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Tricyclic Graph with Minimum Randić Index

Liangwen Tang [1,*,†], Zhumei Song [2,†] and Mugang Lin [3]

1. School of Computer Science and Technology, Hengyang Normal University, Hengyang 421002, China;
2. School of Information and Communication, Shenzhen Institute of Information Technology, Shenzhen 518172, China; songzm@sziit.edu.cn
3. School of Computer Science and Technology, Hengyang Normal University, Hengyang 421002, China; mglin@hynu.edu.cn

* Correspondence: tanglw@hynu.edu.cn

† These authors contributed equally to this work.

Abstract: The Randić index of a graph G is the sum of $(d_G(u)d_G(v))^{-\frac{1}{2}}$ over all edges uv of G, where $d_G(u)$ denotes the degree of vertex u in G. In this paper, we investigate a few graph transformations that decrease Randić index of graph. By applying those transformations, we determine the minimum Randić index on tricyclic graphs, and characterize the corresponding extremal graphs.

Keywords: Randić index; tricyclic graph; graph transformation

1. Introduction

In this paper we are concerned with undirected simple connected graphs, unless otherwise specified. Let $G = (V, E)$ be such a graph with n vertices and m edges, which is addressed as an (n, m)-graph in what follows. A graph is cyclic if it contains at least one cycle, otherwise acyclic. More specifically, an $(n, n + k)$-graph is tree, unicyclic, bicyclic, tricyclic or tetracyclic, if $k = -1, 0, 1, 2, 3$ respectively. Denote $N_G(v)$ the neighbors of vertex v in G, and $d_G(v) = |N_G(v)|$ the degree of v. As usual, let $\Delta(G) = \max\{d_G(v)|v \in V\}$ and $\delta(G) = \min\{d_G(v)|v \in V\}$. A pendant vertex (or leaf) is a vertex of degree one. The star S_n is tree with $n - 1$ pendant vertices, and the path P_n is tree with two pendant vertices. For vertex v, we call $N^1_G(v) = \{u \in N_G(v)|d_G(u) = 1\}$ pendant neighbors of v, and $N^2_G(v) = \{u \in N_G(v)|d_G(u) > 1\}$ non-pendant neighbors. Furthermore, let $d^1_G(v) = |N^1_G(v)|$ and $d^2_G(v) = |N^2_G(v)|$. If $\widetilde{E}$ is an edge set, then $G - \widetilde{E}$ denotes the graph formed from G by deleting edges in $\widetilde{E}$, while $G + \widetilde{E}$ means the graph from G by adding edges in $\widetilde{E}$. If graph G and H are isomorphic, we can write it as $G \cong H$.

The Randić index(or R index for short) of G is defined as

$$R(G) = \sum_{uv \in E} \frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

This structural descriptor was proposed as branching index[11] by Milan Randić in 1975. Since then, mathematical properties of R index have been studied extensively. For a comprehensive survey, see [6,8,9].

From the view of extremal graph theory, Bollobas and Erdoš [1] first proved that S_n is the unique graph with the minimum R index for all n-vertex graphs and n-vertex trees. Trees with the second to the fourth minimum R indices have been determined by Zhao and Li in [10]. Caporossi et al [2] and P. Yu [12] showed that P_n attains the maximum R index in trees of order n. In [2], trees and unicyclic graphs with the first and the second maximum R indices and bicyclic graphs with the maximum R index are also considered. The unique unicyclic graph with the minimum R index has been determined by Gao and Lu in [7]. Du and Zhou [5] investigated more minimum and maximum R indices of trees, unicyclic and bicyclic graphs, for instance the second to the fifth maximum and the second minimum R indices of bicyclic graphs. Furthermore, the tricyclic and tetracyclic graphs with the maximum and the second maximum R indices have been determined in [3,4], while the counterparts with the minimum Randić indices are still unidentified for now.

In this paper, we investigate a few graph transformations which decrease R index of graphs. With the aid of these transformations, we derive tricyclic graph with the minimum Randić index as follows.

Theorem 1. *If G is a tricyclic graph of order $n \geqslant 4$, then $R(G) \geqslant \frac{n-4+\sqrt{3}}{\sqrt{n-1}} + 1$. And the equality holds if and only if $G \cong TR_n^\star$, where $TR_n^\star$ is obtained by attaching $n-4$ pendant vertices to one vertex of a complete graph K_4 shown as Figure 1.*

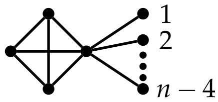

Figure 1. The structure of $TR_n^\star$

2. Preliminaries

The following lemmas are required mainly for characterizing transformations in the next section.

Lemma 1. *Suppose $f(x)$ is twice differentiable, and $f''(x) > 0$. Let $S = f(a) - f(b) - f(c) + f(d)$ with $a + d = b + c$, $a \leqslant b \leqslant d$ and $a \leqslant c \leqslant d$. Then $S \geqslant 0$ with equality holding if and only if $a = b$ or $a = c$.*

Proof. If $a = b$, then $d = c$ from $a + d = b + c$, thereby $S = 0$. Similarly $S = 0$ if $a = c$. Without loss of generality, we may assume that $a < b \leqslant c < d$. Thus we obtain $S = f'(\xi_1)(a-b) + f'(\xi_2)(d-c) = (b-a)(f'(\xi_2) - f'(\xi_1)) = (b-a)(\xi_2 - \xi_1)f''(\eta) > 0$, where $a < \xi_1 < b \leqslant c < \xi_2 < d$ and $\xi_1 < \eta < \xi_2$. Therefore the lemma holds clearly. □

Lemma 2. *If $x \geqslant 2, y \geqslant 2$, then $f(x,y) = \frac{y-2+\frac{2}{\sqrt{x}}}{\sqrt{y}} - \frac{y-1+\frac{1}{\sqrt{x}}}{\sqrt{y+x-1}} > 0$.*

Proof. Let $g(x,y) = f(x,y)\left(\frac{y-2+\frac{2}{\sqrt{x}}}{\sqrt{y}} + \frac{y-1+\frac{1}{\sqrt{x}}}{\sqrt{y+x-1}}\right) y(x+y-1) = (x-3+\frac{2}{\sqrt{x}})y^2 + (4\sqrt{x} - 4x + 7 + \frac{3}{x} - \frac{10}{\sqrt{x}})y + 4x - \frac{4}{x} + \frac{8}{\sqrt{x}} - 8\sqrt{x}$ with $x \geqslant 2$ and $y \geqslant 2$. Note that it suffices to show that $g(x,y) > 0$.

Observe that $(x-3+\frac{2}{\sqrt{x}})' = 1 - \frac{1}{x^{\frac{3}{2}}} > 0$, so $x - 3 + \frac{2}{\sqrt{x}} \geqslant 2 - 3 + \frac{2}{\sqrt{2}} > 0$. Hence $\frac{\partial g(x,y)}{\partial y} = 2y(x-3+\frac{2}{\sqrt{x}}) + (4\sqrt{x} - 4x + 7 + \frac{3}{x} - \frac{10}{\sqrt{x}}) \geqslant 4(x-3+\frac{2}{\sqrt{x}}) + (4\sqrt{x} - 4x + 7 + \frac{3}{x} - \frac{10}{\sqrt{x}}) = 4\sqrt{x} + \frac{3}{x} - \frac{2}{\sqrt{x}} - 5 \geqslant 4\sqrt{2} + \frac{3}{2} - \frac{2}{\sqrt{2}} - 5 > 0$ since $(4\sqrt{x} + \frac{3}{x} - \frac{2}{\sqrt{x}} - 5)' = \frac{2}{\sqrt{x}} + \frac{1}{x^{\frac{3}{2}}} - \frac{3}{x^2} > 0$. As a consequence, $g(x,y) \geqslant g(x,2) = 2 + \frac{2}{x} - \frac{4}{\sqrt{x}} \geqslant 2 + \frac{2}{2} - \frac{4}{\sqrt{2}} > 0$ as $(2 + \frac{2}{x} - \frac{4}{\sqrt{x}})' = \frac{2}{x^{\frac{3}{2}}} - \frac{2}{x^2} > 0$. Hence the lemma holds easily. □

Lemma 3. *if $x \geqslant 3, y \geqslant 3$ and $k \geqslant \frac{2}{\sqrt{5}}$, and let*

$$f(x,y) = \frac{x-3}{\sqrt{x}} + \frac{y-3}{\sqrt{y}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x+y-5}{\sqrt{x+y-1}} + \left(\frac{k}{\sqrt{x}} - \frac{k}{\sqrt{x+y-1}}\right).$$

Then $f(x,y) > 0$ and $f(y,x) > 0$.

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Proof. Since $f(x,y) \geqslant \frac{x-3}{\sqrt{x}} + \frac{y-3}{\sqrt{y}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x+y-5}{\sqrt{x+y-1}} + \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}}\right) = h(x,y)$, then we only have to show $h(x,y) > 0$. Consider the gradient

$$\begin{aligned}\frac{\partial h(x,y)}{\partial y} &= \frac{1}{2}\left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x+y-1}}\right) + \frac{3-\frac{1}{\sqrt{x}}}{2y^{\frac{3}{2}}} - \frac{4-\frac{2\sqrt{5}}{5}}{2\,(x+y-1)^{\frac{3}{2}}} \\ &\geqslant \frac{x-1}{4\eta^{\frac{3}{2}}} + \frac{3-\frac{1}{\sqrt{3}}}{2y^{\frac{3}{2}}} - \frac{4-\frac{2\sqrt{5}}{5}}{2\,(x+y-1)^{\frac{3}{2}}} \\ &\geqslant \frac{(\frac{3-1}{2}+3-\frac{1}{\sqrt{3}}) - (4-\frac{2}{\sqrt{5}})}{2\,(x+y-1)^{\frac{3}{2}}} = \frac{\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{3}}}{2\,(x+y-1)^{\frac{3}{2}}} > 0,\end{aligned}$$

where $y < \eta < x+y-1$, hence $h(x,y) \geqslant h(x,3)$.

