*

Theorem 4 (Hall's Theorem [21])**.** *Let F be a bipartite graph with bipartition W and B. Then F has a perfect matching if and only if $|W| = |B|$ and for any $U \subseteq W$, $|N(U)| \geq |U|$ holds.*

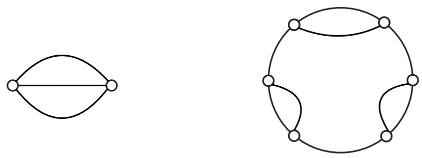
For the connected cubic simple bipartite graph, we can know its anti-Kekulé number is 4 [22].

Theorem 5 ([22])**.** *If G is a connected cubic simple bipartite graph, then $ak(F) = 4$.*

The above result can be used to determine the anti-Kekulé numbers of some interesting graphs, such as, $(4,6)$-fullerenes [22], toroidal fullerenes [22] etc. Theorem 5 is also valid for $(2,6)$-fullerene when $|V(F)| > 6$.

Theorem 6. *Let F be a $(2,6)$-fullerene graph with $|V(F)| > 6$. Then $ak(F) = 4$.*

Proof. Let F be a $(2,6)$-fullerene. For any vertex u in F, if $|N(u)| = 1$, then $F \cong 3 \times K_2$ (see Fig. 1(a) the graph $3 \times K_2$). For any vertex u in F, if $|N(u)| = 2$, then $F \cong F_6$ (see Fig. 1(b) the graph F_6). We can easily know that both $3 \times K_2$ and F_6 cannot exist anti-Kekulé set. On the other hand, if we let n and f_6 be the number of vertices and the hexagons of F, respectively, then by the Euler's formula and the formula of degree sum, we can get $n = 2f_6 + 2$. Thus if $f_6 = 0$, then $n = 2$ and $F \cong 3 \times K_2$. If $f_6 = 1$, then $n = 4$, which is impossible as every hexagonal face must contain six vertices. If $f_6 = 2$, then $n = 6$ and $F \cong F_6$ (see Fig. 1(b) the graph F_6). Therefore, when $|V(F)| \leq 6$, there is no anti-Kekulé set in F.



(a) $3 \times K_2$ (b) F_6

Figure 1. The (2,6)-fullerenes $3 \times K_2$ (a), F_6 (b).

Next, we discuss the anti-Kekulé number of F with $|V(F)| > 6$. Then there is a vertex u in F and $|N(u)| = 3$. Let x, y, z be the three neighbors of u. Let e_1 and e_2 be two edges incident with x other than ux, and let e_3 and e_4 be two edges incident with y other than uy. Since every face of F is 2-length or 6-length and F is 2-edge-connected, the four edges e_1, e_2, e_3, e_4 are pairwise different. We claim that $\{e_1, e_2, e_3, e_4\}$ is an anti-Kekulé set. It is obvious that $F - \{e_1, e_2, e_3, e_4\}$ has no perfect matchings as the two vertices x, y cannot be contained in the same perfect matching. If $F - \{e_1, e_2, e_3, e_4\}$ is no connected, then we obtain a cut edge uz in F, contradicting Lemma 1. Then we find an anti-Kekulé set of size 4 and so $ak(F) \leq 4$.

In the following, we show $ak(F) \geq 3$. Let A be an anti-Kekulé set of size $ak(F)$. Then $F' = F - A$ is connected and has no perfect matchings. According to Theorem 3, there exists $S \subseteq V(F')$ such that $c_0(F' - S) > |S|$. If we choose such an S with the maximum size, then $F' - S$ has no even components. On the contrary, suppose that $F' - S$ has an even component H. For any vertex $v \in V(H)$, $c_0(H - v) \geq 1$. Let $S' = S \cup \{v\}$, then $c_0(F' - S') \geq c_0(F' - S) + 1 > |S| + 1 = |S'|$, which is a contradiction to the choice of S.

