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K. MAHESH KRISHNA *

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Unbounded Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

K. Mahesh Krishna

Post Doctoral Fellow, Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, Karnataka 560 059, India; kmahesh_vs@isibang. ac. in, kmaheshak@gmail.com

Abstract: Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be unbounded continuous 1-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}$, we show that

(1) $\mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x)) \geq \dfrac{1}{\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right)\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right)}.$

where

$\theta_f : \mathcal{D}(\theta_f) \ni x \mapsto \theta_f x \in \mathcal{L}^1(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K},$

$\theta_g : \mathcal{D}(\theta_g) \ni x \mapsto \theta_g x \in \mathcal{L}^1(\Delta, \nu); \quad \theta_g x : \Delta \ni \beta \mapsto (\theta_g x)(\beta) := g_\beta(x) \in \mathbb{K}.$

We call Inequality (1) as **Unbounded Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**. Along with recent **Functional Continuous Uncertainty Principle** (derived in [arXiv:2308.00312v1 [math.FA], 1 August 2023]), Inequality (1) also improves Ricaud-Torrésani uncertainty principle *[IEEE Trans. Inform. Theory, 2013]*. In particular, it improves Elad-Bruckstein uncertainty principle *[IEEE Trans. Inform. Theory, 2002]* and Donoho-Stark uncertainty principle *[SIAM J. Appl. Math., 1989]*.

Keywords: uncertainty principle; frame; banach space

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1. Introduction

Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j\rangle)_{j=1}^n \in \mathbb{K}^n.$$

Most general form of discrete uncertainty principle for finite dimensional Hilbert spaces is the following.

Theorem 1.1. *(**Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**) [1–3] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0 \|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|^2}, \quad \forall h \in \mathcal{H} \setminus \{0\}.$$

Recently, Theorem 1.1 has been derived for Banach spaces using the following notion.

Definition 1.2. *[4] Let (Ω, μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a **continuous p-Schauder frame** for $\mathcal{X}$ ($1 \leq p < \infty$) if the following holds.*

(i) For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.

(ii) *For every* $x \in \mathcal{X}$,

$$\|x\|^p = \int_{\Omega} |f_\alpha(x)|^p \, d\mu(\alpha).$$

(iii) *For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ *is weakly measurable.*
(iv) *For every* $x \in \mathcal{X}$,

$$x = \int_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

where the integral is weak integral.

Given a continuous p-Schauder frame $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ for $\mathcal{X}$, define

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

Theorem 1.3. *(**Functional Continuous Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**) [4–6] Let* (Ω, μ), (Δ, ν) *be measure spaces. Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *and* $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ *be continuous p-Schauder frames for a Banach space* $\mathcal{X}$. *Then*

(i) *for* $p > 1$, *we have*

$$\mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|}, \quad \forall x \in \mathcal{X} \setminus \{0\};$$
$$\nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}, \quad \forall x \in \mathcal{X} \setminus \{0\}.$$

where q *is the conjugate index of* p.
(ii) *for* $p = 1$, *we have*

$$\mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}. \tag{2}$$

In this paper, we derive an unbounded uncertainty principle which contains Theorem 1.1 as a particular case.

2. Unbounded Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

We first generalize Definition 1.2 using unbounded linear functionals. Our motivation to do so is the theory of unbounded frames (also known as semi frames or pseudo frames) for Hilbert and Banach spaces, see [7–10].

Definition 2.1. *Let* (Ω, μ) *be a measure space. Let* $\{\tau_\alpha\}_{\alpha\in\Omega}$ *be a collection in a Banach space* $\mathcal{X}$ *and* $\{f_\alpha\}_{\alpha\in\Omega}$ *be a collection of linear functions on* $\mathcal{X}$ *(which may not be bounded). The pair* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *is said to be a* ***unbounded continuous p-Schauder frame*** *or* ***continuous semi p-Schauder frame*** *for* $\mathcal{X}$ *if the following conditions holds.*

(i) *For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ *is measurable.*
(ii) *The map*

$$\theta_f : \mathcal{D}(\theta_f) \ni x \mapsto \theta_f x \in \mathcal{L}^1(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

is well-defined (need not be bounded).

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(iii) *For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.*
(iv) *For every $x \in \mathcal{D}(\theta_f)$,*

$$x = \int_\Omega f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

where the integral is weak integral.

Theorem 2.2. ***(Unbounded Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle)*** *Let (Ω, μ), (Δ, ν) be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ be unbounded continuous 1-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}$, we have*

$$\mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right)\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right)}. \tag{3}$$

Proof. Let $x \in \mathcal{D}(\theta_f) \setminus \{0\}$. Then

$$\begin{aligned}
\|\theta_f x\| &= \int_\Omega |f_\alpha(x)| \, d\mu(\alpha) = \int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)| \, d\mu(\alpha) = \int_{\operatorname{supp}(\theta_f x)} \left| f_\alpha\left(\int_\Delta g_\beta(x)\omega_\beta \, d\nu(\beta) \right) \right| d\mu(\alpha) \\
&= \int_{\operatorname{supp}(\theta_f x)} \left| \int_\Delta g_\beta(x) f_\alpha(\omega_\beta) \, d\nu(\beta) \right| d\mu(\alpha) = \int_{\operatorname{supp}(\theta_f x)} \left| \int_{\operatorname{supp}(\theta_g x)} g_\beta(x) f_\alpha(\omega_\beta) \, d\nu(\beta) \right| d\mu(\alpha) \\
&\leq \int_{\operatorname{supp}(\theta_f x)} \int_{\operatorname{supp}(\theta_g x)} |g_\beta(x) f_\alpha(\omega_\beta)| \, d\nu(\beta) \, d\mu(\alpha) \\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)| \right) \int_{\operatorname{supp}(\theta_f x)} \int_{\operatorname{supp}(\theta_g x)} |g_\beta(x)| \, d\nu(\beta) \, d\mu(\alpha) \\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)| \right) \mu(\operatorname{supp}(\theta_f x)) \int_{\operatorname{supp}(\theta_g x)} |g_\beta(x)| \, d\nu(\beta) \\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)| \right) \mu(\operatorname{supp}(\theta_f x)) \|\theta_g x\|.
\end{aligned}$$