Let $g(x) = x(x+2)\left(\frac{x-3+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{5}}}{\sqrt{x}} + \frac{x-2+\frac{2}{\sqrt{5}}}{\sqrt{x+2}}\right)h(x,3) = \frac{2\sqrt{3}}{3}x^2 + \frac{4\sqrt{15}+12\sqrt{5}-100-10\sqrt{3}}{15}x + \frac{304+8\sqrt{15}-72\sqrt{5}-60\sqrt{3}}{15}$. Since the axis of symmetry of $g(x)$ is $-\frac{4\sqrt{15}+12\sqrt{5}-100-10\sqrt{3}}{15}/\frac{2\sqrt{3}}{3}/2 \approx 2.16493 < 3$, hence $g(x) \geqslant g(3) = -12\sqrt{5}/5 + 4/15 + 4\sqrt{15}/3 \approx 0.06408 > 0$, implying $h(x,3) > 0$.

Analogously, it can be shown that $f(y,x) > 0$ holds. Therefore the lemma follows immediately. □

Lemma 4. *If $x \geqslant 3, y \geqslant 4$, then*

$$f(x,y) = \frac{x-3}{\sqrt{x}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x-2}{\sqrt{x+y-1}} + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}}\right) \geqslant 0.$$

Proof. Consider the gradient $\frac{\partial f(x,y)}{\partial y} = \frac{x-2+\frac{\sqrt{2}}{2}}{2(x+y-1)^{\frac{3}{2}}} - \frac{1}{2\sqrt{x}y^{\frac{3}{2}}}$. Since

$$\begin{aligned}&\frac{\partial f(x,y)}{\partial y}\left(\frac{x-2+\frac{\sqrt{2}}{2}}{2(x+y-1)^{\frac{3}{2}}} + \frac{1}{2\sqrt{x}y^{\frac{3}{2}}}\right)\left(4\,(x+y-1)^3\,x\right) \\ &= x\left(x-2+\frac{\sqrt{2}}{2}\right)^2 - \frac{(x+y-1)^3}{y^3} \geqslant x\left(x-2+\frac{\sqrt{2}}{2}\right)^2 - \frac{(x+3)^3}{4^3} \\ &= \frac{63(x-3)+(60\sqrt{2}-76)}{64}x^2 + (\frac{252}{64} - 2\sqrt{2})x + \frac{9(x-3)}{64} > 0,\end{aligned}$$

where the last inequality holds by $x \geqslant 3$, and consequently $\frac{\partial f(x,y)}{\partial y} > 0$.

Therefore, it is sufficient to show $f(x,4) \geqslant 0$. Let $q(x) = f(x,4)x(x+3)(\frac{x-3+\frac{1}{2}+\frac{\sqrt{2}}{2}}{\sqrt{x}} + \frac{x-2+\frac{\sqrt{2}}{2}}{\sqrt{x+3}}) = 2x^2 + \frac{10\sqrt{2}-51}{4}x + \frac{81-30\sqrt{2}}{4}$. Since the axis of symmetry of $q(x)$ is $(-\frac{10\sqrt{2}-51}{4})/4 \approx 2.3 < 3$, then $q(x) \geqslant q(3) = 0$. Note that $q(x)$ and $f(x,4)$ have the same sign, so the proof is complete. □

Lemma 5. *Let x_1, x_2, y be integers with $y \geqslant 3$, and let $f(x_1,x_2,y) = \frac{x_1+\frac{1}{\sqrt{2}}+k}{\sqrt{x_1+x_2}} + \frac{y-2+\frac{1}{\sqrt{2}}}{\sqrt{y}} + \frac{1}{\sqrt{y(x_1+x_2)}} - \frac{x_1+y-2+\sqrt{2}+k}{\sqrt{x_1+x_2+y-2}} - \frac{1}{2}$, then $f(x_1,x_2,y) > 0$, if either of the following is satisfied: (1) $x_1 \geqslant 1, 2 \leqslant x_2 \leqslant 10, k = 0$; (2) $x_1 = 0, 2 \leqslant x_2 \leqslant 6, k = \frac{1}{\sqrt{3}}$.*

Proof. **(1)** Let $h(t) = \frac{t-\sqrt{2}+2x_2}{2(t+x_2)^{\frac{3}{2}}}$ with $t \geqslant 1$. Note that $h' = \frac{-\frac{t}{2}-2x_2+\frac{3\sqrt{2}}{2}}{2(t+x_2)^{\frac{5}{2}}} < 0$. Then we have $\frac{\partial f(x_1,x_2,y)}{\partial x_1} = \frac{x_1-\sqrt{2}+2x_2+(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{y}})}{2(x_1+x_2)^{\frac{3}{2}}} - \frac{(x_1+y-2)-\sqrt{2}+2x_2}{2((x_1+y-2)+x_2)^{\frac{3}{2}}} > h(x_1) - h(x_1+y-2) > 0$, hence it suffices to prove $f(1,x_2,y) > 0$.

Let $A = \frac{y-2+\frac{1}{\sqrt{2}}}{\sqrt{y}} + \frac{1}{\sqrt{y(1+x_2)}} + \frac{1+\frac{1}{\sqrt{2}}}{\sqrt{1+x_2}} - \frac{1}{2}$, and $B = \frac{y-1+\sqrt{2}}{\sqrt{x_2+y-1}}$. First note that $\frac{1+\frac{1}{\sqrt{2}}}{\sqrt{1+x_2}} - \frac{1}{2} \geqslant \frac{1+\frac{1}{\sqrt{2}}}{\sqrt{1+10}} - \frac{1}{2} > 0$, which means $A > 0$. Let $g(x_2,y) = f(1,x_2,y)(A+B)y(x_2+1)(x_2+y-1) = (A^2 - B^2)y(x_2+1)(x_2+y-1) = a_1 y^{\frac{5}{2}} + a_2 y^2 + a_3 y^{\frac{3}{2}} + a_4 y + a_5 y^{\frac{1}{2}} + a_6$. It is easy to see $f(1,x_2,y) > 0$ if $g(x_2,y) > 0$.

Now let $b_1 = a_1$, and $b_i = \sqrt{3} b_{i-1} + a_i$ for $i = 2,3,4,5,6$. With assistance of computer, one can get all values of b_i for $2 \leqslant x_2 \leqslant 10$ as shown in Table 1, and conclude that $b_i > 0$ for each i.

Table 1. values of b_i calculated by computer

x_2	b_1	b_2	b_3	b_4	b_5	b_6
2	2.914	1.975	4.250	4.797	6.224	12.32
3	2.828	3.742	9.896	7.282	8.128	19.11
4	2.634	7.311	18.34	10.15	10.90	29.61
5	2.363	12.74	29.43	13.10	14.33	43.61
6	2.033	20.06	43.05	15.92	18.28	60.95
7	1.657	29.30	59.13	18.44	22.61	81.51
8	1.243	40.47	77.60	20.54	27.23	105.2
9	0.7967	53.58	98.40	22.11	32.07	131.9
10	0.3237	68.64	121.5	23.06	37.06	161.5

We find that $g(x_2,y) = b_1 y^{\frac{5}{2}} + a_2 y^2 + a_3 y^{\frac{3}{2}} + a_4 y + a_5 y^{\frac{1}{2}} + a_6 \geqslant \sqrt{3} b_1 y^2 + a_2 y^2 + a_3 y^{\frac{3}{2}} + a_4 y + a_5 y^{\frac{1}{2}} + a_6 = b_2 y^2 + a_3 y^{\frac{3}{2}} + a_4 y + a_5 y^{\frac{1}{2}} + a_6$. By repeating this process, we arrive at $g(x_2,y) \geqslant b_6 > 0$. Therefore $f(x_1,x_2,y) > 0$ holds.

(2) Since the result in this case can be proved by a similar argument as (1), so we omit the details and only give the values of b_i in Table 2.

Table 2. values of b_i calculated by computer

x_2	b_1	b_2	b_3	b_4	b_5	b_6
2	1.633	0.8524	0.5197	1.586	2.748	4.759
3	1.449	1.443	2.912	2.461	3.225	7.122
4	1.138	3.776	7.914	3.697	4.599	13.00
5	0.7443	7.944	15.36	4.898	6.596	22.15
6	0.2925	14.00	25.15	5.812	9.032	34.43

The lemma therefore follows easily. □

Lemma 6. *If $y \geqslant z \geqslant 2$, then $f(y,z) = (\frac{1}{\sqrt{yz}} - \frac{1}{\sqrt{(y+1)(z-1)}}) - (\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{z-1}} - \frac{1}{\sqrt{y+1}}) > 0$.*

Proof. Observe that $f(y,z) = (1-\frac{1}{\sqrt{z}})(1-\frac{1}{\sqrt{y}}) - (1-\frac{1}{\sqrt{z-1}})(1-\frac{1}{\sqrt{y+1}})$. It is obvious that $f(y,2) = \left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{y}}\right) > 0$.

Now we consider the case $y \geqslant z > 2$. Let $h(y,z) = \ln\left(1-\frac{1}{\sqrt{z}}\right)\left(1-\frac{1}{\sqrt{y}}\right) - \ln\left(1-\frac{1}{\sqrt{z-1}}\right)\left(1-\frac{1}{\sqrt{y+1}}\right)$ with $y \geqslant z > 2$. Note that $f(y,z)$ has the same sign with $h(y,z)$ since $\ln(x)$ is increasing for $x > 0$. Let $g(t) = -\ln(1-\frac{1}{\sqrt{t}})$ for $t > 1$, and $g''(t) = \frac{3\sqrt{t}-2}{4t^2(\sqrt{t}-1)^2} > 0$. Hence we obtain $h(y,z) = g(z-1) - g(z) - g(y) + g(y+1) > 0$ by Lemma 1. Therefore the lemma holds easily. □

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Lemma 7. *Let* $f(y,z) = \frac{y+k}{\sqrt{y}} + \frac{z+k}{\sqrt{z}} + \frac{k'}{\sqrt{yz}}$ *with* $y \geqslant z \geqslant 2$*, then* $f(y,z) > f(y+1, z-1)$ *if it meets one of the following conditions: (1)* $k \leqslant 0$ *and* $k' = 0$*; (2)* $k \leqslant -1$, $k' = 1$.