Since $|V(F')|$ is even, then $c_0(F' - S) \geq |S| + 2$ by parity. For any edge $e \in A$, adding e to $F' - S$ will connect at most two odd components, then $c_o(F' + e - S) \geq c_0(F' - S) - 2$. Since A is the smallest anti-Kekulé set of F, then $F' + e$ has a perfect matching for any edge $e \in A$. Hence, by Theorem 3, for the above subset S, $c_o(F' + e - S) \leq |S|$. Therefore, $|S| \geq c_o(F' + e - S) \geq c_0(F' - S) - 2 \geq |S|$. We obtain $c_0(F' - S) = |S| + 2$, and the edge e connects exactly two components of $F' - S$.

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Let F_i be the odd components of $F' - S$, where $1 \leq i \leq |S| + 2$. For $F_i \subseteq F$, let $d(F_i)$ denote the number of the set of edges with one end in F_i and the other end in $F - F_i$. Denote the number of edges between S and the odd components by N. Since F is cubic, S sends out at most $3|S|$ to N. In addition, $\bigcup_{i=1}^{|S|+2} F_i$ sends out exactly $\sum_{i=1}^{|S|+2} d(F_i) - 2ak(F)$ edges to N. Hence

$$N = \sum_{i=1}^{|S|+2} d(F_i) - 2ak(F) \leq 3|S|. \tag{3}$$

Because F is 2-edge-connected, $d(F_i) \geq 2$ for every i. On the other hand, since $d(F_i) = 3|V(F_i)| - 2|E(F_i)|$, which implies that $d(F_i)$ and $|V(F_i)|$ are of the same parity. Every F_i sends odd number edges, hence $d(F_i) \geq 3$. Substituting it into (3), we have

$$3(|S| + 2) - 2ak(F) \leq \sum_{i=1}^{|S|+2} d(F_i) - 2ak(F) \leq 3|S|,$$

the above inequality gives $ak(F) \geq 3$.

We find that the anti-Kekulé number of F is either 3 or 4. Suppose by the contrary that $ak(F) = 3$. Then there exists an anti-Kekulé set $A = \{e_1, e_2, e_3\}$ of cardinality three in F, such that $F - A$ is connected and has no perfect matchings. Assume W and B are the bipartition of F. By Hall's theorem, there exists $U \subseteq W$ such that

$$|N_{F-A}(U)| \leq |U| - 1 \tag{4}$$

where $N_{F-A}(U)$ means $N(U)$ in $F - A$. Moreover, for $e_i \in A$, since A is the smallest anti-Kekulé set, $F - A + e_i$ has a perfect matching. Immediately by Theorem 4, for the above subset U,

$$|U| \leq |N_{F-A+e_i}(U)| \tag{5}$$

for $i = 1, 2$ and 3, where $N_{F-A+e_i}(U)$ means $N(U)$ in $F - A + e_i$. In addition, the neighbors of U will be increased by at most one if we add an edge e_i to $F - A$. Hence

$$|N_{F-A+e_i}(U)| \leq |N_{F-A}(U)| + 1. \tag{6}$$

Combining inequalities (4), (5) and (6), we have $|U| = |N_{F-A}(U)| + 1$, and e_i is incident with the vertices of U and $B - N_{F-A+e_i}(U)$ in $F - A + e_i$. Thus the edges going out from $U \subseteq V(F)$ either goes into A or goes into the edges going out from $N_{F-A}(U)$. Then the number of edges between U and $N_{F-A}(U)$ is $3|U| - 3$. Since $|N_{F-A}(U)| = |U| - 1$, $3|N_{F-A}(U)| = 3(|N_{F-A}(U)| + 1) - 3 = 3|U| - 3$, that is, there is no edge between $N_{F-A}(U)$ and $W - U$ in $F - A$. As a result, A is an edge-cut, which is a contradiction to the definition of anti-Kekulé set.

□

In [23], Grünbaum and Motzkin showed that (5,6)-fullerene and (4,6)-fullerene having n hexagonal faces exist for every non-negative integer n satisfying $n \neq 1$, and gave a similar result for (3,6)-fullerene. Therefore, we consider whether (2,6)-fullerene having n hexagonal faces also exists for any n. We tried to give a positive answer to this question, but we found that the conclusion seems quite elusive. Therefore, in this part, we mainly prove that there exists a $(2,6)$-fullerene F having f_F hexagonal faces where f_F is related to the two parameters n, m.