Therefore

$$\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} \|\theta_f x\| \leq \mu(\operatorname{supp}(\theta_f x)) \|\theta_g x\|. \tag{4}$$

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On the other way, let $x \in \mathcal{D}(\theta_g) \setminus \{0\}$. Then

$$\begin{aligned}
\|\theta_g x\| &= \int_\Delta |g_\beta(x)|\,d\nu(\beta) = \int_{\operatorname{supp}(\theta_g x)} |g_\beta(x)|\,d\nu(\beta) = \int_{\operatorname{supp}(\theta_g x)} \left| g_\beta\left(\int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha)\right)\right| d\nu(\beta)\\
&= \int_{\operatorname{supp}(\theta_g x)} \left|\int_\Omega f_\alpha(x) g_\beta(\tau_\alpha)\,d\mu(\alpha)\right| d\nu(\beta) = \int_{\operatorname{supp}(\theta_g x)} \left|\int_{\operatorname{supp}(\theta_f x)} f_\alpha(x) g_\beta(\tau_\alpha)\,d\mu(\alpha)\right| d\nu(\beta)\\
&\leq \int_{\operatorname{supp}(\theta_g x)} \int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x) g_\beta(\tau_\alpha)|\,d\mu(\alpha)\,d\nu(\beta)\\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \int_{\operatorname{supp}(\theta_g x)} \int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\,d\mu(\alpha)\,d\nu(\beta)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\operatorname{supp}(\theta_g x)) \int_{\operatorname{supp}(\theta_f x)} |f_\alpha(x)|\,d\mu(\alpha)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\operatorname{supp}(\theta_g x))\|\theta_f x\|.
\end{aligned}$$

Therefore

$$\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|}\|\theta_g x\| \leq \nu(\operatorname{supp}(\theta_g x))\|\theta_f x\|. \tag{5}$$

Multiplying Inequalities (4) and (5) we get

$$\frac{1}{\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right)\left(\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right)}\|\theta_f x\|\|\theta_g x\| \leq \mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x))\|\theta_g x\|\|\theta_f x\|,$$
$$\forall x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}.$$

A cancellation of $\|\theta_f x\|\|\theta_g x\|$ gives the required inequality. $\square$

Corollary 2.3. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be collections in a Banach space $\mathcal{X}$ such that*

$$x = \sum_{j=1}^n f_j(x)\tau_j = \sum_{k=1}^m g_k(x)\omega_k, \quad \forall x \in \mathcal{X}.$$

Then for every $x \in \mathcal{X} \setminus \{0\}$,

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{1}{\left(\max\limits_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|\right)\left(\max\limits_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)},$$

where

$$\theta_f : \mathcal{X} \ni x \mapsto (f_j(x))_{j=1}^n \in \ell^1([n]); \quad \theta_g : \mathcal{X} \ni x \mapsto (g_k(x))_{k=1}^m \in \ell^1([m]).$$

Even though by multiplying two inequalities in (2) we get Inequality (3) for continuous 1-Schauder frames, observe that the conclusion in (2) is stronger (with stronger assumption) than that of Theorem 2.2.

Note that proof of Theorem 2.2 does not work for unbounded continuous p-Schauder frames for $p > 1$ (even by using Holder's inequality). We are therefore left over with following problem.

Problem 2.4. *What is the unbounded version of Theorem 2.2 for $p > 1$?*

References

1. Donoho, D.L.; Stark, P.B. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.* **1989**, *49*, 906–931. doi:10.1137/0149053.
2. Elad, M.; Bruckstein, A.M. A generalized uncertainty principle and sparse representation in pairs of bases. *IEEE Trans. Inform. Theory* **2002**, *48*, 2558–2567. doi:10.1109/TIT.2002.801410.
3. Ricaud, B.; Torrésani, B. Refined support and entropic uncertainty inequalities. *IEEE Trans. Inform. Theory* **2013**, *59*, 4272–4279. doi:10.1109/TIT.2013.2249655.
4. Krishna, K.M. Functional Continuous Uncertainty Principle. *arXiv:2308.00312v1 [math.FA] 1 August* **2023**.
5. Krishna, K.M. Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle. *arXiv: 2304. 03324v1, [math.FA] 5 April* **2023**.
6. Krishna, K.M. An Endpoint Functional Continuous Uncertainty Principle. *10.20944/preprints202308.1108.v1, 15 August* **2023**.
7. Liu, B.; Liu, R.; Zheng, B. Parseval p-frames and the Feichtinger conjecture. *J. Math. Anal. Appl.* **2015**, *424*, 248–259. doi:10.1016/j.jmaa.2014.11.021.
8. Christensen, O. Frames and pseudo-inverses. *J. Math. Anal. Appl.* **1995**, *195*, 401–414. doi:10.1006/jmaa.1995.1363.
9. Antoine, J.P.; Balazs, P. Frames, semi-frames, and Hilbert scales. *Numer. Funct. Anal. Optim.* **2012**, *33*, 736–769. doi:10.1080/01630563.2012.682128.
10. Antoine, J.P.; Balazs, P. Frames and semi-frames. *J. Phys. A* **2011**, *44*, 205201, 25. doi:10.1088/1751-8113/44/20/205201.