Proof. **(1)** Let $g(t) = -\frac{t+k}{\sqrt{t}}$ with $t \geqslant 1$, and note that $g''(t) = \frac{t-3k}{4t^{\frac{5}{2}}} > 0$. Then we have $f(y,z) - f(y+1, z-1) = g(z-1) - g(z) - g(y) + g(y+1) > 0$ by Lemma 1.
(2) Let $h(t) = -\frac{t+k+1}{\sqrt{t}}$ with $t \geqslant 1$, and note that $h''(t) = \frac{t-3k-3}{4t^{\frac{5}{2}}} > 0$. By Lemma 6, we have $f(y,z) - f(y+1, z-1) \geqslant \left(\frac{z+k}{\sqrt{z}} + \frac{y+k}{\sqrt{y}} - \frac{z-1+k}{\sqrt{z-1}} - \frac{y+1+k}{\sqrt{y+1}}\right) + \left(\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{z-1}} - \frac{1}{\sqrt{y+1}}\right) = h(z-1) - h(z) - h(y) + h(y+1) > 0$, where the last inequality follows from Lemma 1. □

Lemma 8. *If* $x \geqslant 4, y \geqslant 4$*, then* $f(x,y) = \frac{y-3+\frac{\sqrt{3}}{3}+\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{x}}}{\sqrt{y}} + \frac{x-4+\frac{\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x}} - \frac{x+y-7+\frac{2\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x+y-3}} - (\frac{1}{3} + \frac{1}{\sqrt{6}}) > 0.$

Proof. Since

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \frac{4-\sqrt{2}-\frac{\sqrt{3}}{3}-\frac{1}{\sqrt{y}}}{2x^{\frac{3}{2}}} - \frac{4-\sqrt{2}-\frac{2\sqrt{3}}{3}}{2(x+y-3)^{\frac{3}{2}}} + \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x+y-3}}\right)\\ &> \frac{4-\sqrt{2}-\frac{\sqrt{3}}{3}-\frac{1}{\sqrt{4}}}{2x^{\frac{3}{2}}} - \frac{4-\sqrt{2}-\frac{2\sqrt{3}}{3}}{2(x+y-3)^{\frac{3}{2}}} > \frac{\frac{\sqrt{3}}{3}-\frac{1}{2}}{2(x+y-3)^{\frac{3}{2}}} > 0,\end{aligned}$$

hence $f(x,y) \geqslant f(4,y)$. And moreover,

$$\begin{aligned}\frac{\partial f(4,y)}{\partial y} &= \frac{\frac{5}{2}-\frac{\sqrt{3}}{3}-\frac{\sqrt{2}}{2}}{2y^{\frac{3}{2}}} - \frac{4-\frac{2\sqrt{3}}{3}-\sqrt{2}}{2(y+1)^{\frac{3}{2}}} + \left(\frac{1}{2\sqrt{y}} - \frac{1}{2\sqrt{y+1}}\right)\\ &= \frac{\frac{5}{2}-\frac{\sqrt{3}}{3}-\frac{\sqrt{2}}{2}}{2y^{\frac{3}{2}}} + \frac{1}{4\eta^{\frac{3}{2}}} - \frac{4-\frac{2\sqrt{3}}{3}-\sqrt{2}}{2(y+1)^{\frac{3}{2}}} > \frac{\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}-1}{2(y+1)^{\frac{3}{2}}} > 0,\end{aligned}$$

where $y < \eta < y+1$. Therefore $f(x,y) \geqslant f(4,4) \approx 0.05036 > 0$. □

3. Transformations Decreasing Randić Index

To find tricyclic graphs with small Randić index, we provide some transformations which decrease Randić index of graphs. It is worth noting that all transformations defined here preserve the number of vertices and edges of a graph. For simplicity, we will not repeat this property in the sequel.

Theorem 2 (Transformation I)**.** *Suppose* G *is a graph with given two adjacent vertices* u *and* v *such that* $N_G(v) \cap N_G(u) = \emptyset$*. Let graph* $G' = G - \{vw | w \in N_G(v) \backslash u\} + \{uw | w \in N_G(v) \backslash u\}$*, and we write the transformation as* $G' = \Gamma(G, u, v)$*. Then* $R(G) > R(G')$ *if* G *meets one of the following conditions:*

(1) v *has only one non-pendant neighbor* u *and* $d_G(v) \geqslant 2$*;*
(2) v *has two non-pendant neighbors* u *and* w *with* $d_G(u) \geqslant d_G(w)$*;*
(3) v *has three non-pendant neighbors* u, v_1, v_2 *and* u *has three non-pendant neighbors* v, u_1, u_2 *such that* $\frac{1}{\sqrt{d_G(u_1)}} + \frac{1}{\sqrt{d_G(u_2)}} + \frac{1}{\sqrt{d_G(v_2)}} + \frac{1}{\sqrt{d_G(v_2)}} \geqslant \frac{2}{\sqrt{5}}$*;*
(4) $d_G(v) \geqslant 4$ *and* u *has three non-pendant neighbors* v, u_1, u_2 *with* $\frac{1}{\sqrt{d_G(u_1)}} + \frac{1}{\sqrt{d_G(u_2)}} \geqslant \frac{1}{\sqrt{2}}$.

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Proof. Let $\widetilde{E}$ be the edge set of G that are not incident with u or v, and $\widetilde{R} = \sum_{pq\in\widetilde{E}} \frac{1}{\sqrt{d_G(p)d_G(q)}}$. And let $x = d_G(u), y = d_G(v), \Delta = R(G) - R(G')$. Then

$$R(G) = \widetilde{R} + \sum_{z\in N_G(u)\backslash v} \frac{1}{\sqrt{x d_G(z)}} + \sum_{z\in N_G(v)\backslash u} \frac{1}{\sqrt{y d_G(z)}} + \frac{1}{\sqrt{xy}}$$

$$R(G') = \widetilde{R} + \left(\sum_{z\in N_G(u)\backslash v} \frac{1}{\sqrt{d_G(z)}} + \sum_{z\in N_G(v)\backslash u} \frac{1}{\sqrt{d_G(z)}} + 1 \right) \frac{1}{\sqrt{x+y-1}}.$$

(1) Note that $x, y \geqslant 2$ and v has $y-1$ pendant neighbors. Then

$$\begin{aligned}
\Delta &= \frac{y-1}{\sqrt{y}} + \frac{1}{\sqrt{xy}} - \frac{y}{\sqrt{x+y-1}} + \sum_{z\in N_G(u)\backslash v} \frac{1}{\sqrt{d_G(z)}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}}\right) \\
&\geqslant \frac{y-1}{\sqrt{y}} + \frac{1}{\sqrt{xy}} - \frac{y}{\sqrt{x+y-1}} \\
&= \left(\frac{y-2+\frac{2}{\sqrt{x}}}{\sqrt{y}} - \frac{y-1+\frac{1}{\sqrt{x}}}{\sqrt{x+y-1}} \right) + \left(1 - \frac{1}{\sqrt{x}}\right)\left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x+y-1}}\right) > 0,
\end{aligned}$$

where the last inequality holds by Lemma 2 and $\frac{1}{\sqrt{y}} > \frac{1}{\sqrt{x+y-1}}$.

(2) Notice that $x \geqslant 2, y \geqslant 2$, $d_G(w) \leqslant x$, and v has $y-2$ pendant neighbors in this case. Then

$$\begin{aligned}
\Delta &\geqslant \frac{y-2}{\sqrt{y}} + \frac{1}{\sqrt{xy}} - \frac{y-1}{\sqrt{y+x-1}} + \frac{1}{\sqrt{d_G(w)}} \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+x-1}} \right) \\
&+ \sum_{z\in N_G(u)\backslash v} \frac{1}{\sqrt{d_G(z)}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}}\right) \\
&\geqslant \frac{y-2}{\sqrt{y}} + \frac{1}{\sqrt{xy}} - \frac{y-1}{\sqrt{y+x-1}} + \frac{1}{\sqrt{x}} \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+x-1}} \right) \\
&= \frac{y-2+\frac{2}{\sqrt{x}}}{\sqrt{y}} - \frac{y-1+\frac{1}{\sqrt{x}}}{\sqrt{y+x-1}} > 0,
\end{aligned}$$

where the last inequality holds by Lemma 2.

(3) Let $k_1 = \frac{1}{\sqrt{d_G(u_1)}} + \frac{1}{\sqrt{d_G(u_2)}}$, $k_2 = \frac{1}{\sqrt{d_G(v_1)}} + \frac{1}{\sqrt{d_G(v_2)}}$, and $p(x,y) = \frac{x-3}{\sqrt{x}} + \frac{y-3}{\sqrt{y}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x+y-5}{\sqrt{x+y-1}}$. Note that $x, y \geqslant 3$ and $k_1 + k_2 \geqslant \frac{2}{\sqrt{5}}$ from the condition. Then

$$\begin{aligned}
\Delta &= p(x,y) + \left(\frac{k_1}{\sqrt{x}} - \frac{k_1}{\sqrt{x+y-1}} \right) + \left(\frac{k_2}{\sqrt{y}} - \frac{k_2}{\sqrt{x+y-1}} \right) \\
&= \frac{k_1}{k_1+k_2} \left[p(x,y) + \left(\frac{k_1+k_2}{\sqrt{x}} - \frac{k_1+k_2}{\sqrt{x+y-1}} \right) \right] \\
&+ \frac{k_2}{k_1+k_2} \left[p(x,y) + \left(\frac{k_1+k_2}{\sqrt{y}} - \frac{k_1+k_2}{\sqrt{x+y-1}} \right) \right] > 0,
\end{aligned}$$

where the last inequality holds by Lemma 3.