Let F be a $(2,6)$-fullerene. A *fragment* H of F is a subgraph of F consisting of a cycle together with its interior and every inner face of H is also a face of F. We define $\partial(H)$ as the *boundary* of the exterior face of H. A face f of F is a *neighboring face* of H if f is not a face of H and f has at least one edge in common with H. A path of length k (the number of edges) is called a *k-path*. Denote by f_H the number of hexagons of H.

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Proposition 1. *In all the (2,6)-fullerenes, there exists a fragment, say G_n, such that $f_{G_n} = n^2 + n$, $n \in \mathbb{Z}$.*

Proof. Let G_0 be a 2-length face and f_{11}, f_{12} be two neighboring faces of G_0 (see Fig. 2(a)). Then $f_{G_0} = 0$. Suppose that f_{11}, f_{12} are hexagons. Set $G_1 = G_0 \bigcup \{f_{11}, f_{12}\}$, suppose both f_{11} and f_{12} are inner faces of G_1 and let $f_{21}, f_{22}, f_{23}, f_{24}$ be four neighboring faces of G_1 along the clockwise direction such that f_{21} is incident with the two consecutive 2-degree vertices on $\partial(G_1)$ (see Fig. 2(b)). Then $f_{G_1} = 2$. Suppose that $f_{21}, f_{22}, f_{23}, f_{24}$ are hexagons, pairwise different, and intersecting if and only if $f_{2i}, f_{2,i+1}$ are intersecting at only one edge for $i = 1, 2, 3, 4$, $f_{25} = f_{21}$. Set $G_2 = G_1 \bigcup \{f_{21}, f_{22}, f_{23}, f_{24}\}$. Suppose $f_{21}, f_{22}, f_{23}, f_{24}$ are the inner faces of G_2 and let $f_{31}, f_{32}, f_{33}, f_{34}, f_{35}, f_{36}$ be six neighboring faces of G_2 along the clockwise direction such that f_{31} is incident with the two consecutive 2-degree vertices on $\partial(G_2)$ (see Fig. 2(c)). Then $f_{G_2} = 2 + 4 = 6$.

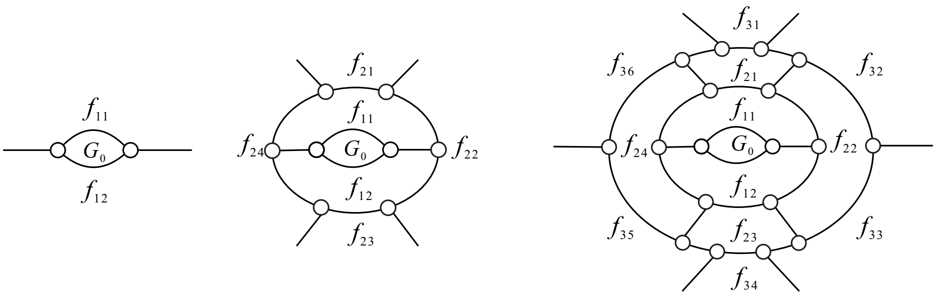

Figure 2. The fragments G_0 (a), G_1 (b) and G_2 (c).

Suppose that the proposition holds for any integer less than n, where $n > 2$. According to the induction hypothesis, $f_{G_{n-1}} = n^2 - n$ and $f_{n1}, f_{n2}, \ldots, f_{n,2n}$ be $2n$ neighboring faces of G_{n-1} along the clockwise direction such that f_{n1} is incident with the two consecutive 2-degree vertices on $\partial(G_{n-1})$. Suppose that $f_{n1}, \ldots, f_{n,2n}$ are hexagons, pairwise different, and intersecting if and only if $f_{ni}, f_{n,i+1}$ are intersecting at only one edge for $i = 1, 2, \ldots, 2n$, $f_{n,2n+1} = f_{n1}$. Set $G_n = G_{n-1} \bigcup \{f_{n1}, \ldots, f_{n,2n}\}$. Suppose $f_{n1}, \ldots, f_{n,2n}$ are all inner faces of G_n (see Fig. 3). Then $f_{G_n} = n^2 - n + 2n = n^2 + n$, $n \in \mathbb{Z}$.