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(4) Obviously $x \geqslant 3, y \geqslant 4$ and $N_G(v)\backslash u \neq \varnothing$. Then

$$\begin{aligned}\Delta = & \frac{x-3}{\sqrt{x}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x-2}{\sqrt{x+y-1}} \\ & + \sum_{z \in N_G(v)\backslash u} \frac{1}{\sqrt{d_G(z)}} \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x+y-1}} \right) \\ & + \left(\frac{1}{\sqrt{d_G(u_1)}} + \frac{1}{\sqrt{d_G(u_2)}} \right) \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}} \right) \\ & > \frac{x-3}{\sqrt{x}} + \frac{1}{\sqrt{x}\sqrt{y}} - \frac{x-2}{\sqrt{x+y-1}} + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+y-1}} \right) \geqslant 0,\end{aligned}$$

where the last inequality holds by Lemma 4. □

Theorem 3 (Transformation II). *Suppose G is a graph, and there is a cycle $C = v_0v_1v_2v_0$ in G with $d_G^1(v_1) \geqslant 1$ and $d_G^2(v_1) = 2$. Let G' arise from G by moving all pendant neighbors from v_1 to v_0. Then $R(G) > R(G')$ if either of the following is satisfied:*

(1) $d_G^1(v_0) \geqslant 1, 2 \leqslant d_G^2(v_0) \leqslant 10$;
(2) $2 \leqslant d_G^2(v_0) \leqslant 6$ and there is a vertex $u \in N_G(v_0)\backslash\{v_1, v_2\}$ with $d_G(u) \leqslant 3$.

Proof. Let $x_1 = d_G^1(v_0)$, $x_2 = d_G^2(v_0)$ and $y = d_G(v_1)$. Let $f(x) = \frac{1}{\sqrt{x}}$, then $f''(x) = \frac{3}{4x^{\frac{5}{2}}} > 0$. And note that $2 + (x_1 + x_2 + y - 2) = y + (x_1 + x_2)$, $2 \leqslant x_1 + x_2 \leqslant (x_1 + x_2 + y - 2)$ and $2 \leqslant y \leqslant (x_1 + x_2 + y - 2)$, then we get $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{x_1+x_2}} - \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{x_1+x_2+y-2}} \geqslant 0$ by Lemma 1. Then

$$\begin{aligned}& R(G) - R(G') \\ = & \frac{x_1}{\sqrt{x_1+x_2}} + \frac{y-2}{\sqrt{y}} + \frac{1}{\sqrt{y\,(x_1+x_2)}} - \frac{x_1 + y - 2 + \frac{1}{\sqrt{2}}}{\sqrt{x_1+x_2+y-2}} \\ & + \sum_{z \in N_G^2(v_0)\backslash\{v_1,v_2\}} \frac{1}{\sqrt{d_G(z)}} \left(\frac{1}{\sqrt{x_1+x_2}} - \frac{1}{\sqrt{x_1+x_2+y-2}} \right) \\ & - \frac{1}{\sqrt{d_G(v_2)}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{x_1+x_2}} - \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{x_1+x_2+y-2}} \right) \\ \geqslant & \frac{x_1}{\sqrt{x_1+x_2}} + \frac{y-2}{\sqrt{y}} + \frac{1}{\sqrt{y\,(x_1+x_2)}} - \frac{x_1 + y - 2 + \frac{1}{\sqrt{2}}}{\sqrt{x_1+x_2+y-2}} \\ & + \sum_{z \in N_G^2(v_0)\backslash\{v_1,v_2\}} \frac{1}{\sqrt{d_G(z)}} \left(\frac{1}{\sqrt{x_1+x_2}} - \frac{1}{\sqrt{x_1+x_2+y-2}} \right) \\ & - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{x_1+x_2}} - \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{x_1+x_2+y-2}} \right),\end{aligned}$$

where the last inequality follows by $d_G(v_2) \geqslant 2$.
(1) Note that $x_1 \geqslant 1, 2 \leqslant x_2 \leqslant 10$ and $y \geqslant 3$. Then

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$$
\begin{aligned}
&R(G)-R(G')\\
&\geqslant \frac{x_1}{\sqrt{x_1+x_2}}+\frac{y-2}{\sqrt{y}}+\frac{1}{\sqrt{y\,(x_1+x_2)}}-\frac{x_1+y-2+\frac{1}{\sqrt{2}}}{\sqrt{x_1+x_2+y-2}}\\
&-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{x_1+x_2}}-\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{x_1+x_2+y-2}}\right)\\
&=\frac{x_1+\frac{1}{\sqrt{2}}}{\sqrt{x_1+x_2}}+\frac{y-2+\frac{1}{\sqrt{2}}}{\sqrt{y}}+\frac{1}{\sqrt{y\,(x_1+x_2)}}-\frac{x_1+y-2+\sqrt{2}}{\sqrt{x_1+x_2+y-2}}-\frac{1}{2}>0,
\end{aligned}
$$

where the last inequality holds by (1) of Lemma 5.
(2) Obviously, the assertion holds if $d_G^1(v_0)\geqslant 1$ by (1). Hence we only need to consider the case $d_G^1(v_0)=0$. Note that $x_1=0, 2\leqslant x_2\leqslant 6, y\geqslant 3, d_G(u)\leqslant 3$, then

$$
\begin{aligned}
&R(G)-R(G')\\
&\geqslant \left(\frac{y-2}{\sqrt{y}}+\frac{1}{\sqrt{yx_2}}\right)-\frac{y-2+\frac{1}{\sqrt{2}}}{\sqrt{x_2+y-2}}+\frac{1}{\sqrt{d_G(u)}}\left(\frac{1}{\sqrt{x_2}}-\frac{1}{\sqrt{x_2+y-2}}\right)\\
&-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{x_2}}-\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{x_2+y-2}}\right)\\
&\geqslant \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}}{\sqrt{x_2}}+\frac{y-2+\frac{1}{\sqrt{2}}}{\sqrt{y}}+\frac{1}{\sqrt{yx_2}}-\frac{y-2+\sqrt{2}+\frac{1}{\sqrt{3}}}{\sqrt{x_2+y-2}}-\frac{1}{2}>0,
\end{aligned}
$$

where the last inequality holds by (2) of Lemma 5. □

Theorem 4 (Transformation III)**.** *Suppose G is a graph with given two vertices u and v such that $d_G(u)\geqslant d_G(v)\geqslant 2$ and $d_G^1(u)\geqslant d_G^1(v)\geqslant 1$. Let G' be graph obtained from G by moving one pendant neighbor of v to u, then $R(G)>R(G')$ if $d_G^2(v)=d_G^2(u)$ and $\sum_{z\in N_G^2(u)\setminus v}\frac{1}{\sqrt{d_G(z)}}=\sum_{z\in N_G^2(v)\setminus u}\frac{1}{\sqrt{d_G(z)}}$.*

Proof. Let $\widetilde{E}$ be the edge set of G that are not incident with u or v, and $x=d_G(u)\geqslant 2, y=d_G(v)\geqslant 2$. And observe that $-|d_G^2(u)|+\sum_{z\in N_G^2(u)\setminus v}\frac{1}{\sqrt{d_G(z)}}=-|d_G^2(v)|+\sum_{z\in N_G^2(v)\setminus u}\frac{1}{\sqrt{d_G(z)}}$, denoted by k.

If u and v are not adjacent, then $k=\sum_{z\in N_G^2(u)}(\frac{1}{\sqrt{d_G(z)}}-1)\leqslant 0$, so

$$
\begin{aligned}
R(G)&=\sum_{pq\in\widetilde{E}}\frac{1}{\sqrt{d_G(p)d_G(q)}}+\sum_{z\in N_G^2(u)}\frac{1}{\sqrt{xd_G(z)}}+\frac{x-|N_G^2(u)|}{\sqrt{x}}\\
&+\sum_{z\in N_G^2(v)}\frac{1}{\sqrt{yd_G(z)}}+\frac{y-|N_G^2(v)|}{\sqrt{y}}\\
&=\sum_{pq\in\widetilde{E}}\frac{1}{\sqrt{d_G(p)d_G(q)}}+\frac{x+k}{\sqrt{x}}+\frac{y+k}{\sqrt{y}}\\
R(G')&=\sum_{pq\in\widetilde{E}}\frac{1}{\sqrt{d_G(p)d_G(q)}}+\frac{x+1+k}{\sqrt{x+1}}+\frac{y-1+k}{\sqrt{y-1}}.
\end{aligned}
$$

Therefore we get $R(G)>R(G')$ by (1) of Lemma 7.

If u and v are adjacent, then $k = -1 + \sum_{z \in N_G^2(u) \setminus v} (\frac{1}{\sqrt{d_G(z)}} - 1) \leqslant -1$, so

$$\begin{aligned} R(G) = & \sum_{pq \in \tilde{E}} \frac{1}{\sqrt{d_G(p) d_G(q)}} + \sum_{z \in N_G^2(u) \setminus v} \frac{1}{\sqrt{x d_G(z)}} + \sum_{z \in N_G^2(v) \setminus u} \frac{1}{\sqrt{y d_G(z)}} \\ & + \frac{x - |N_G^2(u)|}{\sqrt{x}} + \frac{y - |N_G^2(v)|}{\sqrt{y}} + \frac{1}{\sqrt{xy}} \\ & = \sum_{pq \in \tilde{E}} \frac{1}{\sqrt{d_G(p) d_G(q)}} + \frac{x+k}{\sqrt{x}} + \frac{y+k}{\sqrt{y}} + \frac{1}{\sqrt{xy}} \\ R(G') = & \sum_{pq \in \tilde{E}} \frac{1}{\sqrt{d_G(p) d_G(q)}} + \frac{x+1+k}{\sqrt{x+1}} + \frac{y-1+k}{\sqrt{y-1}} + \frac{1}{\sqrt{(x+1)(y-1)}}, \end{aligned}$$

hence we have $R(G) > R(G')$ by (2) of Lemma 7. Thus the proof is complete. □

Lemma 9. *Suppose G is a graph with two vertices u and v such that $N_G^2(u) \setminus v = N_G^2(v) \setminus u$ and $d_G^1(u) > 0, d_G^1(v) > 0$. Let G' be a graph obtained from G by moving all pendant neighbors of v to u, then $R(G) > R(G')$.*

Proof. Observe that if we exchange pendant neighbors of v and u, $R(G)$ does not change. Hence the assertion holds easily from Theorem 4. □

4. Main Results

4.1. Undeletable subgraph and Classification of tricyclic graphs

We need the following important definition to start our analysis.