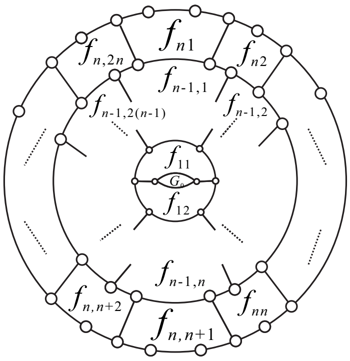

Figure 3. The fragment G_n.

□

Proposition 2. *In all the (2,6)-fullerenes, there exists a fragment, say C_n, such that $f_{C_n} = n$, $n \in \mathbb{Z}$.*

Proof. Let C_0 be a $3 \times K_2$, then $f_{C_0} = 0$. Let d_1 and d_2 be two 2-length faces. Its boundary $\partial(d_i)$ is labelled v_{i1}, v_{i2} $(i = 1, 2)$. Let P_i be a path that connects two vertices v_{1i}, v_{2i} $(i = 1, 2)$ and $V(P_1) \bigcap V(P_2) = \emptyset$. If both P_1, P_2 are 2-paths, then as F is 2-connected, there is a hexagon, say f_1, such that f_1 contains the paths P_1, P_2 and the edges $v_{11}v_{12}$, $v_{21}v_{22}$. Set $C_1 = d_1 \bigcup d_2 \bigcup f_1$, without lose of generality, suppose f_1 is the inner face of C_1 (see Fig. 4(a)). Thus $f_{C_1} = 1$. If both P_1, P_2 are 4-paths, then all of whose internal vertices are denoted by x_1, x_2, x_3 and y_1, y_2, y_3, respectively, such

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that $P_1 = v_{11}x_1x_2x_3v_{21}$, $P_2 = v_{12}y_1y_2y_3v_{22}$. Let $x_2y_2 \in E(F)$, then there are 2 hexagons, denoted by f_1, f_2, such that $\partial(f_1) = v_{11}v_{12}y_1y_2x_2x_1v_{11}$, $\partial(f_2) = v_{21}x_3x_2y_2y_3v_{22}v_{21}$. Set $C_2 = d_1 \bigcup d_2 \bigcup f_1 \bigcup f_2$, also suppose f_1, f_2 are two inner faces of C_2 (see Fig. 4(b)), then $f_{C_2} = 2$. Suppose P_1, P_2 are $2n$-paths, $n \in \mathbb{N}^+$. Let $P_1 = v_{11}x_1 \ldots x_{2n-1}v_{21}$ and $P_2 = v_{12}y_1 \ldots y_{2n-1}v_{22}$. Suppose that $x_iy_i \in E(F)$ $(i = 2, 4, \ldots, 2n-2)$, then there are n hexagons between P_1 and P_2, denoted by $f_1, f_2 \ldots f_n$. Set $C_n = d_1 \bigcup d_2 \bigcup_{i=1}^{n} f_i$ such that $f_1, f_2 \ldots f_n$ are the inner faces of C_n (see Fig. 4(c)). Therefore, C_n is a fragment and $f_{C_n} = n$, $n \in \mathbb{N}^+$. Thus there exists a fragment C_n such that $f_{C_n} = n$, $n \in \mathbb{Z}$.