Definition 1. *Suppose G is a cyclic graph, then the undeletable subgraph $\phi(G)$ of G is defined as a maximum subgraph without pendant vertex, i.e., the subgraph arising from G by deleting all pendant vertices recursively.*

Obviously that $\phi(G)$ is connected and $\delta(\phi(G)) \geqslant 2$. Moreover, undeletable subgraph of a graph is unique. And it is easy to verify that the undeletable subgraph of a unicyclic graph is a cycle.

With the definition and Theorem 2, we are able to prove the following crucial lemma:

Lemma 10. *Suppose G is a cyclic (n, m)-graph with undeletable subgraph $\phi(G)$. Then there exists a (n, m)-graph G' such that $R(G) > R(G')$ if there is a vertex $w \in V(G) \setminus V(\phi(G))$ with $d_G(w) > 1$.*

Proof. Let $\hat{G} = G - E(\phi(G))$ by deleting all edges of $\phi(G)$. By the definition of $\phi(G)$, $\hat{G}$ contains no cycle, i.e., $\hat{G}$ is a forest.

Let T be the tree of $\hat{G}$ containing w. We claim that T contains exactly one vertex of $V(\phi(G))$. First, assume that T contains no vertex of $V(\phi(G))$, then T is not connected with vertices of $V(\phi(G))$ in G, thereby G is disconnected which is a contradiction. Now assume that T contains at least two vertices of $V(\phi(G))$, and denote two of which by x and y, then there is a unique path in T that connects them containing a vertex $z \notin V(\phi(G))$ since $E(T) \bigcap E(\phi(G)) = \varnothing$. And there exists a path in $\phi(G)$ that connects u and v since $\phi(G)$ is connected. Therefore vertex z lies on a cycle of G, implying that it belongs to $V(\phi(G))$, which contradicts the fact $z \notin V(\phi(G))$. So T contains exactly one vertex of $V(\phi(G))$, say v_0.

Let $P = v_0 v_1 \ldots v_h$ be the longest path from v_0 to all other vertices in T. Note that $h \geqslant 2$ because a path from v_0 to a pendant vertex containing w is of length at least two. Hence $d_G(v_{h-1}) \geqslant 2$, $d_G(v_{h-2}) \geqslant 2$, and v_{h-2} is the only non-pendant neighbor of v_{h-1}. Then by (1) of Theorem 2, there is an (n, m)-graph $G' = \Gamma(G, v_{h-2}, v_{h-1})$ such that $R(G) > R(G')$. □

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For a graph G with undeletable subgraph $\phi(G)$, if $d_G(w) = 1$ for each $w \in V(G)\backslash V(\phi(G))$, i.e., each $v \in V(\phi(G))$ only has pendant neighbors in $V(G)\backslash V(\phi(G))$, it is said to be a pendant-maximized graph.

Lemma 11. *Suppose G is a pendant-maximized tricyclic $(n, n+2)$-graph with undeletable subgraph $\phi(G)$. If there is a vertex $v \in V(\phi(G))$ with exactly two non-adjacent neighbors in $\phi(G)$, then there is an $(n, n+2)$-graph G' such that $R(G) > R(G')$; otherwise, $\phi(G)$ must be one of the 15 graphs as shown in Figure 2,3,4 up to isomorphism.*

Proof. **(1)** We first prove the "if" part. Without loss of generality, let the neighbors of v in $\phi(G)$ be u and w with $d_G(u) \geqslant d_G(w)$. Observe that $d^2_G(v) = 2$ because G is pendant-maximized and v has two neighbors in $\phi(G)$. Hence $N_G(v) \cap N_G(u) = \varnothing$ since u and w are non-adjacent. Therefore, there is an $(n, n+2)$-graph $G' = \Gamma(G, u, v)$ such that $R(G) > R(G')$ by (2) of Theorem 2.
(2) Now the "otherwise" part. By the definition of undeletable subgraph, $\phi(G)$ is a tricyclic graph, that is, $|E(\phi(G))| = |V(\phi(G))| + 2$ and $\delta(\phi(G)) \geqslant 2$. In the remaining argument, all degree and neighbors are constrained in $\phi(G)$.

We claim that $4 \leqslant |V(\phi(G))| \leqslant 10$, where the lower bound is obvious by checking graphs of order 1 to 4. We first show there are at most 6 vertices of degree 2 in $\phi(G)$. Notice that each vertex v of degree 2 must lie on a cycle of length 3 because the neighbors of v must be adjacent. Moreover, each cycle of length 3 contains at most 2 vertices of degree 2, otherwise the cycle is disconnected with other parts of $\phi(G)$. Since there are at most 3 edge-disjoint cycles in a tricyclic graph, thereby at most 3 edge-disjoint cycles of length 3. Hence by Handshaking Lemma, we have $2 \times 6 + 3 \times (|V(\phi(G))| - 6) \leqslant (|V(\phi(G))| + 2) \times 2$, implying $|V(\phi(G))| \leqslant 10$.

Therefore we can list all possible graph structures for $\phi(G)$ with $4 \leqslant |V(\phi(G))| \leqslant 10$, which are exactly those shown in Figures 2,3,4. Thus the proof is complete. □

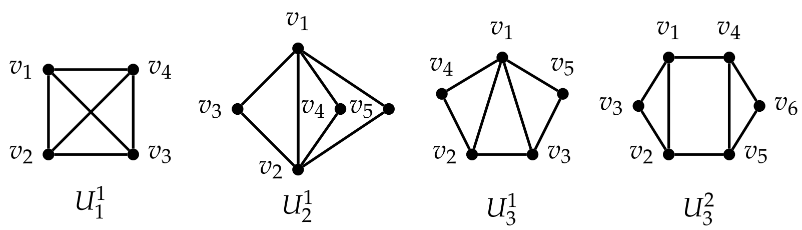

Figure 2. graphs containing no edge-disjoint cycle

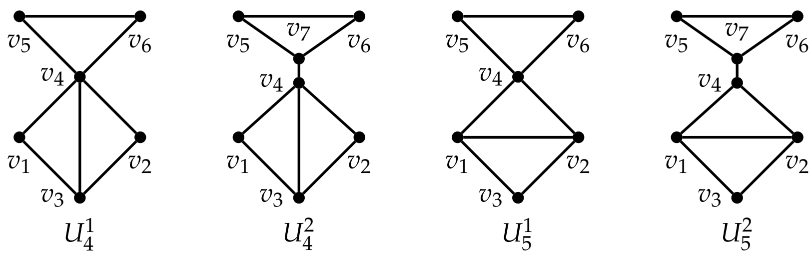

Figure 3. graphs containing one edge-disjoint cycle

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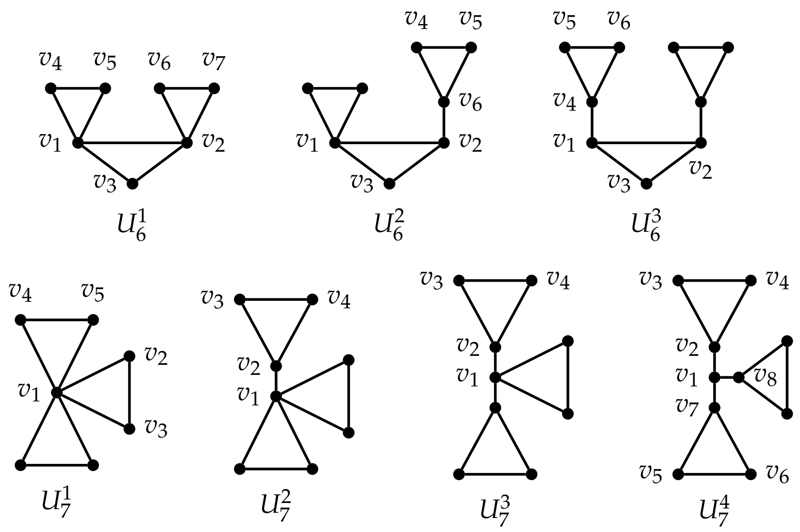

Figure 4. graphs containing three edge-disjoint cycles

Let $\mathbb{TR}_i^j(n)$ be the set of n-vertex tricyclic graphs obtained from U_i^j by attaching $n - |V(U_i^j)|$ pendant vertices to U_i^j. It is evident that graphs belonging to $\mathbb{TR}_i^j(n)$ are pendant-maximized. On the other hand, if a pendant-maximized tricyclic graph G with $\phi(G) \cong U_i^j$, then $G \in \mathbb{TR}_i^j(n)$.

4.2. *Relations between* $\mathbb{TR}_i^j(n)$

Suppose $\mathbb{A}$ and $\mathbb{B}$ are two graph sets, and if for any graph $G \in \mathbb{A}$, there is a graph $G' \in \mathbb{B}$ such that $R(G) > R(G')$, then this relation is written as $R(\mathbb{A}) > R(\mathbb{B})$ or $R(\mathbb{B}) < R(\mathbb{A})$. Our remaining task is to figure out the above described relations between all $\mathbb{TR}_i^j(n)$.

Lemma 12. $R(\mathbb{TR}_7^4(n)) > R(\mathbb{TR}_7^3(n)) > R(\mathbb{TR}_7^2(n)) > R(\mathbb{TR}_7^1(n))$.

Proof. We prove the results in the order of left to right.
(1) Suppose G is a graph in $\mathbb{TR}_7^4(n)$ with undeletable subgraph labelled as U_7^4 in Figure 4. It may be assumed that $d_G^1(v_4) = 0$; otherwise, by Lemma 9, pendant neighbors of v_4 can be moved to v_3 without increasing $R(G)$ since $N_G^2(v_3)\backslash v_4 = N_G^2(v_4)\backslash v_3$. Moreover, we may assume $d_G^1(v_6) = 0$ by similarly reasoning. Then clearly, $d_G(v_4) = d_G(v_6) = 2$.

Consider first when $d_G^1(v_2) > 0$. If $d_G^1(v_3) > 0$, by (1) of Theorem 3, moving pendant neighbors of v_3 to v_2 reduces $R(G)$. So we may assume $d_G^1(v_3) = 0$. Since $d_G^2(v_2) = d_G^2(v_1) = 3$, and $\frac{1}{\sqrt{v_3}} + \frac{1}{\sqrt{v_4}} + \frac{1}{\sqrt{v_7}} + \frac{1}{\sqrt{v_8}} > \frac{2}{\sqrt{2}} > \frac{2}{\sqrt{5}}$, therefore graph $G' = \Gamma(G, v_1, v_2)$ satisfies $R(G) > R(G')$ appealing to (3) of Theorem 2.

Now we turn to the case of $d_G^1(v_2) = 0$, that is, $d_G(v_2) = 3$. Note that $d_G^2(v_7) = d_G^2(v_1) = 3$, and $\frac{1}{\sqrt{v_2}} + \frac{1}{\sqrt{v_8}} + \frac{1}{\sqrt{v_5}} + \frac{1}{\sqrt{v_6}} > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > \frac{2}{\sqrt{5}}$, thus graph $G'' = \Gamma(G, v_1, v_7)$ satisfies $R(G) > R(G'')$ again by (3) of Theorem 2. It is not difficult to check that G' and G'' both belong to $\mathbb{TR}_7^3(n)$. Thus we have $R(\mathbb{TR}_7^4(n)) > R(\mathbb{TR}_7^3(n))$.
(2) Let $G \in \mathbb{TR}_7^3(n)$ be a graph with undeletable subgraph as U_7^3 in Figure 4. As before, we may assume $d_G^1(v_4) = 0$. Let $G' = \Gamma(G, v_2, v_1)$, and obviously $G' \in \mathbb{TR}_7^2(n)$. Since $d_G(v_1) \geqslant 4$, $d_G^2(v_2) = 3$ and $\frac{1}{\sqrt{v_3}} + \frac{1}{\sqrt{v_4}} \geqslant \frac{1}{\sqrt{2}}$, then we get $R(G) > R(G')$ by (4) of Theorem 2. Thus $R(\mathbb{TR}_7^3(n)) > R(\mathbb{TR}_7^2(n))$ holds.
(3) Using similar arguments as (2), there is a graph $G' = \Gamma(G, v_2, v_1) \in \mathbb{TR}_7^1(n)$ such that $R(G) > R(G')$. Hence $R(\mathbb{TR}_7^3(n)) > R(\mathbb{TR}_7^2(n))$. □

Lemma 13. $R(\mathbb{TR}_6^3(n)) > R(\mathbb{TR}_6^2(n)) > R(\mathbb{TR}_6^1(n))$.

Proof. We prove the relations from left to right.
(1) Let $G \in \mathbb{TR}_6^3(n)$ be a graph with $\phi(G)$ as U_6^3 in Figure 4. As before, we assume that $d_G^1(v_6) = 0$.

Let $S = \frac{1}{\sqrt{v_2}} + \frac{1}{\sqrt{v_3}} + \frac{1}{\sqrt{v_5}} + \frac{1}{\sqrt{v_6}}$. If $d_G^1(v_2) = 0$, then $S > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > \frac{2}{\sqrt{5}}$. And if $d_G^1(v_2) > 0$, we may assume $d_G^1(v_3) = 0$; otherwise $R(G)$ can be reduced by moving pendant neighbors of v_3 to v_2 according to (1) of Theorem 3. Then we have $S > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > \frac{2}{\sqrt{5}}$. Moreover, $d_G^2(v_1) = d_G^2(v_4) = 3$, hence there is a graph $G' = \Gamma(G, v_1, v_4)$ with $R(G) > R(G')$ from (3) of Theorem 2. And it is evident that $G' \in \mathbb{TR}_6^2(n)$, thus we obtain $R(\mathbb{TR}_6^3(n)) > R(\mathbb{TR}_6^2(n))$.
(2) Suppose G is a graph in $\mathbb{TR}_6^2(n)$ with undeletable subgraph as U_6^2 in Figure 4. Using analogous arguments as (1), we can show that $\frac{1}{\sqrt{v_1}} + \frac{1}{\sqrt{v_3}} + \frac{1}{\sqrt{v_4}} + \frac{1}{\sqrt{v_5}} > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} > \frac{2}{\sqrt{5}}$. Additionally, note that $d_G^2(v_2) = d_G^2(v_6) = 3$. By (3) of Theorem 2, there is a graph $G' = \Gamma(G, v_2, v_6) \in \mathbb{TR}_6^1(n)$ such that $R(G) > R(G')$. So it follows $R(\mathbb{TR}_6^2(n)) > R(\mathbb{TR}_6^1(n))$ easily. □

Lemma 14. *$R(\mathbb{TR}_4^2(n)) > R(\mathbb{TR}_4^1(n))$, $R(\mathbb{TR}_7^1(n)) > R(\mathbb{TR}_4^1(n))$, $R(\mathbb{TR}_6^1(n)) > R(\mathbb{TR}_4^1(n))$.*

Proof. We prove the three relations in the order of left to right.
(1) Let $G \in \mathbb{TR}_4^2(n)$ with undeletable subgraph as U_4^2 in Figure 3. As before, we assume $d_G^1(v_6) = 0$, i.e., $d_G(v_6) = 2$. And notice that $d_G(v_4) \geqslant 4, d_G^2(v_7) = 3$. Then by (4) of Theorem 2, graph $G' = \Gamma(G, v_4, v_7) \in \mathbb{TR}_4^1(n)$ satisfies $R(G) > R(G')$. Thus $R(\mathbb{TR}_4^2(n)) > R(\mathbb{TR}_4^1(n))$ holds clearly.
(2) Let $G \in \mathbb{TR}_7^1(n)$ with undeletable subgraph as U_7^1 in Figure 4. As before, we assume $d_G^1(v_3) = d_G^1(v_5) = 0$. Note that $d_G^2(v_1) = 6$ and $v_5 \in N_G(v_1) \backslash \{v_2, v_3\}$. If $d_G^1(v_2) > 0$, then by (2) of Theorem 3, $R(G)$ can be reduced by moving pendant neighbors of v_2 to v_1. Similarly, it holds for v_4. So we may assume $d_G^1(v_2) = d_G^1(v_4) = 0$.

Let $d_i = d_G(v_i)$, and we have $d_2 = d_3 = d_4 = d_5 = 2$ and $d_1 \geqslant 6$. Let $G' = G - v_2v_3 + v_2v_5$, then $R(G) - R(G') = (\frac{1}{\sqrt{d_1d_3}} + \frac{1}{\sqrt{d_1d_5}} + \frac{1}{\sqrt{d_2d_3}} + \frac{1}{\sqrt{d_4d_5}}) - (\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_1(d_5+1)}} + \frac{1}{\sqrt{d_2(d_5+1)}} + \frac{1}{\sqrt{d_4(d_5+1)}}) = \frac{1}{\sqrt{d_1}}(\sqrt{2} - 1 - \frac{1}{\sqrt{3}}) + 1 - \frac{2}{\sqrt{6}} \geqslant \frac{1}{\sqrt{6}}(\sqrt{2} - 1 - \frac{1}{\sqrt{3}}) + 1 - \frac{2}{\sqrt{6}} \approx 0.1169 > 0$. It is easy to check that $G' \in \mathbb{TR}_4^1$. Thus $R(\mathbb{TR}_7^1(n)) > R(\mathbb{TR}_4^1(n))$ follows.
(3) Let $G \in \mathbb{TR}_6^1(n)$ with undeletable subgraph as U_6^1 in Figure 4. As before, we assume $d_G^1(v_5) = d_G^1(v_7) = 0$. Moreover, we may assume $d_G^1(v_3) = 0$; Otherwise, note that $d_G^2(v_1) = 4$ and $v_5 \in N_G(v_1) \backslash \{v_2, v_3\}$ with $d_G(v_5) = 2$, then appealing to (2) of Theorem 3, moving pendant neighbors of v_3 to v_1 will decrease $R(G)$.

Now notice that $d_G^2(v_2) = 4$, $d_G(v_3) = 2$ and $v_3 \in N_G(v_2) \backslash \{v_6, v_7\}$, we can decrease $R(G)$ by moving pendant neighbors of v_6 to v_2 if $d_G^1(v_6) > 0$. Hence we only have to consider the case $d_G(v_6) = d_G(v_7) = 2$.