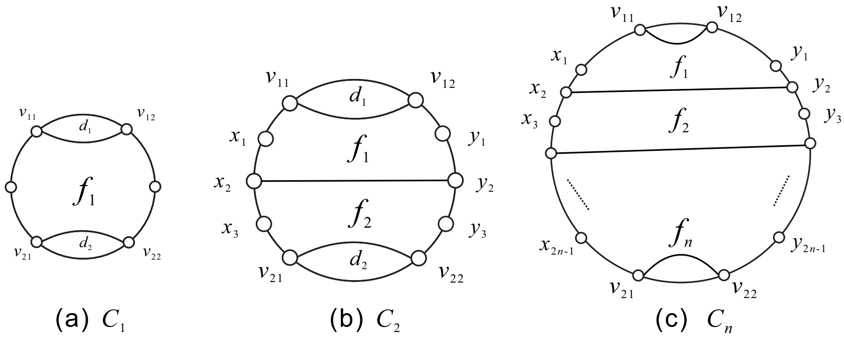

Figure 4. The fragments C_1 (a), C_2 (b) and C_n (c).

□

Proposition 3. *In all the (2,6)-fullerenes, there exists a fragment, say L_n^m, such that $f_{L_n^m} = 2n^2 + (m+3)n$, $n \in \mathbb{N}^+$, $m \in \mathbb{Z}$.*

Proof. Let $G_n^{'}$, $G_n^{''}$ be two fragments as indicated in Fig. 3. By Proposition 1, we know that $G_n^{'}$ and $G_n^{''}$ both have $n^2 + n$ hexagons. Suppose n is a positive integer. Since there are $2n+2$ 2-degree vertices on $\partial(G_n^{'})$, we can record them clockwise as $u_1, u_2, \ldots, u_{2n+2}$ such that u_1 and u_{2n+2} are adjacent. Similarly, $2n+2$ 2-degree vertices on $\partial(G_n^{''})$ are denoted by $v_1, v_2, \ldots, v_{2n+2}$ along the anticlockwise direction of $G_n^{''}$ such that v_1 and v_{2n+2} are adjacent. For $G_1^{'}$ and $G_1^{''}$. Let $e_1 = u_1v_1, e_2 = u_2v_2$, then e_1 and e_2 are contained in the hexagon, say f_1. Set $K_1 = G_1^{'} \bigcup G_1^{''} \bigcup f_1$ (see Fig. 5(a)), then $f_{K_1} = 5$. For $G_n^{'}$ and $G_n^{''}$. Let $e_i = u_iv_i$, $i = 1, 2 \ldots n+1$, then e_i and e_{i+1} are contained in the hexagon, say f_i, $i = 1, 2 \ldots n$. Set $K_n = G_n^{'} \bigcup G_n^{''} \bigcup_{i=1}^{n} f_i$, suppose all of $f_1 \ldots f_n$ are the inner faces of K_n (see Fig. 5(b), the embedding of K_n), then $f_{K_n} = 2(n^2+n) + n = 2n^2 + 3n$.

Next, we construct the fragment L_n^m from K_n as follows. We replace each edge $e_i = u_iv_i$ by a path P_i such that $P_i = u_ix_{i1}x_{i2} \ldots x_{i,2m}v_i$, $i = 1, 2 \ldots n+1$, $m \in \mathbb{Z}$. Suppose that $x_{i2}x_{i+1,1}, x_{i4}x_{i+1,3} \ldots x_{i,2m}x_{i+1,2m-1}$ be the edges of F, $i = 1, 2 \ldots n$. Therefore, there are $m+1$ hexagons between P_i and P_{i+1}, denoted by $f_{i1}, f_{i2} \ldots f_{i,m+1}$, $i = 1, 2 \ldots n$. Set $L_n^m = G_n^{'} \bigcup G_n^{''} \bigcup_{i=1}^{n} \{f_{i1}, f_{i2} \ldots f_{i,m+1}\}$, $m \in \mathbb{Z}$ (see Fig. 5(c), the embedding of L_n^m). Therefore, L_n^m is a fragment and $f_{L_n^m} = 2(n^2+n) + (m+1)n = 2n^2 + (m+3)n$, $n \in \mathbb{N}^+$, $m \in \mathbb{Z}$.