Let $d_i = d_G(v_i)$, and notice that $d_1 \geqslant 4, d_2 \geqslant 4$. Let $G' = G - v_6v_7 + v_6v_1$, then $R(G) - R(G') > (\frac{1}{\sqrt{d_2d_7}} + \frac{1}{\sqrt{d_6d_7}}) - (\frac{1}{\sqrt{d_2}} + \frac{1}{\sqrt{d_6(d_1+1)}}) = \frac{1}{\sqrt{d_2}}(\frac{1}{\sqrt{2}} - 1) + \frac{1}{2} - \frac{1}{\sqrt{2(d_1+1)}} \geqslant \frac{1}{\sqrt{4}}(\frac{1}{\sqrt{2}} - 1) + \frac{1}{2} - \frac{1}{\sqrt{10}} > 0$. It is easy to see that $G' \in \mathbb{TR}_4^1(n)$, thus $R(\mathbb{TR}_6^1(n)) > R(\mathbb{TR}_4^1(n))$ holds. □

Lemma 15. *$R(\mathbb{TR}_5^2(n)) > R(\mathbb{TR}_5^1(n)) > R(\mathbb{TR}_3^1(n))$, $R(\mathbb{TR}_4^1(n)) > R(\mathbb{TR}_3^1(n))$, $R(\mathbb{TR}_3^2(n)) > R(\mathbb{TR}_3^1(n))$.*

Proof. We prove the assertions from left to right.
(1) Let $G \in \mathbb{TR}_5^2(n)$ with undeletable subgraph as U_5^2 in Figure 3. As before, we assume $d_G^1(v_6) = 0$. Observe that $N_G^2(v_1) \backslash v_2 = N_G^2(v_2) \backslash v_1 = \{v_3, v_4\}$. By Lemma 9, we can move pendant neighbors of v_2 to v_1 and do not increase $R(G)$ if $d_G^1(v_2) > 0$. Therefore we may assume that $d_G^1(v_2) = 0$. Notice that $d_G^2(v_4) = d_G^2(v_7) = 3$ and $\frac{1}{\sqrt{v_1}} + \frac{1}{\sqrt{v_2}} + \frac{1}{\sqrt{v_5}} + \frac{1}{\sqrt{v_6}} > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > \frac{2}{\sqrt{5}}$, hence there is $G' = \Gamma(G, v_4, v_7) \in \mathbb{TR}_5^1(n)$ such that $R(G) > R(G')$ from (3) of Theorem 2. Thus we obtain $R(\mathbb{TR}_5^2(n)) > R(\mathbb{TR}_5^1(n))$.
(2) Let $G \in \mathbb{TR}_5^1(n)$ with undeletable subgraph as U_5^1 in Figure 3. By analogous arguments as (1), we may assume that $d_G^1(v_6) = d_G^1(v_2) = 0$, that is, $d_G(v_6) = 2, d_G(v_2) = 3$. And note that $d_G^2(v_4) = 4$

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and $v_2 \in N_G(v_4) \backslash \{v_5, v_6\}$. Then according to (2) of Theorem 3, moving pendant neighbors of v_5 to v_4 reduces $R(G)$ if $d_G^1(v_5) > 0$. So we may assume that $d_G^1(v_5) = 0$.

Let $G' = G - v_5v_6 + v_5v_1$, and let $d_i = d_G(v_i)$. Note that $d_1 \geqslant 3, d_4 \geqslant 4, d_G(v_5) = d_G(v_6) = 2$. Then $R(G) - R(G') > (\frac{1}{\sqrt{d_4d_6}} + \frac{1}{\sqrt{d_5d_6}}) - (\frac{1}{\sqrt{d_4}} + \frac{1}{\sqrt{(d_1+1)d_5}}) = \frac{1}{\sqrt{d_4}}(\frac{1}{\sqrt{2}} - 1) + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{2(d_1+1)}} \geqslant \frac{1}{\sqrt{4}}(\frac{1}{\sqrt{2}} - 1) + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{2(3+1)}} = 0$. It is easy to verify that $G' \in \mathbb{TR}_3^1(n)$. Therefore $R(\mathbb{TR}_5^1(n)) > R(\mathbb{TR}_3^1(n))$ holds.
(3) Let $G \in \mathbb{TR}_4^1(n)$ with undeletable subgraph as U_4^1 in Figure 3. Using similar arguments as (2), we may assume that $d_G^1(v_2) = d_G^1(v_5) = d_G^1(v_6) = 0$, i.e., $d_G(v_2) = d_G(v_5) = d_G(v_6) = 2$.

Let $G' = G - v_3v_2 + v_3v_6$, and let $d_i = d_G(v_i)$. Note that $d_4 \geqslant 5$. Then $R(G) - R(G') > (\frac{1}{\sqrt{d_2d_4}} + \frac{1}{\sqrt{d_4d_6}} + \frac{1}{\sqrt{d_2d_3}} + \frac{1}{\sqrt{d_5d_6}}) - (\frac{1}{\sqrt{d_4}} + \frac{1}{\sqrt{d_4(d_6+1)}} + \frac{1}{\sqrt{d_3(d_6+1)}} + \frac{1}{\sqrt{d_5(d_6+1)}}) = \frac{1}{\sqrt{d_4}}(\sqrt{2} - 1 - \frac{1}{\sqrt{3}}) + \frac{1}{\sqrt{d_3}}(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}} \geqslant \frac{1}{\sqrt{5}}(\sqrt{2} - 1 - \frac{1}{\sqrt{3}}) + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}} \approx 0.01879 > 0$. It is clear that $G' \in \mathbb{TR}_3^1(n)$. Therefore we obtain $R(\mathbb{TR}_4^1(n)) > R(\mathbb{TR}_3^1(n))$ as desired.
(4) Let $G \in \mathbb{TR}_3^2(n)$ with undeletable subgraph as U_3^2 in Figure 2. Let $S = \frac{1}{\sqrt{v_2}} + \frac{1}{\sqrt{v_3}}$. If $d_G^1(v_2) = 0$, i.e., $d_G(v_2) = 3$, then clearly $S \geqslant \frac{1}{\sqrt{3}}$. Otherwise, note that $d_G^2(v_2) = 3, d_G^1(v_2) > 0$, according to (1) of Theorem 3, moving pendant neighbors of v_3 to v_2 decreases $R(G)$ if $d_G^1(v_2) > 0$. So $S \geqslant \frac{1}{\sqrt{2}}$ in this case. As a consequence, we have $S \geqslant \frac{1}{\sqrt{3}}$ by the above argument. Similarly, $\frac{1}{\sqrt{v_5}} + \frac{1}{\sqrt{v_6}} \geqslant \frac{1}{\sqrt{3}}$, implying that $\frac{1}{\sqrt{v_2}} + \frac{1}{\sqrt{v_3}} + \frac{1}{\sqrt{v_5}} + \frac{1}{\sqrt{v_6}} > \frac{2}{\sqrt{5}}$. And notice that $d_G^2(v_1) = d_G^2(v_4) = 3$ and $N_G(v_1) \bigcap N_G(v_4) = \varnothing$. Appealing to (3) of Theorem 2, graph $G' = \Gamma(G, v_1, v_4)$ satisfies $R(G) > R(G')$ with $G' \in \mathbb{TR}_3^1(n)$. Therefore we have $R(\mathbb{TR}_3^2(n)) > R(\mathbb{TR}_3^1(n))$. □

Before proceeding with more relations, let us define some essential functions and graph classes. Let

$$F_1(n) = \frac{n - 4 + \sqrt{3}}{\sqrt{n-1}} + 1,\; F_2(n) = \frac{n - \frac{9}{2} + \frac{3\sqrt{2}}{2}}{\sqrt{n-1}} + \frac{3\sqrt{2}}{4},$$
$$F_3(n) = \frac{n - 5 + \frac{2\sqrt{3}}{3} + \sqrt{2}}{\sqrt{n-1}} + \frac{1+\sqrt{6}}{3}.$$

For $i = 1, 2, 3$, let $TR_i^*(n)$ be n-vertex graphs in $\mathbb{TR}_i^1(n)$ with a vertex of degree $n-1$. It is worth noting that all pendant vertices of $TR_i^*(n)$ are adjacent to a single vertex. Further, it can be verified easily that $TR_1^*(n) \cong TR_n^{\bigstar}$.

Lemma 16. *If $G \in \mathbb{TR}_1^1(n)$, then $R(G) \geqslant F_1(n)$ with equality if and only if $G \cong TR_1^*(n)$.*

Proof. If $G \cong TR_1^*(n)$, then clearly $R(G) = F_1(n)$. If $G \ncong TR_1^*(n)$, at least two vertices of $\phi(G)$ have pendant neighbors. Suppose the undeletable subgraph $\phi(G)$ is labelled as U_1^1 in Figure 2. Without loss of generality, we may assume that $d_G^1(v_1) \geqslant d_G^1(v_2) \geqslant d_G^1(v_3) \geqslant d_G^1(v_4)$. Note that $N_G^2(v_1) \backslash v_2 = N_G^2(v_2) \backslash v_1 = \{v_3, v_4\}$. By Lemma 9, moving pendant neighbors of v_2 to v_1 will decrease $R(G)$ if $d_G^1(v_2) > 0$. Similarly, this holds for v_3 and v_4. So we can conclude that $R(G) > F_1(n)$ if $G \ncong TR_1^*(n)$. So the proof is complete. □

Lemma 17. *If $G \in \mathbb{TR}_2^1(n)$, then $R(G) \geqslant F_2(n)$ with equality if and only if $G \cong TR_2^*(n)$.*

Proof. Suppose the undeletable subgraph $\phi(G)$ is labelled as U_2^1 in Figure 2. If $G \cong TR_2^*(n)$, i.e., one of v_1, v_2 is adjacent to all pendant neighbors, then obviously $R(G) = F_2(n)$. Then let us consider the case $G \ncong TR_2^*(n)$.
Case 1. $d_G^1(v_1) > 0, d_G^1(v_2) > 0, d_G^1(v_3) = d_G^1(v_4) = d_G^1(v_5) = 0$.

It is easy to see that $N_G^2(v_1) \backslash v_2 = N_G^2(v_2) \backslash v_1 = \{v_3, v_4, v_5\}$. By Lemma 9, $R(G)$ can be reduced by moving pendant neighbors of v_2 to v_1, implying $R(G) > F_2(n)$.

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Case 2. one of v_3, v_4, v_5 has pendant neighbors.

We may assume that $d^1_G(v_3) > 0, d^1_G(v_4) = d^1_G(v_5) = 0$. Notice that $d^2_G(v_1) = 4, d_G(v_4) = 2$, and $v_4 \in N_G(v_1)\backslash\{v_2, v_3\}$. By (2) of Theorem 3, we can move pendant neighbors of v_3 to v_1 to reduce $R(G)$. Thus we have $R(G) > F_2(n)$.

Case 3. at least two of v_3, v_4, v_5 have pendant neighbors.

Suppose $d^1_G(v_3) > 0, d^1_G(v_4) > 0$ without loss of generality. Note that $N^2_G(v_3)\backslash v_4 = N^2_G(v_4)\backslash v_3 = \{v_1, v_2\}$. Then again appealing to Lemma 9, pendant neighbors of v_4 can be moved to v_3 with $R(G)$ decreased. Then we arrive at *Case 2*, thus $R(G) > F_2(n)$.

Therefore, it completes the proof. □

Lemma 18. *If $G \in \mathbb{TR}^1_3(n)$, then $R(G) \geqslant F_3(n)$ with equality if and only if $G \cong TR^*_3(n)$.*

Proof. Suppose the undeletable subgraph $\phi(G)$ is labelled as U^1_3 in Figure 2. If $G \cong TR^*_3(n)$, i.e., v_1 is adjacent to all pendant vertices, then obviously $R(G) = F_3(n)$. So we suppose that $G \ncong TR^*_3(n)$.

Case 1. $d^1_G(v_2) = d^1_G(v_3) = 0$.

Consider first $d^1_G(v_5) > 0$. And observe that $d^2_G(v_1) = 4, d_G(v_2) = 2$ and $v_2 \in N_G(v_1)\backslash\{v_3, v_5\}$. Then by (2) of Theorem 3, we can move pendant neighbors of v_5 to v_1 and get $R(G)$ smaller. Likewise, this holds for $d^1_G(v_4) > 0$. Therefore we obtain $R(G) > F_3(n)$.

Case 2. one of $d^1_G(v_2) > 0, d^1_G(v_3) > 0$ holds. Without loss of generality, suppose $d^1_G(v_2) > 0, d^1_G(v_3) = 0$.

Subcase 2.1. $d^1_G(v_4) = d^1_G(v_5) = 0$. Let G' be obtained from G by moving pendant neighbors of v_2 to v_1. Observe that $d_G(v_4) = d_G(v_5) = 2, d_G(v_3) = 3$. Let $x = d_G(v_1) \geqslant 4$ and let $y = d_G(v_2) = d^1_G(v_2) + d^1_G(v_2) \geqslant 4$. So $R(G) - R(G') = (\frac{\sqrt{6}}{6} + \frac{y-3+\frac{\sqrt{3}}{3}+\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{x}}}{\sqrt{y}} + \frac{x-4+\frac{\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x}}) - (\frac{\sqrt{6}}{3} + \frac{1}{3} + \frac{x+y-7+\frac{2\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x+y-3}}) = \frac{y-3+\frac{\sqrt{3}}{3}+\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{x}}}{\sqrt{y}} + \frac{x-4+\frac{\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x}}$ $- \frac{x+y-7+\frac{2\sqrt{3}}{3}+\sqrt{2}}{\sqrt{x+y-3}} - (\frac{1}{3} + \frac{1}{\sqrt{6}}) > 0$, where the last inequality holds by Lemma 8. Thus we obtain $R(G) > F_3(n)$.

Subcase 2.2. $d^1_G(v_4) > 0, d^1_G(v_5) = 0$. Observe that $d^2_G(v_4) = 2, d^2_G(v_2) = 3$, moving pendant neighbors of v_4 to v_2 will reduce $R(G)$ from (1) of Theorem 3. Then we arrive at *Subcase 2.1*.

Subcase 2.3. $d^1_G(v_4) = 0, d^1_G(v_5) > 0$. By analogous argument as *Case 1*, we can move pendant neighbors of v_5 to v_1, so we get to *Subcase 2.1*.

Subcase 2.4. $d^1_G(v_4) > 0, d^1_G(v_5) > 0$. Similarly as *Subcase 2.2*, pendant neighbors of v_4 can be moved to v_2 and $R(G)$ will decrease. Then we arrive at *Subcase 2.3*.

According to the 4 subcases, we obtain $R(G) > F_3(n)$ in this case.

Case 3. $d^1_G(v_2) > 0, d^1_G(v_3) > 0$.

Subcase 3.1. $d^1_G(v_4) = d^1_G(v_5) = 0$. Note that $d^2_G(v_2) = d^2_G(v_3) = 3$ and $\sum_{u\in N^2_G(v_2)\backslash v_3} \frac{1}{\sqrt{d_G(u)}} = \sum_{u\in N^2_G(v_3)\backslash v_2} \frac{1}{\sqrt{d_G(u)}} = \frac{1}{\sqrt{d_G(v_4)}} + \frac{1}{\sqrt{d_G(v_1)}}$. By Theorem 9, $R(G)$ can be reduced by moving pendant neighbors of v_3 to v_2. Then we get the *Subcase 2.1*.

Subcase 3.2. $d^1_G(v_4) + d^1_G(v_5) > 0$. Notice that $d^2_G(v_2) = 3, d^2_G(v_4) = 2$. By (1) of Theorem 3, pendant neighbors of v_4 can be moved to v_2 with $R(G)$ decreased if $d^1_G(v_4) > 0$. Analogously, this holds for v_5. Thus we arrive at *Subcase 3.1*.

Now, we can conclude that $R(G) > F_3(n)$ if vertices other than v_1 of $\phi(G)$ have pendant neighbors. Thus the proof is complete. □

Lemma 19. $R(\mathbb{TR}^1_3(n)) > R(\mathbb{TR}^1_2(n)) > R(\mathbb{TR}^1_1(n))$.

Proof. Suppose $\mathbb{A}, \mathbb{B}$ are two graph sets with $\min_{G\in\mathbb{A}} R(G) < \min_{G\in\mathbb{B}} R(G)$. Let $G_A \in \mathbb{A}$ be a graph satisfying $R(G_A) = \min_{G\in\mathbb{A}} R(G)$. Then for any graph $G_B \in \mathbb{B}$, we have $R(G_B) \geqslant \min_{G\in\mathbb{B}} R(G) > R(G_A)$, that is, $R(\mathbb{A}) < R(\mathbb{B})$.

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So by Lemma 16, 17, 18, it suffices to show that $F_3(n) > F_2(n) > F_1(n)$. Observe that $n \geqslant 5$ for $G \in \mathbb{TR}_3^1(n)$ or $G \in \mathbb{TR}_2^1(n)$. Hence $F_3(n) - F_2(n) = \frac{1+\sqrt{6}}{3} - \frac{3\sqrt{2}}{4} + \frac{\frac{2}{\sqrt{3}} - \frac{1}{2} - \frac{1}{\sqrt{2}}}{\sqrt{n-1}} \geqslant \frac{1+\sqrt{6}}{3} - \frac{3\sqrt{2}}{4} + \frac{\frac{2}{\sqrt{3}} - \frac{1}{2} - \frac{1}{\sqrt{2}}}{\sqrt{4}} \approx 0.06297 > 0$. And $F_2(n) - F_1(n) = \frac{3\sqrt{2}}{4} - 1 + \frac{\frac{3}{\sqrt{2}} - \frac{1}{2} - \sqrt{3}}{\sqrt{n-1}} \geqslant \frac{3\sqrt{2}}{4} - 1 + \frac{\frac{3}{\sqrt{2}} - \frac{1}{2} - \sqrt{3}}{\sqrt{4}} \approx 0.005295 > 0$. Therefore the assertion holds clearly. □

We draw all the relations mentioned here in Figure 5, in which $\mathbb{A} \to \mathbb{B}$ represents $R(\mathbb{A}) < R(\mathbb{B})$.

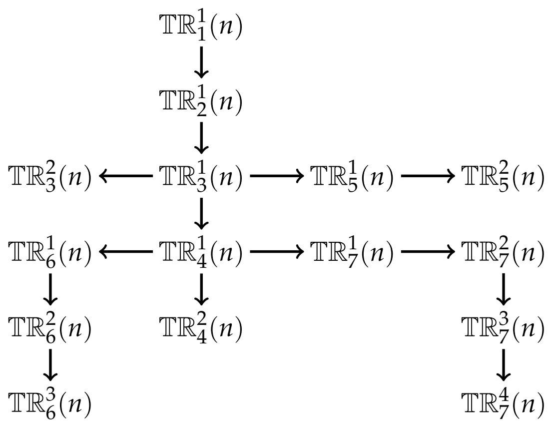

Figure 5. Relations between all $\mathbb{TR}_i^j(n)$

4.3. The proof of Theorem 1

Now we are ready to prove our main result.

Proof of Theorem 1. First note that $F_1(n) = \frac{n-4+\sqrt{3}}{\sqrt{n-1}} + 1$ and $TR_1^*(n) \cong TR_n^\bigstar$, so it is equivalent to show that $R(G) \geqslant F_1(n)$ with equality if and only if $G \cong TR_1^*(n)$. Let $\mathcal{F} = \bigcup \mathbb{TR}_i^j(n)$ be the union of all $\mathbb{TR}_i^j(n)$.

If $G \cong TR_1^*(n)$, it is clear that $R(G) = F_1(n)$.

If $G \in \mathbb{TR}_1^1(n) \backslash \{TR_1^*(n)\}$, we have $R(G) > F_1(n)$ by Lemma 16.

If $G \in \mathcal{F} \backslash \mathbb{TR}_1^1(n)$, by Lemma 12, 13, 14, 15, 19 together, we obtain $R(G) > R(TR_1^*(n)) = F_1(n)$.

If $G \notin \mathcal{F}$ and G is pendant-maximized, by Lemma 11, we can find a graph $G' \in \mathcal{F}$ such that $R(G) > R(G') \geqslant F_1(n)$.

If $G \notin \mathcal{F}$ and G is not pendant-maximized, by Lemma 10 and 11, we will again find a graph $G' \in \mathcal{F}$ such that $R(G) > R(G') \geqslant F_1(n)$.

Therefore the theorem holds clearly. □

5. Conclusions

In the current work, we investigate three kinds of graph transformations decreasing Randić index of graphs, which may be valuable for studying relations between Randić index and structure of graphs. For instance, Theorem 4 implies that the pendant neighbors of two vertices of a graph connects to its Randić index predictably. By applying these transformations systematicaly, the minimum Randić index of tricyclic graphs is determined with the corresponding extremal graphs. In fact, the minimum Randić index of trees, unicyclic and bicyclic graphs could be obtained by the analogous method without much effort.

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