□

Proposition 4. *In all the (2,6)-fullerenes, there exists a fragment, say H_n, such that $f_{H_n} = 2n+2$, $n \in \mathbb{Z}$.*

Proof. Let $H_0 \cong F_6$ be the $(2,6)$-fullerene with six vertices. Without lose of generality, suppose the exterior face of H_0 is a 2-length face with the boundary $u_1v_1u_1$, and the remaining two 2-length faces are connected by an edge u_2v_2 (see Fig. 6(a) the embedding of H_0 and the labelling of u_1, v_1, u_2, v_2). Next, we construct the fragment H_n from H_0 as follows: we replace the two parallel edges u_1v_1 and one edge u_2v_2 by two paths P_1, P_3 and one path P_2 such that $P_i = u_ix_{i1}x_{i2} \ldots x_{i,2n}v_i$, $i = 1, 2$, and $P_3 = u_1x_{31}x_{32} \ldots x_{3,2n}v_1$, $n \in \mathbb{N}^+$. Suppose that $x_{i2}x_{i+1,1}, x_{i4}x_{i+1,3} \ldots x_{i,2n}x_{i+1,2n-1}$ be the edges of F, $i = 1, 2$. We construct $n+1$ hexagons between P_i and P_{i+1}, denoted by $f_{i1}, f_{i2} \ldots f_{i,n+1}$, $i = 1, 2$. Set $H_n = H_0 - \{u_1v_1, u_2v_2\} \bigcup_{i=1}^{2} \{f_{i1}, f_{i2} \ldots f_{i,n+1}\}$ such that $f_{i1}, f_{i2} \ldots f_{i,n+1}$ are the inner faces of H_n,

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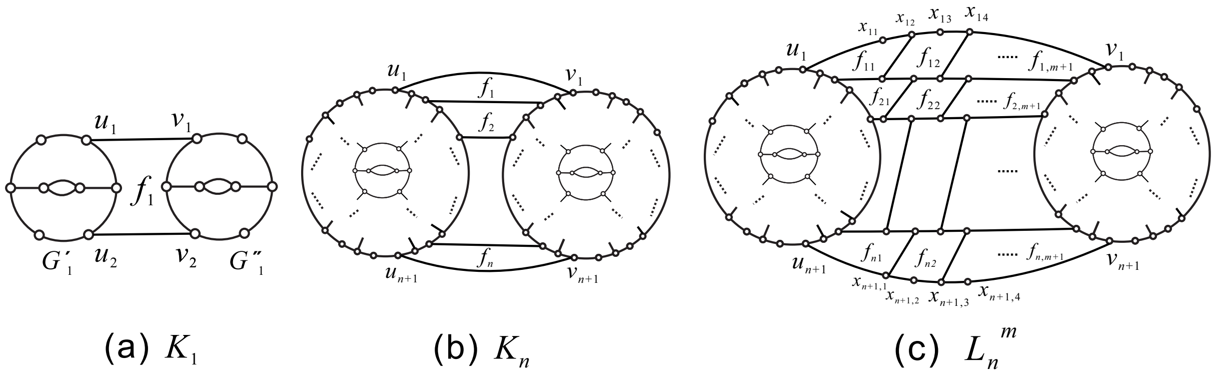

Figure 5. The fragments K_1 (a), K_n (b) and L_n^m (c).

$n \in \mathbb{N}^+$ (see Fig. 6(b)). Therefore, H_n is a fragment and $f_{H_n} = 2(n+1) = 2n+2$, $n \in \mathbb{N}^+$. Thus there exists a fragment H_n such that $f_{H_n} = 2n+2$, $n \in \mathbb{Z}$.

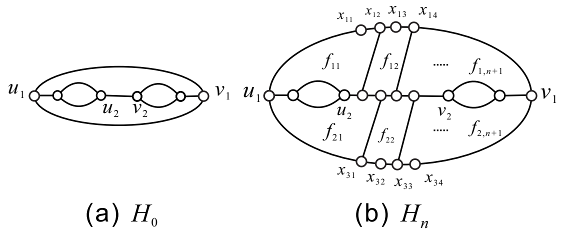

Figure 6. The fragments H_0 (a), H_n (b).

□

By Propositions 1–4, we can find a $(2,6)$-fullerene F having f_F hexagonal faces which related to the parameters n, m.

Theorem 7. *There exists a* $(2,6)$*-fullerene* F *such that* $f_F = n^2 + 2n$, $n \in \mathbb{Z}$.

Proof. Let G_n be a fragment of F as shown in Fig. 3. Its boundary $\partial(G_n)$ is labelled $u_1, u_2, \ldots, u_{4n+2}$ along the clockwise direction, where u_1 and u_2 are two consecutive 2-degree vertices. Let C_n be a fragment of F as shown in Fig. 4(c). Its boundary $\partial(C_n)$ is labelled $v_1, v_2, \ldots, v_{4n+2}$ along the clockwise direction, where v_1 and v_2 are two consecutive 3-degree vertices. Next assume the graphs G_n and C_n each drawn on a hemisphere, with the boundary as equator. If $\partial(G_n) = \partial(C_n)$, then set $F = G_n \bigcup C_n$. By Propositions 1 and 2, then $f_F = f_{G_n} + f_{C_n} = n^2 + 2n$, $n \in \mathbb{Z}$.
□

Theorem 8. *There exists a* $(2,6)$*-fullerene* F *such that* $f_F = 3n^2 + m^2 + 3mn + 6n + 3m + 2$, $n \in \mathbb{N}^+$, $m \in \mathbb{Z}$.

Proof. Let G_{m+n+1} be a fragment of F as shown in Fig. 3. Its boundary $\partial(G_{m+n+1})$ is labelled $u_1, u_2, \ldots, u_{4m+4n+6}$ along the clockwise direction, where u_1 and u_2 are two consecutive 2-degree vertices. Let L_n^m be a fragment of F as shown in Fig. 5(c). Its boundary $\partial(L_n^m)$ is labelled $v_1, v_2, \ldots, v_{4m+4n+6}$ along the clockwise direction, where v_1 and v_2 are two consecutive 3-degree vertices. Next assume the graphs G_{m+n+1} and L_n^m each drawn on a hemisphere, with the boundary as equator. If $\partial(G_{m+n+1}) = \partial(L_n^m)$, then set $F = G_{m+n+1} \bigcup L_n^m$. By Propositions 1 and 3, then $f_F = f_{G_{m+n+1}} + f_{L_n^m} = 3n^2 + m^2 + 3mn + 6n + 3m + 2$, $n \in \mathbb{N}^+$, $m \in \mathbb{Z}$.
□

Theorem 9. *There exists a* $(2,6)$*-fullerene* F *such that* $f_F = n^2 + 3n + 2$, $n \in \mathbb{Z}$.

Proof. Let G_n be a fragment of F as shown in Fig. 3. Its boundary $\partial(G_n)$ is labelled $u_1, u_2, \ldots, u_{4n+2}$ along the clockwise direction, where u_1 and u_2 are two consecutive 2-degree vertices. Let H_n be a fragment of F as shown in Fig. 6(b). Its boundary $\partial(H_n)$ is labelled $v_1, v_2, \ldots, v_{4n+2}$ along the clockwise direction, where v_1 and v_2 are two consecutive 3-degree vertices. Next assume the graphs G_n and H_n each drawn on a hemisphere, with the boundary as equator. If $\partial(G_n) = \partial(H_n)$, then set $F = G_n \bigcup H_n$. By Propositions 1 and 4, then $f_F = f_{G_n} + f_{H_n} = n^2 + 3n + 2, n \in \mathbb{Z}$.
□

3. Conclusion

In this paper, we have obtained that every $(2,6)$-fullerene F is 1-extendable but not 2-extendable, 1-resonant and has the anti-Kekulé number 4 with $|V(F)| > 6$. Moreover, every $(2,6)$-fullerene isn't bicritical. At last, we prove that there exists a $(2,6)$-fullerene F having f_F hexagonal faces, where f_F is related to the two parameters n, m. There are, however, still several important open questions.

Problem 1. *Whether every $(2,6)$-fullerene is k-resonant $(k \geq 2)$?*

Problem 2. *Whether for every $n \neq 1$ there exists a $(2,6)$-fullerene F such that $f_F = n$?*

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